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TABLEAUX IN THE WHITNEY MODULE OF A MATROID

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Abstract. The Whitney module of a matroid is a natural analogue of the tensor algebra of the exterior algebra of a vector space that takes into account the dependencies of the matroid.

In this paper we indicate the role that tableaux can play in describing the Whitney module. We will use our results to describe a basis of the Whitney module of a certain class of matroids known as freedom (also known as Schubert, or shifted) matroids. The doubly multilinear submodule of the Whitney module is a representation of the symmetric group. We will describe a formula for the multiplicity hook shapes in this representation in terms of the no broken circuit sets.

1. Introduction and Motivation

If V is a complex vector space of dimension k we let $\bigwedge V$ denote the exterior algebra of V, $T^n(V)$ the n-fold tensor product $V^{\otimes n}$ and $T(V)$ the tensor algebra $\bigotimes_{n\geq 0} T^n(V)$. For the moment we will only be concerned with the $\mathrm{GL}(V)$-*module* structure of $T(\bigwedge V)$. We begin by seeing how tableaux describe a basis for the $\mathbb{C}$-vector space $T(\bigwedge V)$.

Recall that the irreducible polynomial representations of $\mathrm{GL}(V)$ are indexed by partitions λ with length at most $\dim V$. We denote the irreducible representation with highest weight λ by $\mathbb{S}^\lambda(V)$. It follows from the Weyl character formula that the dimension of $\mathbb{S}^\lambda(V)$ is the number of column strict tableaux of shape λ with entries in $[\dim V] := \{1, 2, \ldots, \dim V\}$. Using Young's Rule we obtain the $\mathrm{GL}(V)$-module decomposition of the tensor product of exterior products:

$$\bigwedge^{\mu_1} V \otimes \bigwedge^{\mu_2} V \otimes \cdots \otimes \bigwedge^{\mu_\ell} V = \bigoplus_{\lambda:\ell(\lambda)\leq k} (\mathbb{S}^\lambda V)^{\oplus K_{\lambda',\mu}}.$$

Here $K_{\lambda',\mu}$ is the number of column strict tableaux of shape λ' (the conjugate partition of λ) that contain μ_i i's. From this, one easily deduces the following.

Theorem 1.1. *The tensor algebra $T(\bigwedge V)$ has a basis indexed by pairs of tableaux (T_r, T_c) of the same shape where T_r has strictly increasing rows, weakly increasing columns and entries in $[\dim V]$ and T_c is column strict with arbitrary entries.*

The Whitney module of a matroid M, $W(M)$, will be a quotient of a certain letter-place algebra that mimics $T(\bigwedge V)$, but takes into account the dependencies of the matroid M. Its definition is slightly more natural than the closely related Whitney algebra of a matroid, which was defined by Crapo, Rota and Schmitt in [6]. In the final

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section of [6] the Whitney module of a matroid is roughly described in passing. The goal of this paper is to begin to investigate how tableaux play a role in describing the structure of $W(M)$. Our main result is that the obvious spanning set of $W(M)$ is a basis when M is a *freedom matroid* (also known as Schubert, or shifted matroids). We will elaborate on and prove the following result.

Theorem 1.2. *Let M be a freedom matroid on n elements. There is a basis for its Whitney module indexed by pairs of tableaux (T_r, T_c) where T_r and T_c have the same shape and*

1. *T_r is row strict with entries in $[n]$,*
2. *every row of T_r indexes an independent set of M, and*
3. *T_c is column strict.*

We will also precisely state and prove the following result.

Theorem 1.3. *In the complexified doubly multilinear submodule of the Whitney module of M, a basis for the hook shaped isotypic components are determined by the no broken circuit complex of M.*

The paper is organized as follows. First we define the super algebra $\text{Super}(L^-|P^+)$ and recall the standard basis theorem of Grosshans–Rota–Stein [9]. We then define the Whitney algebra and Whitney module of a matroid and see that the latter is spanned by certain elements indexed by pairs of tableaux, as in the standard basis theorem, with the additional condition that the rows of the row strict tableau index independent subsets of the matroid. We will then define freedom matroids and prove that the given spanning set for its Whitney module forms a basis. Following that, we will define the doubly multilinear submodule of the Whitney module and describe an action of the symmetric group on it. After complexifying this submodule, we will give a formula for the multiplicity of irreducible symmetric group modules indexed by hook shapes. The formula will be in terms of the certain no broken circuit subsets of M.

2. Letter-Place Algebras

In this section we show how to view the tensor algebra of the exterior algebra of a finite set as a letter-place superalgebra. All of the definitions come from [9] and we use the main result there, the standard basis theorem, to describe a combinatorial basis of this object in terms of pairs of tableaux. This gives a concrete explanation of our motivating result Theorem 1.1.

2.1. **Exterior Algebra.** Let E be a finite set and $\bigwedge E$ be the exterior algebra of the free $\mathbb{Z}$-module with basis E. We will write decomposable elements as $w = e_{i_1}e_{i_2}\dots e_{i_k}$, to avoid notational clutter. This is a graded commutative algebra, which means that $\bigwedge E$ is the direct of the i-fold exterior products

$$\bigwedge E = \bigoplus_{0\le i} \bigwedge^i E$$

and if $w \in \bigwedge^i E$ and $w' \in \bigwedge^j E$ then

$$ww' = (-1)^{ij} w'w.$$

If $w \in \bigwedge E$ is homogeneous then we denote the degree of the piece which w is in by $|w|$. The tensor algebra of the graded algebra $\bigwedge E$ is the direct sum of the tensor products $T^1(\bigwedge E)$, $T^2(\bigwedge E)$, ... Each of the summands has its own product, called the internal product, which is induced by the rule

$$(w \otimes w') \times (u \otimes u') = (-1)^{|w'||u|}(wu \otimes w'u').$$

The exterior algebra of E is a graded commutative Hopf algebra, with coproduct

$$\delta : \bigwedge E \to \bigwedge E \otimes \bigwedge E$$

induced by the rule $\delta(e) = 1 \otimes e + e \otimes 1$. We will not need the definitions of the counit or antipode here. Given an element $w \in \bigwedge E$ we write its coproduct using Sweedler notation

$$\delta(w) = \sum_w w_{(1)} \otimes w_{(2)}.$$

The iterated coproduct $\delta^{(n)} : \bigwedge E \to T^n(\bigwedge E)$ is defined by the conditions that $\delta^{(1)}$ equals the identity map, $\delta^{(2)} = \delta$ and $\delta^{(n)} = (\delta \otimes 1) \circ \delta^{(n-1)}$. The iterated coproduct $\delta^{(n)}$ is the sum of its homogeneous pieces

$$\delta^{(\alpha)} : \bigwedge E \to (\bigwedge^{\alpha_1} E) \otimes (\bigwedge^{\alpha_2} E) \otimes \cdots \otimes (\bigwedge^{\alpha_n} E)$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a composition with n parts. The image w under $\delta^{(\alpha)}$ is the α-th coproduct slice of w.

2.2. **Letter-Place Algebras.** The goal of this section is to view $T(\bigwedge E)$ as one of the letter-place algebras of Grosshans–Rota–Stein [9]. This will be done by constructing two algebras, one of non-commutative letters the other of commutative places, and from these defining a new algebra of letter-place pairs.

We will not review the complete definition of the letter-place algebras, since this some amount of work to do precisely. Instead, we will take those pieces of the definitions suited for our needs, hinting at the form of the general definitions.

In this section we will declare the elements of E to be negatively signed and refer to them as *negative letters*. To emphasize that we are viewing E in this way, we will denote it by L^- (or sometimes L^-_E when we must emphasize the set E). Let $P^+ = \{p_1, p_2, p_3, \ldots\}$ denote the infinite set of *positively signed places*. We associate to L^- and P^+ two algebras, the exterior algebra and the divided power algebra, respectively, and associate to these the so-called letter-place algebra. Will write the exterior algebra of L^- as $\text{Super}(L^-)$, which is constructed exactly as in the previous section.

We recall the definition of the divided power algebra. Associate to each element $p \in P^+$ an infinite sequence of *divided powers*

$$p^{(0)}, p^{(1)}, p^{(2)}, \ldots.$$

We let $\mathbb{Z}\langle P^+_d \rangle$ denote the free algebra generated by the divided powers. Let $\text{Super}(P^+)$ be the quotient of $\mathbb{Z}\langle P^+_d \rangle$ by the two-sided ideal generated by the elements of the form

$$p^{(j)}p^{(k)} - \binom{j+k}{k} p^{(j+k)}, \quad p^{(j)}q^{(k)} - q^{(k)}p^{(j)}, \quad p^{(0)} - 1,$$

for any $p, q \in P^+$, $p \neq q$. The elements $p^{(j)}$ are meant to behave like $p^i/i!$ in a symmetric algebra. We can endow $\mathrm{Super}(P^+)$ with a coalgebra structure by defining the coproduct of $p^{(j)}$, $p \in P$ as

$$\delta(p^{(j)}) = 1 \otimes p^{(j)} + p^{(1)} \otimes p^{(j-1)} + \cdots + p^{(j-1)} \otimes p^{(1)} + p^{(j)} \otimes 1.$$

As before we write the coproduct of an arbitrary element of $q \in \mathrm{Super}(P^+)$ in Sweedler notation,

$$\delta(q) = \sum_q q_{(1)} \otimes q_{(2)}.$$

It is clear that $\mathrm{Super}(P^+)$ is graded and commutative in the usual sense of commutative algebra.

Finally we are in a positive to define the letter-place algebra $\mathrm{Super}(L^-|P^+)$. Let $(L^-|P^+)$ denote the set of letter-place pairs:

$$(L^-|P^+) = \{(e|p) : e \in L^-, p \in P^+\}.$$

Since L^- consists of negatively signed variables and P^+ consists of positively signed variables, we declare the letter-place pairs to be negatively signed. We define $\mathrm{Super}(L^-|P^+)$ to be the exterior algebra of the set $(L^-|P^+)$.

To describe a standard basis of $\mathrm{Super}(L^-|P^+)$ we define a certain bilinear map

$$\Omega : \mathrm{Super}(L^-) \times \mathrm{Super}(P^+) \to \mathrm{Super}(L^-|P^+),$$

called the *Laplace pairing*, according to a sequence of rules.

R1. $\Omega(1, 1) = 1$.
R2. If $e \in L^-$ and $p \in P^+$ then $\Omega(e, p^{(1)}) = (e|p)$.
R3. If w and p are not in the same graded piece of $\mathrm{Super}(L^-)$ and $\mathrm{Super}(P^+)$, respectively, then $\Omega(w, p) = 0$.
R4. If $\delta(p) = \sum_{(p)} p_{(1)} \otimes p_{(2)}$ then

$$\Omega(ww', p) = \sum_{(p)} \Omega(w, p_{(1)})\Omega(w', p_{(2)}).$$

R4′. If $\delta(w) = \sum_{(w)} w_{(1)} \otimes w_{(2)}$ then

$$\Omega(w, pp') = \sum_{(w)} \Omega(w_{(1)}, p)\Omega(w_{(2)}, p').$$

That the rules R4 and R4′ are equivalent follows from a series of technical checks, which was done [9]. There they give a more general definition of the Laplace pairing was given that includes the possibility of both positively and negatively signed letters and places. In the future we will denote the Laplace pairing of w and q by $(w|q)$, so that elements of the form $(efg|p_1^{(2)}p_2)$ make sense.

Proposition 2.1. *Viewing $T(\bigwedge E)$ as a $\mathbb{Z}$-algebra with the internal product, there is a surjection of $\mathbb{Z}$-algebras*

$$T(\bigwedge E) \to \mathrm{Super}(L^-, P^+),$$

that maps a tensor $w_1 \otimes w_2 \otimes \cdots \otimes w_n$ to the product

$$(w_1|p_1^{|w_1|})(w_2|p_2^{|w_2|}) \ldots (w_n|p_1^{|w_n|}).$$

Example 2.2. Suppose that $e, f \in E = L^-$. Then

$$e \otimes e \otimes f \mapsto (e|p_1)(e|p_2)(f|p_3).$$

We can verify that $ef + fe \mapsto 0$. We know that

$$\delta(p_1^{(2)}) = 1 \otimes p_1^{(2)} + p_1^{(1)} \otimes p_1^{(1)} + p_1^{(2)} \otimes 1,$$

hence

$$ef \mapsto (e|1)(f|p_1^{(2)}) + (e|p_1)(f|p_1) + (e|p_1^{(2)})(f|1) = (e|p_1)(f|p_1) = -(f|p_1)(e|p_1),$$

since $\mathrm{Super}(L^-|P^+)$ is an exterior algebra.

The preimage of $(e|p_2)$ consists of elements of the form

$$1 \otimes e \otimes 1 \otimes \cdots \otimes 1$$

since, e.g.,

$$1 \otimes e \mapsto (1|p_1^{(0)})(e|p_2) = (1|p_1^{(0)})(e|p_2) = (e|p_2).$$

Given a composition $\alpha = (\alpha_1, \dots, \alpha_n)$ we define

$$p^{(\alpha)} := p_1^{(\alpha_1)} p_2^{(\alpha_2)} \dots p_n^{(\alpha_n)}$$

Proposition 2.3. *Let $w \in \bigwedge E$ be a decomposable element. The image of the coproduct slice $\delta^{(\alpha)}(w)$ in $\mathrm{Super}(L^-|P^+)$ is the Laplace pairing $(w|p^{(\alpha)})$.*

Proof. The result is easy to verify if $w = e \in L^-$. If $|w| > 1$ then we may write $w = w'w''$. By the homogeneity of the coproduct and induction we have

$$\delta(w'w'') = \sum_{\beta+\gamma=\alpha} \delta^{(\beta)}(w')\delta^{(\gamma)}(w'') \mapsto \sum_{\beta+\gamma=\alpha} (w'|p^{(\beta)})(w''|p^{(\gamma)}).$$

Since we have $\delta(p^{(\alpha)}) = \sum_{\beta+\gamma=\alpha} p^{(\beta)} \otimes p^{(\gamma)}$, we can rule R4 to write this as $(w'w''|p^{(\alpha)})$. $\square$

2.3. **The Standard Basis Theorem.** From the computation of the $\mathrm{GL}(V)$-module structure of $T(\bigwedge V)$ in the introduction, we expect $\mathrm{Super}(L^-|P^+)$ to have a basis indexed by pairs of tableaux of the same shape where one is row strict and the other is column strict. This is the case, and in this section we recall how to construct this basis.

Let λ be a decreasing sequence of non-negative integers; *a partition*. The *length* of λ is the number of positive integers in the sequence. We will identify λ with its *Young frame*, which is a collection of boxes, north-east justified, the number of boxes in the i-th row being equal to λ_i. Denote the total number of boxes in the Young frame of λ by $|\lambda|$. A *tableau* T is a filling of the elements of A into the boxes of a partition λ. If T is a tableau we will call the partition λ the *shape* of T and write $sh(T)$. For example

3	2	5	4	2
3	3	3	2	
1	2			
1	2			

is a tableau on $\{1, 2, 3, 4, 5\}$ whose shape is $(5, 4, 2, 2)$. The *content* of a tableau is the number of 1's, the number of 2's, ... that appear in the filling. We will write the content of a tableau as a composition whose i-th part is the number of i's in the filling of the tableau. Thus the tableau above content $(2, 5, 4, 1, 1, 0, 0, \dots)$. A *column strict tableau* is a tableau where the numbers in each row weakly increase and the numbers in each column strictly increase. A *row strict tableau* is a tableau where the numbers in each column weakly increase and the numbers in each row strictly increase. We will call a tableaux T a standard Young tableaux if it is both row and column strict and has entries in $\{1, 2, \dots, |sh(T)|\}$.

Let T and S be tableau of the same shape λ and length ℓ. Let the numbers in the i-th row of T be $t_1, \dots, t_{\lambda_i}$, in order. Define w_i to be the product in $\mathrm{Super}(L^-)$ of the elements indexed by $t_1, \dots, t_{\lambda_i}$, i.e., $w_i = e_{t_1} \dots e_{t_{\lambda_i}}$. Let $s_1, \dots, s_{\lambda_i}$ be the elements in the i-th row of S, in order. Define q_i to be the product in $\mathrm{Super}(P^+)$ of the elements indexed by $s_1, \dots, s_{\lambda_i}$, where if $s_j = \dots = s_{j+k-1}$ is a maximal string of equal entries then we take $p_{s_j}^{(k)}$ instead of the product $p_{s_j} \dots p_{s_{j+k-1}}$. For example, if

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 1 & 2 & 3 & \\ \cline{1-3} 3 & & & \\ \cline{1-1}\end{array} \qquad S = \begin{array}{|c|c|c|c|}\hline 3 & 3 & 4 & 3 \\ \hline 5 & 5 & 5 & \\ \cline{1-3} 6 & & & \\ \cline{1-1}\end{array}$$

then

$$w_1 = w_2 = e_1 e_2 e_3, \quad w_2 = e_3, \qquad q_1 = p_3^{(2)} p_4^{(1)} p_3, \quad q_2 = p_5^{(3)}, \quad q_3 = p_6^{(1)}.$$

We define $\mathrm{tab}(T|S)$ by the formula

$$\mathrm{tab}(T|S) = (w_1|q_1)(w_2|q_2) \dots (w_\ell|q_\ell) \in \mathrm{Super}(L^-|P^+),$$

which makes sense according to our definition of the Laplace pairing. We call such an element a tableaux in $\mathrm{Super}(L^-|P^+)$. We are finally in a position to state the main result of this section.

Theorem 2.4 (Grosshans–Rota–Stein [9])**.** *The elements* $\mathrm{tab}(T_r|T_c)$*, where*

(1) T_r *and* T_c *are tableaux of the same shape,*
(2) T_r *is row strict with entries in* $[n]$*,*
(3) T_c *is column strict,*

form a basis for the free module $\mathrm{Super}(L^-|P^+)$*. We will call such tableaux standard.*

In the expansion of $\mathrm{tab}(T|S)$ *as a sum of standard tableaux* $\sum_i c_i \mathrm{tab}(T_i|S_i)$ *we have that the shape of each* T_i *is larger than or equal to the shape of* T*, in dominance order. Further, the content of every* T_i *is equal to the content of* T *and the content of every* S_i *is equal to the content of* S*.*

3. The Whitney Algebra and Module of a Matroid

We assume that the reader is familiar with the basic concepts in matroid theory (see, e.g., [11]).

In this section we define the Whitney algebra and Whitey module of matroid. We then show that if M is realizable over $\mathbb{C}$ then the standard tableau pairs of the previous section have nonzero image in the Whitney module of M if and only if the rows of the first tableau index independent sets of M.

3.1. **Definitions.** Let M be a matroid on E of rank $r(M)$. Decomposable elements of $\bigwedge E$ are given by words on E. We say that a decomposable element $e_{i_1} e_{i_2} \dots e_{i_k} \in \bigwedge E$ is a dependent word if $\{i_1, i_2, \dots, i_k\}$ is a dependent set in M. Likewise we define independent words.

Definition 3.1 (Crapo–Schmitt [6])**.** The Whitney algebra of a matroid M, denoted $\mathcal{W}(M)$, is the quotient of $T(\bigwedge E)$ by the ideal generated by the elements

$$\delta_\alpha(w)$$

where w is a dependent word in M and α is a composition of $|w|$.

The following definition was also given by Brini and Regonati (unpublished, [4]).

Definition 3.2. The Whitney module of a matroid M, denoted $W(M)$, is defined to be the quotient of $\mathrm{Super}(L^-|P^+)$ by the two-sided ideal generated by the elements

$$(w|p^{(\alpha)})$$

where w is a dependent word in M and α is a composition of $|w|$.

Using rule R4 it is clear that it is sufficient to take w to be the word of a circuit of M in the definition of $W(M)$.

Proposition 2.3 implies that there is a surjective map $\mathcal{W}(M) \to W(M)$ that takes the internal product of $\mathcal{W}(M)$ to the product that $W(M)$ inherits as a quotient of an exterior algebra. One can think of $W(M)$ as being obtained from $\mathcal{W}(M)$ by appending a half-infinite string of the form $1 \otimes 1 \otimes 1 \otimes \dots$ to the right of every element of $\mathcal{W}(M)$ (see the comments at the end of [6]).

Since $\mathrm{Super}(L^-|P^+)$ is a graded commutative algebra and the ideal defining $W(M)$ is homogeneous, $W(M)$ inherits a grading. Each of the graded pieces is a finitely generated $\mathbb{Z}$-module, and hence can be written as the direct sum of a free part and a torsion part.

Proposition 3.3. *There is a direct sum decomposition*

$$W(M) = W(M)_{free} \oplus W(M)_{tor}$$

where $W(M)_{free}$ is free and $W(M)_{tor}$ is torsion.

It is a basic example of Crapo and Schmitt [6] that if M is not realizable over a field of characteristic zero then $W(M)_{tor}$ can be non-zero. It is unknown if $W(M)_{tor}$ is zero when M is realizable over a field of characteristic zero.

3.2. **Tableaux in the Whitney Module.** Let T be a tableau with entries in $[n]$, and S an arbitrary tableau of the same shape. Since $W(M)$ is a quotient of $\mathrm{Super}(L^-|P^+)$ we can project the elements $\mathrm{tab}(T|S)$ of Section 2.3 into $W(M)$. Abusing notation, we denote the image of $\mathrm{tab}(T|S)$ in $W(M)$ by $\mathrm{tab}(T|S)$.

Note that every standard tableaux $\mathrm{tab}(T|S)$ can be written as

$$(w_1|p^{(\alpha^1)})(w_2|p^{(\alpha^2)})\dots(w_\ell|p^{(\alpha^\ell)}),$$

where α^i is a composition of $|w_i|$ and $|w_1| \geq |w_2| \geq \dots \geq |w_\ell|$.

Proposition 3.4. *The image of an arbitrary tableaux* $\mathrm{tab}(T|S)$ *in $W(M)$ is zero if some row of T indexes a dependent set of M.*

Proof. This follows since each tableau is a product of elements of the form $(w|p^{(\alpha)})$, and we know that this element is zero in the Whitney module if w is a dependent word. □

The main theorem of this section is the following result.

Theorem 3.5. *Let S be a column strict tableau. If M is realizable over $\mathbb{C}$, the image of image of the tableaux* $\mathrm{tab}(T|S)$ *in $W(M)$ is non-zero if and only if the rows of T index independent sets of M.*

Before we proceed with the proof we set up a nice corollary, that gives us a simple check of whether such a tableaux exists, having prescribed the content and shape of T. The content of the T determines a parallel extension of the labeled matroid M. Indeed if the content of T is μ (a composition with n parts) then the parallel extension is M_μ, which has the μ_i copies of the element e_i.

The *rank partition* of a matroid M is the sequence of numbers $\rho(M) = (\rho_1, \rho_2, \dots)$ determined by the condition that

$$\rho_1 + \rho_2 + \cdots + \rho_k$$

is the size of the largest union of k independent subsets of M. This definition was first given by Dias da Silva in [7] where he proved the following result.

Theorem 3.6 (Dias da Silva [7])**.** *The rank partition of matroid is a partition. There is a partition of the ground set of a loopless matroid M into independent sets of size $\lambda_1 \geq \lambda_2 \geq \dots$ if and only if $\lambda \leq \rho(M)$ in dominance order.*

The following corollary is now immediate from the theorem.

Corollary 3.7. *There is a non-zero tableaux* $\mathrm{tab}(T|S) \in W(M)$ *of shape λ, where S is columns strict and T has content μ, if and only if $\lambda \leq \rho(M_\mu)$ in dominance order.*

To prove Theorem 3.5 we need a lemma.

Lemma 3.8. *Suppose that S is a fixed column strict tableaux of shape λ whose first row gives rise to the element $p^{(\alpha)} \in \mathrm{Super}(P^+)$. Let S' denote S with its first row removed.*

Define two vector spaces: X is the subspace of $\mathrm{Super}(L^-|P^+)\otimes\mathbb{C}$ *spanned by standard tableaux* $\mathrm{tab}(T|S)$ *where T has first row equal containing the numbers $\{1, 2, \dots, \lambda_1\}$. The second vector space X' is the subspace of* $\mathrm{Super}(L^-|P^+)\otimes\mathbb{C}$ *spanned by any standard tableaux* $\mathrm{tab}(T'|S')$.

Then multiplication by

$$(e_1 e_2 \dots e_{\lambda_1} | p^{(\alpha)})$$

induces an isomorphism of vector spaces $X' \to X$.

Proof. This follows directly from the standard basis theorem, since this map takes bases to bases. □

Proof of Theorem 3.5. Let $f : E \to V$ is a realization of M, where V is a complex n-dimensional vector space. After choosing a basis for V, we can identify $\bigwedge V$ with $\mathrm{Super}(L^-)$ and since f is a realization of M, the mapping $(e|p) \mapsto (f(e)|p)$ gives rise to a map of algebras

$$f : W(M) \to \mathrm{Super}(L^-|P^+) \otimes \mathbb{C}, \qquad (w|p^{(\alpha)}) \mapsto (f(w)|p^{(\alpha)}),$$

(compare Proposition 6.3 in [6]). This map will almost always fail to be surjective since M will typically not have rank n. Note that $\mathrm{Super}(L^-|P^+)$ comes with a left $\mathrm{GL}_n(\mathbb{C})$ action, induced by the natural action of $\mathrm{GL}_n(\mathbb{C})$ on $\mathrm{Super}(L^-) \otimes \mathbb{C}$. Taking a limit, there is a corresponding action of $n \times n$ complex matrices on $\mathrm{Super}(L^-|P^+)$.

Suppose that we have the tableau

$$\mathrm{tab}(T|S) = (w_1|p^{(\alpha^1)})(w_2|p^{(\alpha^2)})\dots(w_\ell|p^{(\alpha^\ell)}) \in W(M)$$

where S is column strict of shape λ, length ℓ and $|w_i| = \lambda_i$. Applying the map f from above we obtain

$$f(\mathrm{tab}(T|S)) = (f(w_1)|p^{(\alpha^1)})(f(w_2)|p^{(\alpha^2)})\dots(f(w_\ell)|p^{(\alpha^\ell)}) \in \mathrm{Super}(L^-|P^+)\otimes\mathbb{C}.$$

We will prove by induction on the length of λ that $f(\mathrm{tab}(T|S))$ is not zero provided that $f(w_i) \neq 0$. Since the image of $\mathrm{tab}(T|S)$ is not zero, it must be that $\mathrm{tab}(T|S) \neq 0$ in $W(M)$.

Let A be a generic matrix such that

$$Af(w_1) = e_1e_2\dots e_{\lambda_1}$$

Since A is generic, each of element $A(f(w_i))$ is not zero and decomposable in $\mathrm{Super}(L^-)\otimes\mathbb{C}$. By Lemma 3.8, we have that

$$(e_1e_2\dots e_{\lambda_1}|p^{(\alpha^1)})(Af(w_2)|p^{(\alpha^2)})\dots(A(f(w_\ell))|p^{(\alpha^\ell)})$$

is not zero if and only if

$$(Af(w_2)|p^{(\alpha^2)})\dots(A(f(w_\ell))|p^{(\alpha^\ell)})$$

is not zero. This is not zero by induction. It only remains to check the basis step. This follows since

$$(e_{i_1}\dots e_{i_k}|p^{(\alpha)}),$$

$i_1 < \cdots < i_k$, is a basis element of $\mathrm{Super}(L^-|P^+)\otimes\mathbb{C}$, according to the standard basis theorem. □

Remark 3.9. One cannot remove the hypothesis that S is column strict. For example if M is a boolean matroid (i.e., $W(M) = \mathrm{Super}(L^-|P^+)$) and

$$T = S = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 1 & 2 & 3 \\ \hline \end{array}$$

then

$$\mathrm{tab}(T|S) = (e_1e_2e_3|p_1^{(1)}p_2^{(1)}p_3^{(1)})^2 = 0$$

even though $(e_1e_2e_3|p_1^{(1)}p_2^{(1)}p_3^{(1)}) \neq 0$ in $W(M)$.

It is unknown if Theorem 3.5 holds for any realizable matroid.

4. Freedom Matroids

In this section we define *freedom matroids* and show that the obvious spanning set for their Whitney modules are bases.

4.1. **Definition of a Freedom Matroid.**

Definition 4.1. We will denote the *direct sum of a matroid M with the one element rank one matroid* on the set $\{e\}$ by $M \oplus e$.

Let M be a matroid of rank larger than 0. The *truncation of M to rank $k \leq r(M)$* is the matroid whose bases are those independent sets of M with size k. The truncation of M to rank $r(M) - 1$ will be denoted $T(M)$.

The *principal extension of a matroid M along the improper flat* is the matroid $M + e$ obtained by truncating the direct sum $M \oplus e$ to the the rank of M. That is $M + e = T(M \oplus e)$.

We think of $M + e$ as adding a new element generically to M without increasing its rank.

Definition 4.2. For $i \in \{0, 1\}$ define $M_{(i)}$ to be the rank i matroid on one element e_1. Let s be a binary sequence of length $n > 1$ and s' be the sequence obtained by deleting its last entry s_n. Define a matroid M_s on the set $\{e_1, e_2, \dots, e_n\}$ by setting

$$M_s = \begin{cases} M_{s'} \oplus e_n & s_n = 1, \\ M_{s'} + e_n & s_n = 0. \end{cases}$$

A *freedom matroid* is a labeled matroid of the form M_s for some binary sequence s.

Example 4.3. The freedom matroid associated with the sequence $(1, 0, 1, 0, 1, 0)$ is

$$((((M_1 + e_2) \oplus e_3) + e_4) \oplus e_5) + e_6$$

where M_1 is the one element rank one matroid with ground set $\{e_1\}$. It is represented linearly by the columns of the matrix

$$\begin{bmatrix} 1 & 1 & * & * & * & * \\ & & 1 & 1 & * & * \\ & & & & 1 & 1 \end{bmatrix}$$

where the blank entries are zero and the $*$'d entries are generic elements of $\mathbb{C}$.

Freedom matroids arise in many contexts. They are the matroids associated to a generic element of a Schubert strata of a Grassmannian. They are known to be the matroids whose independence complexes are shifted. They are special cases of lattice-path matroids.

4.2. **The Whitney Module of a Freedom Matroid.**

We can now state and prove our first main theorem.

Theorem 4.4. *Let M be a freedom matroid (in particular, the ground set of M is ordered). The Whitney module $W(M)$ is free and a basis consists of those standard tableaux* $\operatorname{tab}(T_r|T_c)$ *where the rows of T_r index independent sets of M.*

We already know that these tableaux span $W(M)$, so it suffices to prove that $W(M)$ is free and the stated elements are linearly independent when M is a freedom matroid.

This result does not hold in general, as we will see in Section 5. Since uniform matroids are a special case of freedom matroids (they are associated to sequences of the form $(1, 1, \dots, 1, 0, 0, \dots, 0)$) we see that the above theorem describes the Whitney module of uniform matroids, a result obtained in 2000 by Crapo and Schmitt [5].

Lemma 4.5. *Let M and N be matroids. There is an isomorphism of graded algebras*

$$W(M \oplus N) \approx W(M) \otimes W(N),$$

where the $\otimes$ is the super tensor product of graded algebras.

Recall that the super tensor product of two graded $\mathbb{Z}$-algebras H and L has module structure given by the $\mathbb{Z}$-module $H \otimes L$ and algebra structure induced by the formula for multiplying homogeneous elements

$$(h \otimes l)(h' \otimes l') = (-1)^{|h'||l|} hh' \otimes ll'.$$

Proof. We use the following fact: If H and L are graded algebras and I and J are homogeneous ideals of H and L, respectively, then as graded algebras

$$H/I \otimes L/J \approx \frac{H \otimes L}{H \otimes J + I \otimes L}. \tag{1}$$

In the present situation, we suppose that M is a matroid on E and N is a matroid on F. Let L_E^- denote the set of negatively signed letters E and L_F^- denote the set of negatively signed letters F. It is well known that

$$\mathrm{Super}(L_E^-|P^+) \otimes \mathrm{Super}(L_F^-|P^+) \approx \mathrm{Super}(L_E^- \cup L_F^-|P^+). \tag{2}$$

If I is an ideal of $\mathrm{Super}(L_E^-|P^+)$ then under this isomorphism we have

$$I \otimes \mathrm{Super}(L_F^-|P^+) \to \mathrm{Super}(L_E^- \cup L_F^-|P^+) \cdot I, \tag{3}$$

i.e., the ideal $I \otimes \mathrm{Super}(L_F^-|P^+)$ maps to the ideal generated by I in $\mathrm{Super}(L_E^- \cup L_F^-|P^+)$.

To complete the proof all we need to note is that a circuit of $M \oplus N$ is either a circuit of M or a circuit of N. Thus,

$$\begin{aligned} \langle (w|p^\alpha) : w \text{ is a circuit of } M \oplus N \rangle = \langle (w|p^\alpha) &: w \text{ is a circuit of } M \rangle \\ + \langle (w|p^\alpha) &: w \text{ is a circuit of } N \rangle \end{aligned} \tag{4}$$

where here $\langle - \rangle$ denotes taking the ideal in $\mathrm{Super}(L_E^- \cup L_F^-|P^+)$ generated by the elements $-$.

Combining equations (1)—(4) with the definition of the Whitney module we have $W(M) \otimes W(N) \approx W(M \oplus N)$. □

Corollary 4.6. *If $W(M)$ and $W(N)$ are free $\mathbb{Z}$-modules, then so is $W(M \oplus N)$.*

Lemma 4.7. *Suppose that M is a matroid of rank larger than 0 and $W(M)$ is free, so that $W(M)$ has a basis $\mathcal{B}$ consisting of some standard tableaux. Then $W(T(M))$ is free and has a consisting of those standard tableaux in $\mathcal{B}$ whose first row has length less than $r(M)$.*

Proof. It is easy to convince oneself that

$$W(T(M)) = W(M)/\langle \mathrm{tab}(T|S) : \mathrm{sh}(T) = \lambda, \lambda_1 = r(M) \rangle$$

where $\langle - \rangle$ denotes taking the two sided ideal in $W(M)$ generated by the elements $-$. Now, by the standard basis theorem we know that for non-standard tableaux $\mathrm{tab}(T|S)$,

$$\mathrm{tab}(T|S) = \sum_{i=1}^{m} c_i \mathrm{tab}(T_i|S_i)$$

where $\mathrm{sh}(T_i) \geq \mathrm{sh}(T)$ in dominance order and each $(T_i|S_i) \in \mathcal{B}$. This implies that if the first row of T has length $r(M)$ then the first row of every T_i has length at least $r(M)$. It follows that

$$W(T(M)) = W(M)/\langle \mathrm{tab}(T|S) : \mathrm{sh}(T) = \lambda, \lambda_1 = r(M), \mathrm{tab}(T|S) \in \mathcal{B} \rangle$$

Since $W(M)$ is free, the claim now follows. □

Proof of Theorem 4.4. The theorem will follow by induction. We will prove that if the result holds for a matroid M on $\{e_1 < e_2 < \cdots < e_n\}$ then it also holds for $M \oplus e_{n+1}$ and $T(M)$. Since freedom matroids are closed under these operations, it is sufficient to prove the result for the two one element matroids, which is trivial.

For a positive integer m let $W(M)_{\leq m}$ be the subalgebra of $W(M)$ generated by letter place pairs $(e|p)$ where $p \in \{p_1, p_2, \ldots, p_m\}$. By our assumption on $W(M)$ we see that a basis for $W(M)_{\leq m}$ consists of those tableaux $\mathrm{tab}(T_r|T_c)$ where T_r is row strict, T_c is column strict, every row of T_r indexes an independent set of M and every entry of T_c is at most m. This is a module of finite rank, since the shape of every tableau appearing must fit into a m-by-$r(M)$ box.

In light of Lemma 4.5, it is straightforward to convince oneself that

$$W(M \oplus e_{n+1})_{\leq m} = W(M)_{\leq m} \otimes W(\{e_{n+1}\})_{\leq m},$$

where $\{e_{n+1}\}$ denotes the rank one element one element matroid. It follows that the the rank of $W(M)_{\leq m} \otimes W(\{e_{n+1}\})_{\leq m}$ as a $\mathbb{Z}$-module is the number of pairs

$$((T_r, T_c) \quad , \quad (e_{n+1}|p_{i_1})(e_{n+1}|p_{i_2}) \ldots (e_{n+1}|p_{i_k}))$$

where T_r is row strict, T_c is column strict, every row of T_r indexes an independent set of M, the entries of T_c are at most m, and $1 \leq i_1 < i_2 < \cdots < i_k \leq m$. From this pair we produce two new tableaux T_r' and T_c' where

$$T_c' = (((T_c \leftarrow i_1) \leftarrow i_2) \leftarrow \cdots \leftarrow i_k)$$

is obtained by the usual Robinson–Schensted row insertion and T_r' is obtained from T_r by recording the new boxes of T_c' with $n+1$'s.

It follows from the Super RSK Correspondence [2, 10] that this map is bijective and its image consists of pairs of tableaux (T, S) of the same shape where T is row strict with entries in $[n+1]$, S is column strict with entries in $[m]$ and the rows of T index independent subsets of $M \oplus e_{n+1}$. This is because e_{n+1} is in no circuit of the direct sum. This complete the proof of the induction for direct sums.

Suppose now that $W(M)$ is free and a basis consists of those standard tableaux $(T_r|T_c)$ such that every row of T_r indexes an independent set of M. Then Lemma 4.7 proves that the same statement holds for $W(T(M))$. □

5. The Doubly Multilinear Submodule of $W(M)$

The doubly multilinear submodule of $W(M)$ is the submodule generated by elements of the form

$$(e_1|p_{\sigma(1)})(e_2|p_{\sigma(2)}) \ldots (e_n|p_{\sigma(n)}) \in W(M)$$

where σ is any permutation in the symmetric group $\mathfrak{S}_n$. We denote this submodule by $U(M)$. There is a right action of $\mathfrak{S}_n$ on $U(M)$, by permuting places. We will primarily be interested in the complexified version $\mathbb{C} \otimes U(M)$, where M is a matroid realizable over the complex numbers.

The module $U(M)$ arose independently in the thesis of the author, where it was related to the smallest symmetric group (or general linear group) representation containing a fixed decomposable tensor. Recall that $\mathfrak{S}_n$ acts on the right of $V^{\otimes n}$ via place permutation. If $u \in V^{\otimes n}$ is any tensor let $\mathfrak{S}(u)$ be the smallest $\mathfrak{S}_n$-representation in $V^{\otimes n}$ containing u. If $f : E \to V$ is a realization of M, then it is easy to see that the map

$$(e_1|p_1)(e_2|p_2)\dots(e_n|p_n) \mapsto f(e_1) \otimes f(e_2) \otimes \cdots \otimes f(e_n) \in V^{\otimes n}$$

extends to a unique surjective map of $\mathbb{C}\mathfrak{S}_n$-modules

$$\mathbb{C} \otimes U(M) \to \mathfrak{S}(f(e_1) \otimes f(e_2) \otimes \cdots \otimes f(e_n))$$

The latter representation is a subtle projective invariant of the vector configuration $f(E) \subset V$. For example, there is no known example of two different realizations $f, g : E \to V$ of the same matroid such that

$$\mathfrak{S}(f(e_1) \otimes f(e_2) \otimes \cdots \otimes f(e_n)) \not\approx \mathfrak{S}(g(e_1) \otimes g(e_2) \otimes \cdots \otimes g(e_n)),$$

where $\approx$ is isomorphism of $\mathfrak{S}_n$-modules.

5.1. **Which Irreducible Submodules Can Appear in $\mathbb{C} \otimes U(M)$.** The irreducible representations of a the symmetric group $\mathfrak{S}_n$ are parametrized by partitions of n. Given a partition λ and a tableaux T of shape λ with content $(1, 1, \dots, 1)$ we can construct the irreducible $\mathfrak{S}_n$-representation indexed by λ by taking the left or right ideal in $\mathbb{C}\mathfrak{S}_n$ generated by the *Young symmetrizer*

$$c_T = \left(\sum_{\sigma \in R_T} \operatorname{sign}(\sigma)\sigma\right)\left(\sum_{\tau \in C_T} \tau\right)$$

where R_T (respectively, C_T) is the subgroup of $\mathfrak{S}_n$ preserving the rows (respectively the columns) of T. We will say that the partition λ appears in a representation of $\mathfrak{S}_n$ if it contains a submodule isomorphic to the right ideal in $\mathbb{C}\mathfrak{S}_n$ generated by c_T. We say that a partition λ has multiplicity m in $U(M)$ if the $c_T\mathbb{C}\mathfrak{S}_n$-isotypic component of $U(M)$ is isomorphic to a direct sum of exactly m copies of $c_T\mathbb{C}\mathfrak{S}_n$.

Remark 5.1. Our indexing of the irreducible representations of $\mathfrak{S}_n$ is the conjugate of the usual indexing. For example, (n) corresponds to the sign representation and (1^n) corresponds to the trivial representation.

It follows from the our discussion above that if $f : E \to V$ is realization of M in a complex vector space V, and λ appears in

$$\mathfrak{S}(f(e_1) \otimes f(e_2) \otimes \cdots \otimes f(e_n))$$

then λ appears in $\mathbb{C} \otimes U(M)$ with positive multiplicity.

The following result is equivalent to Gamas' Theorem on the vanishing of symmetrized tensors (see [1]).

Theorem 5.2 (Berget [1])**.** *Let $f : E \to V$ be a realization of a matroid M in a complex vector space V. A partition λ appears in*

$$\mathfrak{S}(f(e_1) \otimes f(e_2) \otimes \cdots \otimes f(e_n))$$

if and only if there is a set partition of E into independent sets whose sizes are the part sizes of λ.

Corollary 5.3. *Let M be a matroid realizable over $\mathbb{C}$. The partition λ appears in $U(M)$ if and only if $\lambda \leq \rho(M)$ in dominance order.*

Proof. One direction follows immediately from the previous theorem and Dias da Silva's Theorem 3.6. It remains to prove the converse. The image of an antisymmetrizer

$$\sum_{\sigma \in \mathfrak{S}_k} \operatorname{sign} \sigma \in \mathbb{C}\mathfrak{S}_n,$$

where $k \leq n$, on $(e_1|p_1)(e_2|p_2)\dots(e_n|p_n)$ is

$$(e_1 e_2 \dots e_k | p_1 p_2 \dots p_k)(e_{k+1}|p_{k+1})\dots(e_n|p_n).$$

It follows that if some row of T indexes a dependent set of M then c_T applied to $(e_1|p_1)(e_2|p_2)\dots(e_n|p_n)$ is zero. Since the projector of an $\mathfrak{S}_n$-module to its λ-th isotypic component is, up to a scalar,

$$\sum_{\sigma \in \mathfrak{S}_n} \sigma c_T \sigma^{-1}$$

we conclude that if every tableaux of shape λ has a row indexing a dependent set of M then λ cannot appear in $\mathbb{C} \otimes U(M)$. □

5.2. **Multiplicities of Hook Shapes.** A hook is a partition with at most one part not equal to one. Let λ^k denote the hook whose first part is k, and all other parts are equal to one. In this subsection, we show how the multiplicities of the irreducible $\mathfrak{S}_n$ representations indexed by hook shapes are related to the no broken circuit complex of M. The results of this subsection hold for any matroid, regardless of realizability.

To ease notation for the rest of this section, we assume that the ground set of M is $E = \{1, 2, \dots, n\}$. We define a *broken circuit* of M as a circuit with its smallest element deleted. A subset of the ground set of M is said to be *nbc* if it contains no broken circuits of M. The collection of nbc sets of M is a simplicial complex called the *nbc complex* of M.

Theorem 5.4. *The multiplicity λ^k in $U(M)$ is the number of nbc sets of M of size k which contain the ground set element* 1*.*

Example 5.5. Let M be the matroid realizable over $\mathbb{C}$ be the columns of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Label the columns $1, 2, \dots, 6$, left to right. The circuits of size three of M are $124, 136$, 235, where ijk denotes $\{i, j, k\}$, hence the broken circuits of M are $24, 36, 35$. It follows that the no broken circuit sets of size 3 are

$$123, 125, 126, 134, 145, 146, 156$$

and so the multiplicity of λ^3 in $\mathbb{C} \otimes U(M)$ is 7. For any ordering of the ground set of M, the smallest element is in at most two dependent sets of size 3. It follows that for any ordering of the ground set, the number of standard Young tableaux of shape λ^3 whose first row indexes an independent set of M is at least 8.

Remark 5.6. Even for matroids realizable over $\mathbb{C}$ the standard basis theorem of Theorem 4.4 does not hold. Indeed the previous example proves that the obvious spanning set of hook shaped tableaux $\mathrm{tab}(T_r|T_c) \in W(M)$ where T_r has independent rows cannot be linearly independent.

The nbc sets of M with size k that contain 1 are precisely the nbc bases of the truncation of M to rank k. By the proof of Lemma 4.7 it is sufficient to prove the result when k is equal to the rank of M.

Definition 5.7. If D is a subset of $[n]$ we let $b_D \in \mathbb{C}\mathfrak{S}_n$ denote the antisymmetrizer of the set D, i.e., $b_D = \sum_{\sigma \in \mathfrak{S}_D} \mathrm{sign}(\sigma)\sigma$.

For a given set $B \subset [n]$ we let c_B denote the Young symmetrizer of the tableaux of shape $\lambda^{|B|}$ that has the elements of B in its first row and the remaining elements $[n]-B$ in the rows rows.

It follows directly from the definition of the Young symmetrizer that b_B is a left factor of c_B.

Proposition 5.8. *Let $\langle - \rangle$ denote taking the right ideal in $\mathbb{C}\mathfrak{S}_n$ generated by the elements $-$. There is an isomorphism of $\mathbb{C}\mathfrak{S}_n$-modules*

$$U(M) \approx \mathbb{C}\mathfrak{S}_n / \langle b_D : D \text{ indexes a dependent set of } M \rangle$$

induced by the map that sends $(e_1|p_1)(e_2|p_2)\dots(e_n|p_n)$ to the image of 1 in the quotient.

Proof. The map that sends $1 \in \mathbb{C}\mathfrak{S}_n$ to $(e_1|p_1)(e_2|p_2)\dots(e_n|p_n) \in \mathrm{Super}(L^-|P^+)$ is an isomorphism onto its image. One then verifies that, up to a sign, the image of the antisymmetrizer b_D is the tableaux $\mathrm{tab}(T|S)$ of hook shape where the first row of T and S consists of the numbers in D. □

By taking the $\lambda^{r(M)}$-th isotypic component of the quotient in the above proposition we immediately have the following result.

Corollary 5.9. *Let $\langle - \rangle$ denote taking the right ideal in $\mathbb{C}\mathfrak{S}_n$ generated by the elements $-$. There is an $\mathbb{C}\mathfrak{S}_n$-isomorphism between the $\lambda^{r(M)}$-th isotypic component of $\mathbb{C} \otimes U(M)$ and the quotient*

$$\langle c_B : B \text{ is any set of size } r(M) \rangle / \langle c_D : D \text{ is a dependent set of } M \rangle$$

Lemma 5.10. *Let S be any set of $[n]$, from which we form the Young symmetrizer of the tableaux of shape $\lambda^{|S|}$ that has the elements of S in its first row. In $\mathbb{C}\mathfrak{S}_n$ we have the equality*

$$c_S = \sum_{f \in S} c_{S \cup e - f}(ef)$$

where e is any element not in S.

Proof. We have $c_{S \cup e - f}(ef) = (ef)c_S$ in $\mathbb{C}\mathfrak{S}_n$. It follows that we can write this statement as

$$\left(1 - \sum_{e \in S}(ef)\right) c_S = 0$$

We can write this as

$$b_{S \cup e} c_S / |S|!$$

where $b_{S\cup e}$ is the antisymmetrizer of the set $S \cup e$, since the antisymmetrizer is near idempotent. It is well known that every irreducible that appears in the left or right ideal generated by $b_{S\cup e}$ is larger than $\lambda^{|S|+1}$ in dominance order. Since c_S generates an irreducible of shape $\lambda^{|S|} < \lambda^{|S|+1}$, the product of the two elements must be zero. □

Corollary 5.11. *The quotient*

$$\langle c_B : B \text{ is any set of size } r(M)\rangle / \langle c_D : D \text{ is a dependent set of } M\rangle$$

is generated by the Young symmetrizers of the nbc bases of M.

Proof. Inducting on the number broken circuits contained in a given base, the proof follows at once from the lemma. □

To prove the remainder of the theorem, we will show that the ideal generated by the Young symmetrizers of nbc bases does not meet the ideal generated by the Young symmetrizers of dependent sets. That is, we prove that

$$\langle c_B : B \text{ is an nbc basis of } M\rangle \cap \langle c_D : D \text{ is a dependent set of } M\rangle = 0.$$

This will be done by straightening the latter Young symmetrizers into sums of Young symmetrizers of standard Young tableaux, which in turn is accomplished by a series of somewhat tedious reductions. In the end, the proof comes down to the well known fact that the right ideals generated by Young symmetrizers of standard Young tableaux have intersection equal to zero.

For the rest of this section D will denote a dependent set of M.

Claim 5.12. *We have*

$$\langle c_D : D \text{ contains two circuits}\rangle \subset \langle c_D : 1 \in D\rangle.$$

Proof. If $1 \notin D$ and D contains more than one circuit then each of the sets $D - e \cup 1$ is dependent since $D - e$ is. We have $c_D = \sum_{e\in D} c_{D-e\cup 1}(1e)$ which proves the result. □

The remainder of the proof is adapted from Las Vergnas and Forge [8]. We call a set unicyclic if it contains a unique circuit. Using the circuit elimination axioms it can be shown that D is is unicyclic if and only if it contains an element e such that $D - e$ is independent. The proof of the following claim is exactly the same as the proof of the previous one.

Claim 5.13. *Let $cl(D)$ denote the closure of D in M. We have the inclusion,*

$$\langle c_D : D \text{ unicyclic}, 1 \in cl(D)\rangle \subset \langle c_D : 1 \in D \text{ dependent}\rangle.$$

For the unicyclic sets where $1 \notin cl(D)$ note that we can write $D = I \cup e$, where I is independent and e is the smallest element of the unique circuit of D. For a general independent set I, though, it is possible to choose many elements e such that e is the minimum element of a circuit of $I \cup e$. The external activity of an independent set I, denoted $ex(I)$, is the number of elements e such that $I \cup e$ contains a unique circuit and e is the minimum element of that circuit. Let $Ex(I)$ denote the set of elements e such that e is the minimum element of a circuit of $I \cup e$.

Claim 5.14. *We have the inclusion,*

$$\begin{aligned}\langle c_D : 1 \notin cl(D), D \text{ unicyclic}\rangle \subset{}& \langle c_D : 1 \in D \text{ dependent}\rangle \\ &+ \langle c_{I\cup Ex(I)} : ex(I) = 1, 1 \notin cl(I)\rangle\end{aligned}$$

Proof. Let $I \cup e$ be a unicyclic set that has $ex(I) > 1$ but $1 \notin Ex(I)$. Then there is an element $f \in Ex(I) - e$. We have

$$c_{I\cup e} = \sum_{g\in I} c_{(I-g\cup f)\cup e}(fg)$$

We see that for all $g \in I$, $(I - g \cup f) \cup e$ is unicyclic, does not contain 1 in its closure and is lexicographically smaller than $I \cup e$. Assuming inductively that $c_{(I-g\cup f)\cup e}$ is in the ideal

$$\langle c_D : 1 \in D \text{ dependent}\rangle + \langle c_{I\cup Ex(I)} : ex(I) = 1, 1 \notin cl(I)\rangle$$

we have that $c_{I\cup e}$ is in this ideal too. □

We now straighten the generators of the ideal

$$\langle c_{I\cup Ex(I)} : ex(I) = 1, 1 \notin cl(I)\rangle.$$

If I is independent of rank $r(M) - 1$, has external activity equal to one but does not contain 1 in its closure then $I \cup 1$ is a broken circuit base. However, for all elements $g \in I$, $(I - g \cup 1) \cup Ex(I)$ is a no broken circuit base of M. Since a Young symmetrizer c_S, $|S| = r(M)$ is that of a standard Young tableau if and only if $1 \in S$, we have proved

$$\begin{aligned}\langle c_B : B \text{ an nbc base}\rangle \cap \langle c_D : D \text{ dependent}\rangle \\ = \langle c_B : B \text{ an nbc base}\rangle \cap \langle c_{I\cup ex(I)} : ex(I) = 1, 1 \notin cl(I)\rangle.\end{aligned}$$

Finally, every Young symmetrizer on the last ideal has support on a unique broken circuit base containing 1, so this intersection must be zero. This completes the proof of the theorem.

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