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THE DESCENT STATISTIC ON 123-AVOIDING PERMUTATIONS

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ABSTRACT. We exploit Krattenthaler's bijection between 123-avoiding permutations and Dyck paths to determine the Eulerian distribution over the set $S_n(123)$ of 123-avoiding permutations in S_n. In particular, we show that the descents of a permutation correspond to valleys and triple ascents of the associated Dyck path. We get the Eulerian numbers of $S_n(123)$ by studying the joint distribution of these two statistics on Dyck paths.

1. INTRODUCTION

A permutation $\sigma \in S_n$ *avoids a pattern* $\tau \in S_k$ if σ does not contain a subsequence that is order-isomorphic to τ. The subset of S_n of all permutations avoiding a pattern τ is denoted by $S_n(\tau)$. Pattern avoiding permutations have been intensively studied in recent years from many points of view (see e.g. [7], [4], [1], and references therein).

In the case $\tau \in S_3$, it has been shown that the cardinality of $S_n(\tau)$ equals the n-th Catalan number, for every pattern τ (see e.g. [3] and [7]), and hence the set $S_n(\tau)$ is in bijection with the set of Dyck paths of semilength n. Indeed, the six patterns in S_3 are related as follows:

- $321 = 123^{rev}$,
- $231 = 132^{rev}$,
- $312 = 132^{c}$,
- $213 = (132^{c})^{rev}$,

where *rev* and c denote the usual reverse and complement operations. Hence, in order to determine the distribution of the descent statistic over $S_n(\tau)$, for every $\tau \in S_3$, it is sufficient to examine the distribution of descents over two sets, say $S_n(132)$ and $S_n(123)$.

In both cases, the two bijections due to Krattenthaler [4] (see also [2]) allow one to translate the descent statistic into some appropriate statistics on Dyck paths.

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In the case $\tau = 132$, the bijection relies upon the fact that every permutation in $S_(132)$ is uniquely determined by the values and positions of its left-to-right minima. It is straightforward to check that the descents of a permutation $\sigma \in S_n(132)$ coincide with the positions immediately preceding a left-to-right minimum, not counting the first such left-to-right minimum. These positions are in one-to-one correspondence with the valleys of the associated Dyck path, via Krattenthaler's map. Hence, in this case, the descent distribution is described by the Narayana numbers.

In this paper we investigate the case $\tau = 123$. In particular, we exploit Krattenthaler's map to translate the descents of a permutation $\sigma \in S_n(123)$ into peculiar subconfigurations of the associated Dyck path, namely, valleys and triple ascents.

For that reason, we study the joint distribution of valleys and triple ascents over the set $\mathcal{P}_n$ of Dyck paths of semilength n, and we give an explicit expression for its trivariate generating function

$$A(x,y,z) = \sum_{n\geq 0} \sum_{\mathscr{D}\in\mathcal{P}_n} x^n y^{v(\mathscr{D})} z^{ta(\mathscr{D})} = \sum_{n,p,q\geq 0} a_{n,p,q} x^n y^p z^q,$$

where $v(\mathscr{D})$ denotes the number of valleys in $\mathscr{D}$ and $ta(\mathscr{D})$ denotes the number of triple ascents in $\mathscr{D}$. This series specializes to some well known generating functions, such as the generating function of Catalan numbers, Motzkin numbers, Narayana numbers, and sequence A092107 in [8] (see also [5]).

2. Dyck paths

A *Dyck path* of semilength n is a lattice path in the integer lattice $\mathbb{N} \times \mathbb{N}$ starting from the origin, consisting of n up-steps $U = (1,1)$ and n down steps $D = (1,-1)$, never passing below the x-axis.

A *return* of a Dyck path is a down step ending on the x-axis, not counting the last step of the Dyck path. An *irreducible* Dyck path is a Dyck path with no return.

We note that a Dyck path $\mathscr{D}$ can be decomposed according to its last return (*last return decomposition*) into the juxtaposition of a (possibly empty) Dyck path $\mathscr{D}'$ of shorter length and an irreducible Dyck path $\mathscr{D}''$.

For example, the Dyck path $\mathscr{D} = U^5D^2UD^4UDU^3DUD^3$ decomposes into $\mathscr{D}' \bigoplus \mathscr{D}''$, where $\mathscr{D}' = U^5D^2UD^4UD$ and $\mathscr{D}'' = U^3DUD^3$, as shown in Figure 1.

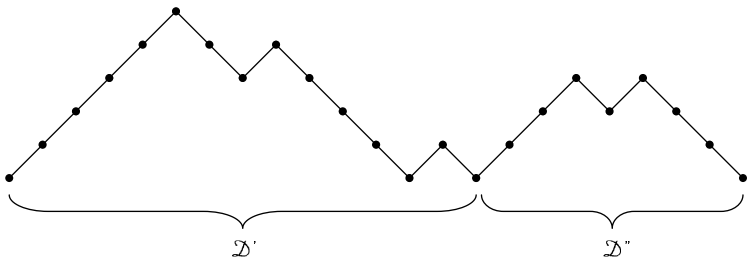

FIGURE 1. The last return decomposition of the Dyck path $\mathscr{D} = U^5D^2UD^4UDU^3DUD^3$.

3. Krattenthaler's bijection

In [4], Krattenthaler describes a bijection between the set $S_n(123)$ and the set $\mathcal{P}_n$ of Dyck paths of semilength n.

Let $\sigma = \sigma(1)\ldots\sigma(n)$ be a 123-avoiding permutation. Recall that a *right-to-left maximum* of σ is an element $\sigma(i)$ which is larger than $\sigma(j)$ for all j with $j > i$ (note that the last entry $\sigma(n)$ is a right-to-left maximum). Let $x_s, \ldots, x_1$ be the right-to-left maxima in σ. Then, we can write

$$\sigma = w_s\, x_s\, \ldots\, w_1\, x_1, \tag{1}$$

where w_i are (possibly empty) words. Moreover, since σ avoids 123, the word $w_s\, w_{s-1}\, \ldots\, w_1$ must be decreasing.

In order to construct the Dyck path $\kappa(\sigma)$ corresponding to σ, read the decomposition (1) from right to left. Any right-to-left maximum x_i is translated into $x_i - x_{i-1}$ up steps (with the convention $x_0 = 0$) and any subword w_i is translated into $l_i + 1$ down steps, where l_i denotes the number of elements in w_i. Then, reflect the constructed path in a vertical line.

For example, the permutation $\sigma = 6\,4\,7\,3\,2\,5\,1$ in $S_7(123)$ corresponds to the path in Figure 2.

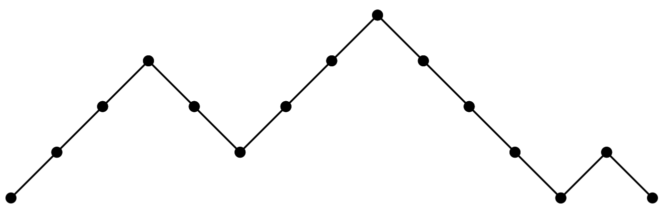

FIGURE 2. The Dyck path $\kappa(\sigma)$, with $\sigma = 6\,4\,7\,3\,2\,5\,1$.

4. The descent statistic

We say that a permutation σ has a *descent* at position i if $\sigma(i) > \sigma(i+1)$. We denote by $\mathrm{des}(\sigma)$ the number of descents of the permutation σ.

In this section we determine the generating function

$$E(x,y) = \sum_{n\geq 0} \sum_{\sigma\in S_n(123)} x^n y^{\mathrm{des}(\sigma)} = \sum_{n\geq 0}\sum_{k\geq 0} e_{n,k} x^n y^k,$$

where $e_{n,k}$ denotes the number of permutations in $S_n(123)$ with k descents.

Proposition 1. *Let σ be a permutation in $S_n(123)$, and $\mathscr{D} = \kappa(\sigma)$. The number of descents of σ is*

$$des(\sigma) = v(\mathscr{D}) + ta(\mathscr{D}),$$

where $v(\mathscr{D})$ is the number of valleys (the number of occurrences of DU) in $\mathscr{D}$ and $ta(\mathscr{D})$ is the number of triple ascents *(the number of occurrences of UUU) in $\mathscr{D}$.*

Proof. Let $\sigma = w_s\, x_s \ldots w_1\, x_1$ be a 123-avoiding permutation. The descents of σ occur precisely in the following positions:

1. between two consecutive symbols in the same word w_i (we have $l_i - 1$ of such descents),
2. after every right-to-left maximum x_i, except for the last one.

The proof is completed as soon as we observe that:

1. every word w_i is mapped into an ascending run of $\kappa(\sigma)$ of length l_i+1. Such an ascending run contains l_i-1 triple ascents, these in their turn are in bijection with the descents contained in w_i,
2. every right-to-left maximum x_i with $i \geq 2$ corresponds to a valley in $\kappa(\sigma)$.

□

The preceding result implies that we can switch our attention from permutations in $S_n(123)$ with k descents to Dyck paths of semilength n with k valleys and triple ascents. Hence, we study the joint distribution of valleys and triple ascents over $\mathcal{P}_n$, namely, we analyze the generating function

$$A(x,y,z) = \sum_{n\geq 0}\sum_{\mathscr{D}\in\mathcal{P}_n} x^n y^{v(\mathscr{D})} z^{ta(\mathscr{D})} = \sum_{n,p,q\geq 0} a_{n,p,q} x^n y^p z^q.$$

We determine the relation between the function $A(x, y, z)$ and the generating function

$$B(x,y,z) = \sum_{n\geq 0} \sum_{\mathscr{D}\in \mathcal{IP}_n} x^n y^{v(\mathscr{D})} z^{ta(\mathscr{D})} = \sum_{n,p,q\geq 0} b_{n,p,q} x^n y^p z^q$$

of the same joint distribution over the set $\mathcal{IP}_n$ of irreducible Dyck paths in $\mathcal{P}_n$.

Proposition 2. *For every $n > 2$, we have:*

$$b_{n,p,q} = a_{n-1,p,q-1} - a_{n-2,p-1,q-1} + a_{n-2,p-1,q}. \tag{2}$$

Proof. An irreducible Dyck path of semilength n with p valleys and q triple ascents can be obtained by prepending U and appending D to a Dyck path of semilength $n-1$ of one of the two following types:

1. a Dyck path with p valleys and q triple ascents, starting with UD,
2. a Dyck path with p valleys and $q-1$ triple ascents, which does not start with UD.

We observe that:

1. The paths of the first kind are in bijection with Dyck paths of semilength $n-2$ with $p-1$ valleys and q triple ascents, enumerated by $a_{n-2,p-1,q}$.

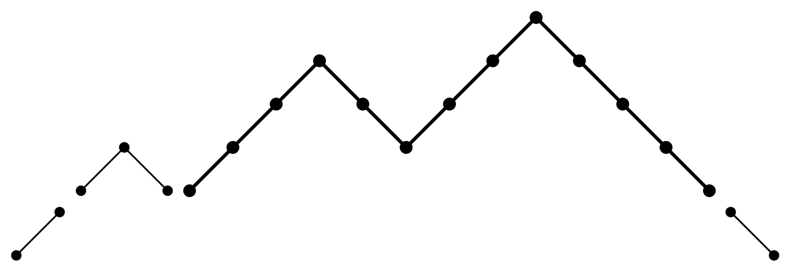

FIGURE 3. The Dyck path $U^2DU^3D^2U^3D^5$ with 2 valleys and 2 triple ascents is obtained by prepending UD to the path $U^3D^2U^3D^4$ with 1 valley and 2 triple ascents, and then elevating.

2. In order to enumerate the paths of the second kind we have to subtract from the integer $a_{n-1,p,q-1}$ the number of Dyck paths of semilength $n-1$ with p valleys and $q-1$ triple ascents, starting with UD. Dyck paths of this kind are in bijection with Dyck paths of semilength $n-2$ with $p-1$ valleys and $q-1$ triple ascents, enumerated by $a_{n-2,p-1,q-1}$.

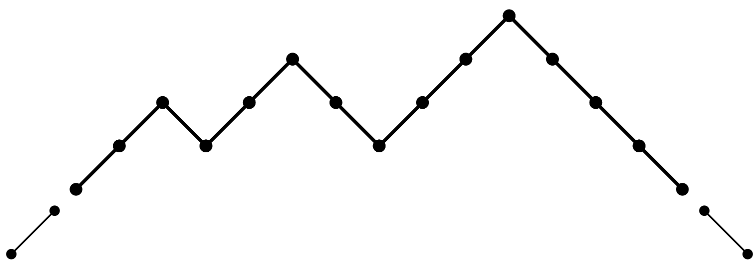

FIGURE 4. The Dyck path $U^3DU^2D^2U^3D^5$ with 2 valleys and 2 triple ascents is obtained by elevating the path $U^2DU^2D^2U^3D^4$ with 2 valleys and 1 triple ascent.

□

Proposition 3. *For every $n > 0$, we have:*

$$a_{n,p,q} = b_{n,p,q} + \sum_{i=1}^{n-1} \sum_{j,s \geq 0} b_{i,j,s} a_{n-i,p-j-1,q-s}. \tag{3}$$

Proof. Let $\mathscr{D}$ be a Dyck path of semilength n and consider its last return decomposition $\mathscr{D} = \mathscr{D}' \bigoplus \mathscr{D}''$. If $\mathscr{D}'$ is empty, then $\mathscr{D}$ is irreducible. Otherwise, we have

- $v(\mathscr{D}) = v(\mathscr{D}') + v(\mathscr{D}'') + 1$,
- $ta(\mathscr{D}) = ta(\mathscr{D}') + ta(\mathscr{D}'')$.

□

Identities (2) and (3) yield the following relations between the two generating functions $A(x, y, z)$ and $B(x, y, z)$.

Proposition 4. *We have*

$$B(x, y, z) = (A(x, y, z) - 1)(xz + x^2y - x^2yz) + 1 + x + x^2 - x^2z \tag{4}$$

and

$$A(x, y, z) = B(x, y, z) + y(B(x, y, z) - 1)(A(x, y, z) - 1). \tag{5}$$

Proof. Note that recurrence (2) holds for $n > 2$. This fact gives rise to the correction terms of x-degree less than 3 in Formula (4). □

Combining Formulae (4) and (5) we obtain the following result.

Theorem 5. *We have:*

$$\begin{aligned} A(x,y,z) = \frac{1}{2xy(xyz-z-xy)} \Big(& -1 + xy + 2x^2y \\ & - 2x^2y^2 + xz - 2xyz - 2x^2yz + 2x^2y^2z \\ & + \sqrt{1-2xy-4x^2y+x^2y^2-2xz+2x^2yz+x^2z^2} \Big). \end{aligned} \tag{6}$$

This last result allows us to determine the generating function $E(x,y)$ of the Eulerian distribution over $S_n(123)$. In fact, the previous arguments show that

$$E(x,y) = A(x,y,y).$$

Hence, we obtain the following explicit expression for $E(x,y)$.

Theorem 6. *We have:*

$$E(x,y) = \frac{-1+2xy+2x^2y-2xy^2-4x^2y^2+2x^2y^3 + \sqrt{1-4xy-4x^2y+4x^2y^2}}{2xy^2(xy-1-x)}.$$

The first values of the sequence $e_{n,d}$ are shown in the following table:

n/d	0	1	2	3	4	5	6
0	1						
1	1						
2	1	1					
3	0	4	1				
4	0	2	11	1			
5	0	0	15	26	1		
6	0	0	5	69	57	1	
7	0	0	0	56	252	120	1

Needless to say, the series $A(x,y,z)$ specializes to some well known generating functions. In particular, $A(x,1,1)$ is the generating function of Catalan numbers, $A(x,1,0)$ the generating function of Motzkin numbers, $yA(x,y,1)$ the generating function of Narayana numbers, and $A(x,1,z)$ the generating function of seq. A092107 in [8].

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