A Graph Model of the Heisenberg-Weyl algebra

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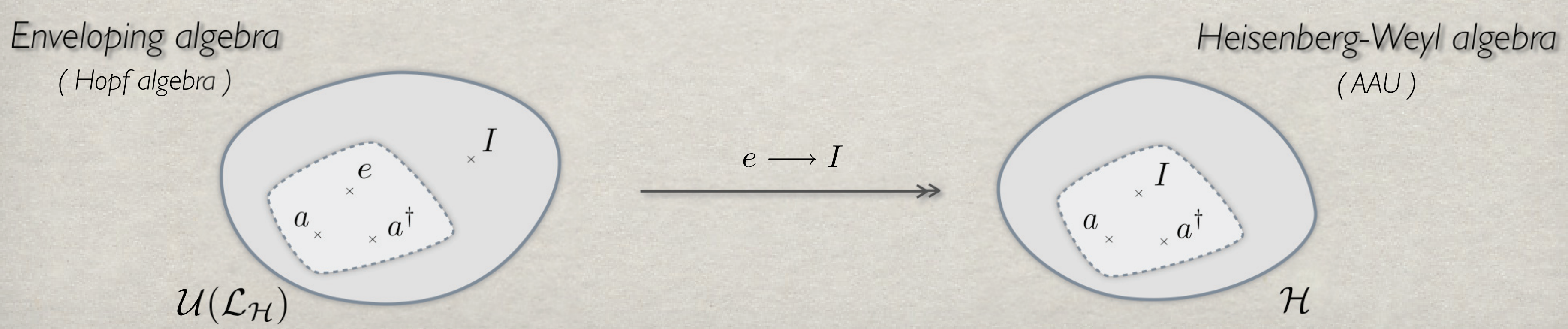
Abstract

The Heisenberg-Weyl algebra, underlying virtually all physical representations of Quantum Theory, is considered from a combinatorial point of view. We construct a concrete model of the algebra in terms of graphs endowed with intuitive concepts of composition and decomposition leading to a rich Hopf algebra structure. The latter encompasses the Heisenberg-Weyl algebra which gains a straightforward interpretation as a shadow of natural constructions on graphs. In this way, by drawing attention to the algebraic structure of Quantum Theory we intend to shed light on the combinatorial nature hidden behind its formalism.



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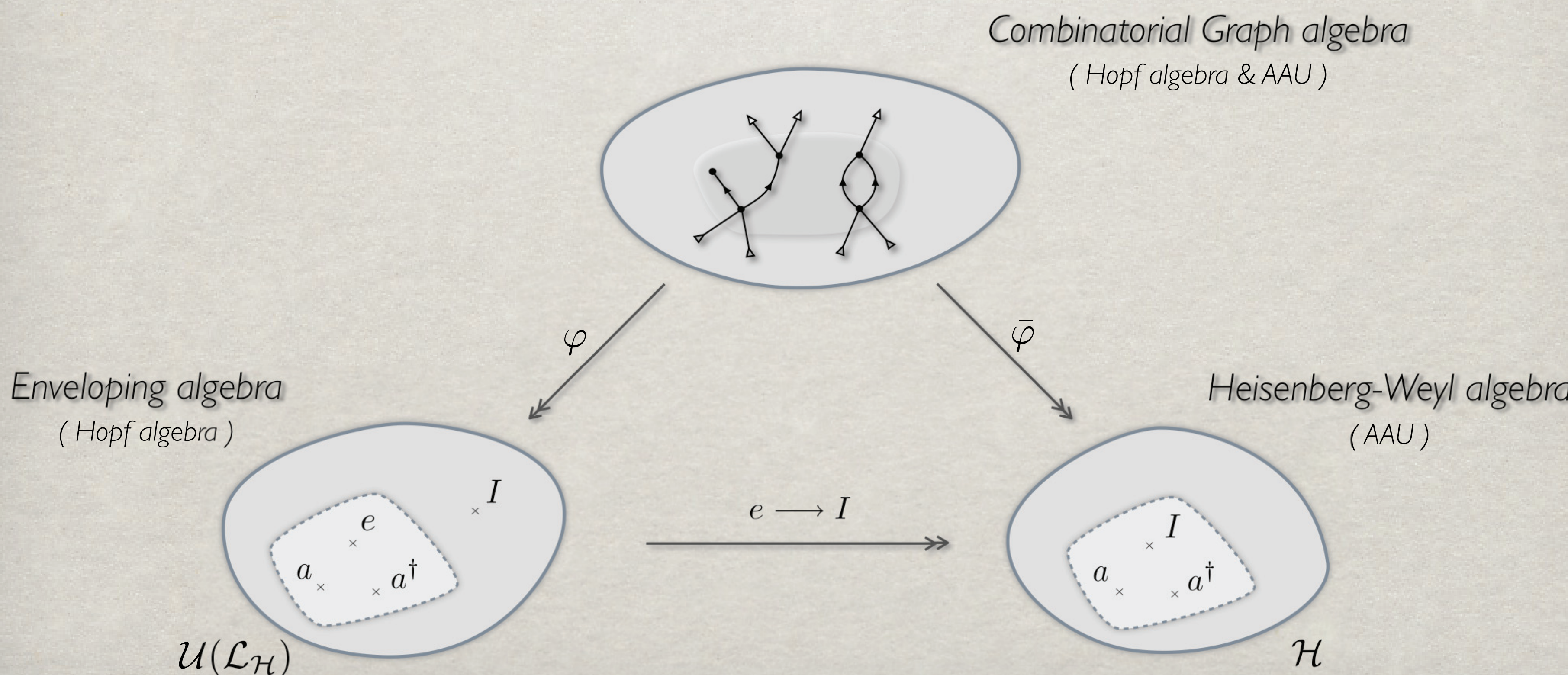


Heisenberg - Weyl algebra revisited

- Generators: $a \, , \, a^\dagger$
- Relation: $a\, a^\dagger = a^\dagger a + I$

Heisenberg - Weyl algebra

$$\mathcal{H} = \mathbb{K}\,\langle a, a^\dagger\rangle \Big/_{[\,a\,,\,a^\dagger\,] = I}$$

- Basis in $\mathcal{H}$: $a^{\dagger\, r} a^s$

$$\underbrace{\mathcal{H} \ni \sum_{\substack{r_1,\ldots,r_k\\ s_1,\ldots s_k}} \alpha_{\substack{r_1,\ldots,r_k\\ s_1,\ldots,s_k}} \; a^{\dagger\, r_1} a^{s_1} \ldots a^{\dagger\, r_k} a^{s_k}}_{\text{ambiguous}} \quad \xRightarrow{\text{Normal order}} \quad \underbrace{\mathcal{H} \ni \sum_{r,s} \alpha_{\,r,s} \; a^{\dagger\, r} a^s}_{\text{unique}}$$

- It is an associative algebra with unit (AAU)
- Structure constants :

$$a^{\dagger\, p} a^q \, a^{\dagger\, k} a^l = \sum_i \binom{q}{i}\binom{k}{i}\, i! \;\; a^{\dagger\, p+k-i} a^{q+l-i}$$

Enveloping algebra

- Generators: $a \, , \, a^\dagger \, , \, e$
- Relations: $a\, a^\dagger = a^\dagger a + e$

  $a\, e = e\, a \, , \quad a^\dagger e = e\, a^\dagger$

Enveloping algebra

$$\mathcal{U}(\mathcal{L}_\mathcal{H}) = \mathbb{K}\,\langle a, a^\dagger, e\rangle \Big/ {\scriptstyle [a, a^\dagger] = e \atop [a\,, e] = [a^\dagger, e] = 0}$$

- Basis in $\mathcal{U}(\mathcal{L}_\mathcal{H})$: $a^{\dagger\, p}\, a^q\, e^r$
- $\mathcal{U}(\mathcal{L}_\mathcal{H})$ is a Hopf algebra.

Co-product $\Delta : \mathcal{U}(\mathcal{L}_\mathcal{H}) \longrightarrow \mathcal{U}(\mathcal{L}_\mathcal{H}) \otimes \mathcal{U}(\mathcal{L}_\mathcal{H})$, s.t. on generators $\Delta\,(x) = x \otimes I + I \otimes x$:

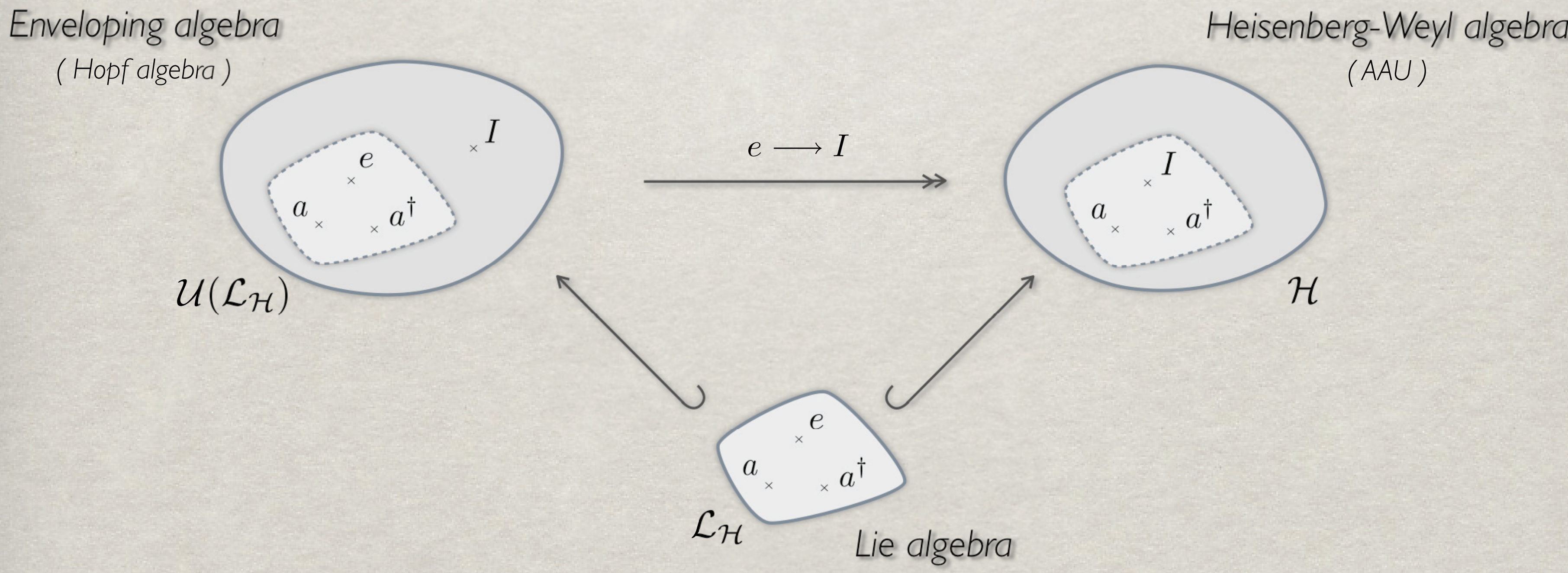
$$\Delta\left(a^{\dagger\, p} a^q\, e^r\right) = \sum_{i,j,k} \binom{p}{i}\binom{q}{j}\binom{r}{k}\; a^{\dagger\, i} a^j\, e^k \otimes a^{\dagger\, p-i} a^{q-j}\, e^{r-k}$$

Co-unit $\varepsilon : \mathcal{U}(\mathcal{L}_\mathcal{H}) \longrightarrow \mathbb{K}$, given by: $\varepsilon\left(a^{\dagger\, p} a^q\, e^r\right) = \begin{cases} 1 & \text{if} \quad p, q, r = 0 \,, \\ 0 & \text{otherwise} \,. \end{cases}$

Antipode $S : \mathcal{U}(\mathcal{L}_\mathcal{H}) \longrightarrow \mathcal{U}(\mathcal{L}_\mathcal{H})$, s.t. for generators $S\,(x) = -x$:

$$S\left(a^{\dagger\, p} a^q\, e^r\right) = (-1)^{p+q+r}\; e^r\, a^q\, a^{\dagger\, p}$$

Algebraic "picture"



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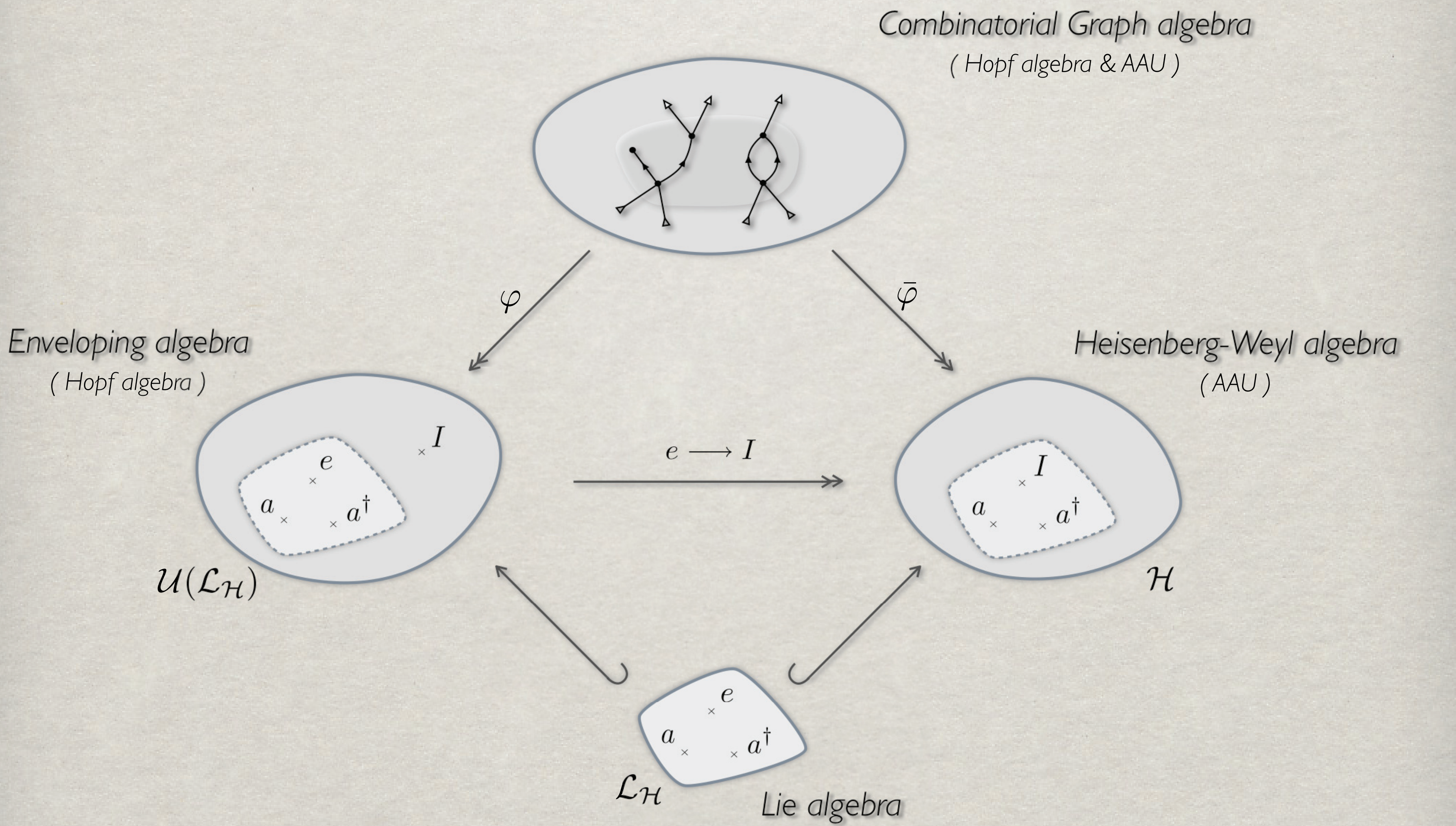

Combinatorial Concepts

A *directed graph* is a collection of edges E and vertices V together with two mappings $h, t : E \longrightarrow V$ prescribing how the *head* and *tail* of each edge is attached to vertices.

Example:

- We shall consider *classes of graphs* up to isomorphism, i.e. simply pictures
- Graphs embedded in a plane are called *planar graphs*
- Following a *cycle* in a graph one ends at the starting point

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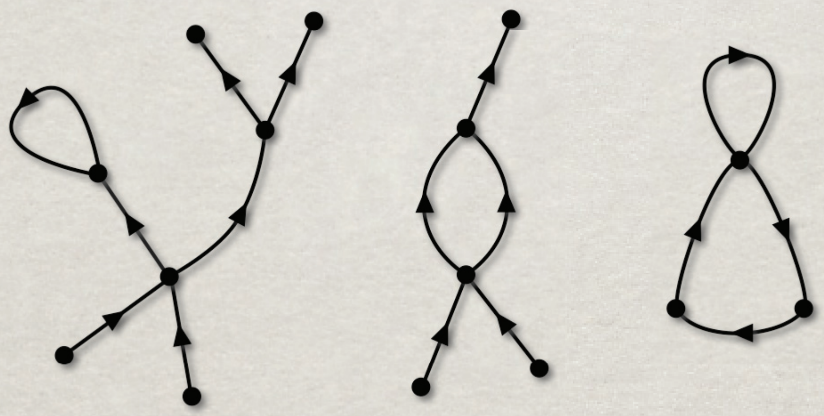

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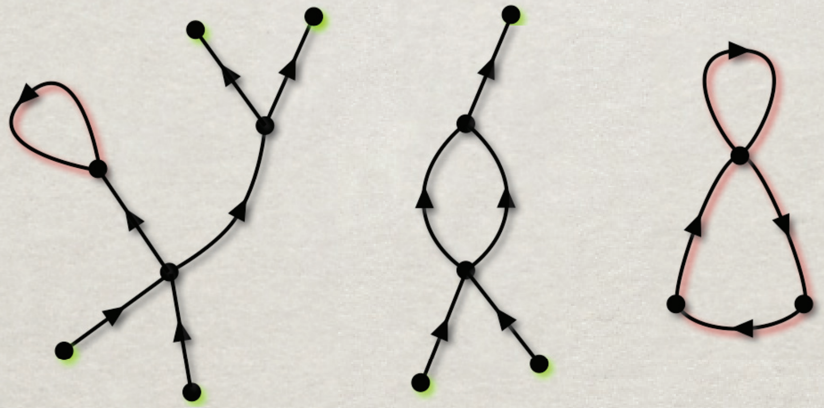
Heisenberg-Weyl graphs

Definition

Combinatorial class of Heisenberg - Weyl graphs consists of planar directed graphs Γ

which do not have cycles and may be partially-defined.



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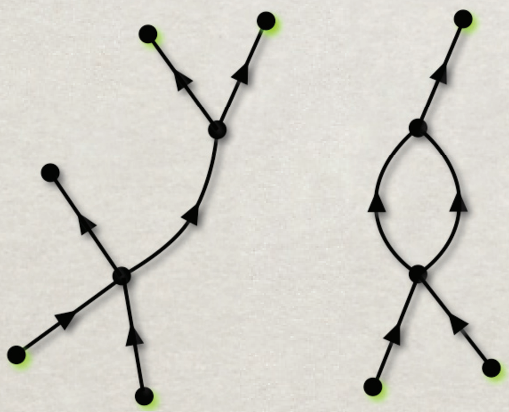

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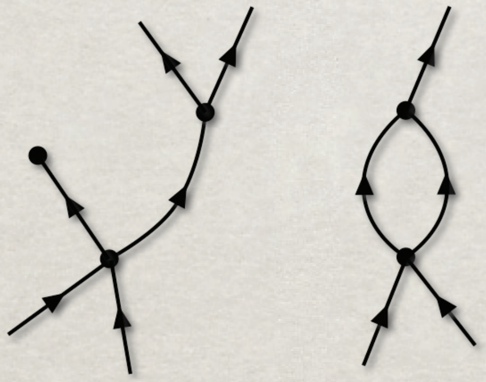

- *Edges in a graph may have one of the ends free (but not both)*

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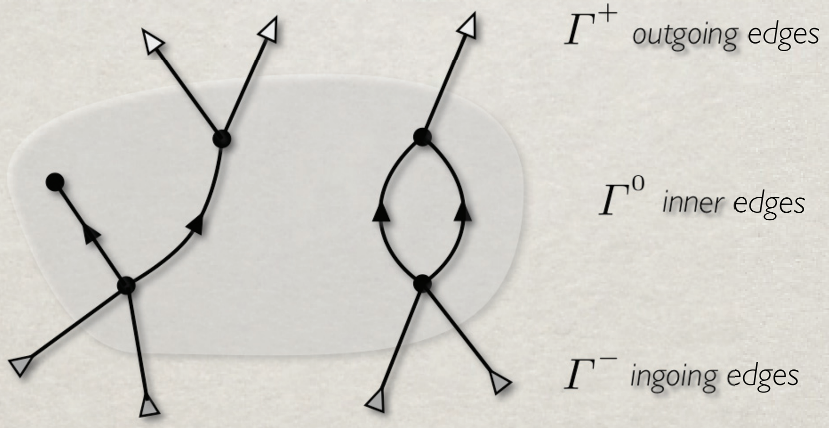

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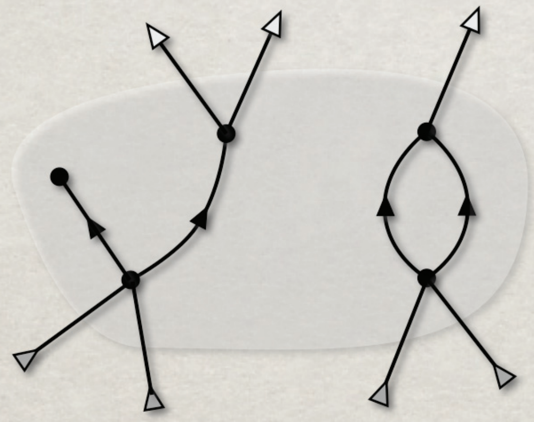


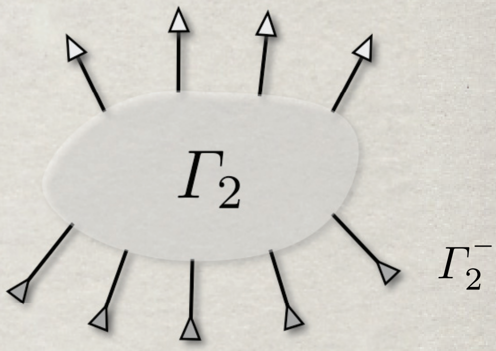
- Edges in a graph may have one of the ends free (but not both)
- It has three sorts of edges: inner, ingoing and outgoing ones
- Size of a graph:

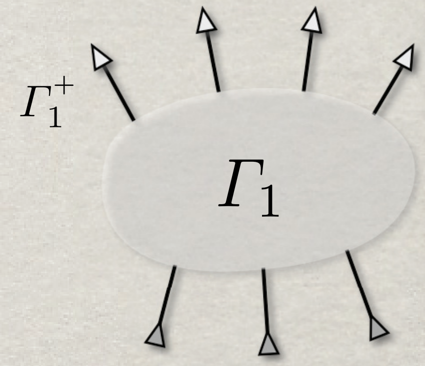
$$d(\Gamma) = 2\,|\Gamma^{0}| + |\Gamma^{+}| + |\Gamma^{-}|$$

Graph composition

Definition

For two graphs Γ_2 and Γ_1 and a matching $m \in \Gamma_2^- \lll \Gamma_1^+$.
the composite graph, denoted as $\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1$, is constructed by joining the edges coupled by the matching m.





- A matching $A \lll B$ of two sets A and B is a choice of pairs $(a, b) \in A \times B$ such that no component appear twice.

- The number of matchings consisting of i pairs (of edges) is given by

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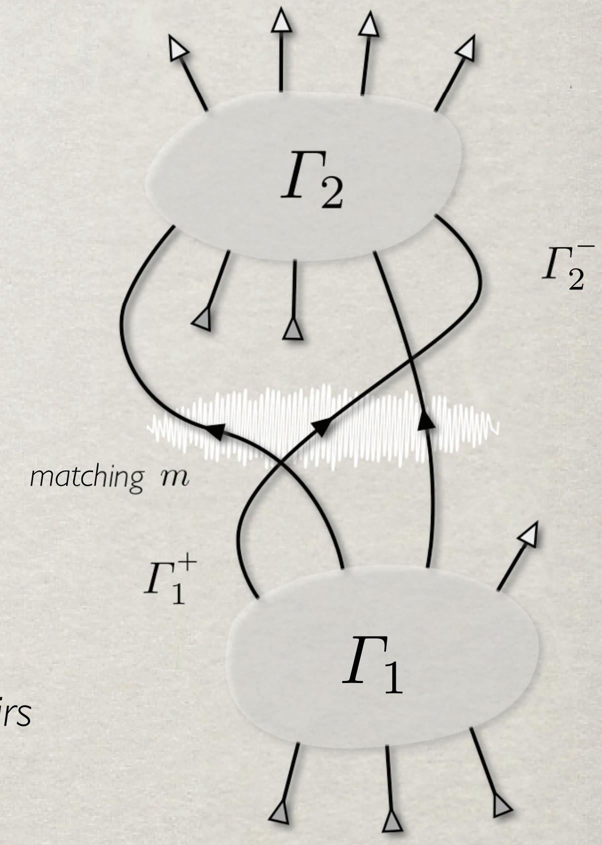

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Graph composition - Properties

Let $\Gamma_2 \blacktriangleleft \Gamma_1$ denote the set of all possible compositions of the graph Γ_2 with Γ_1, i.e.

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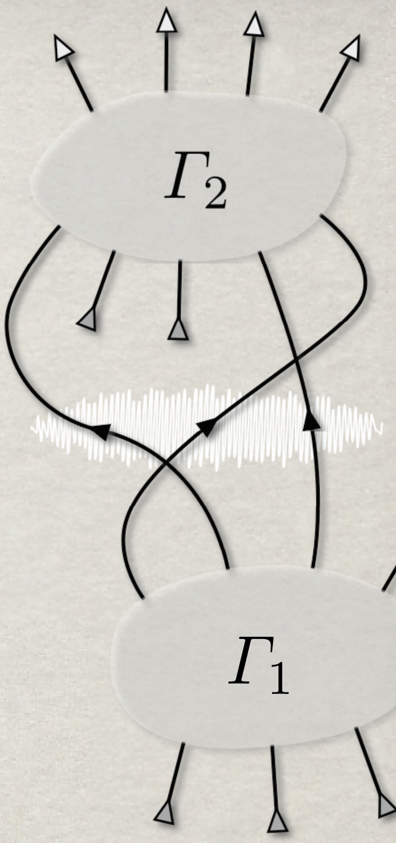

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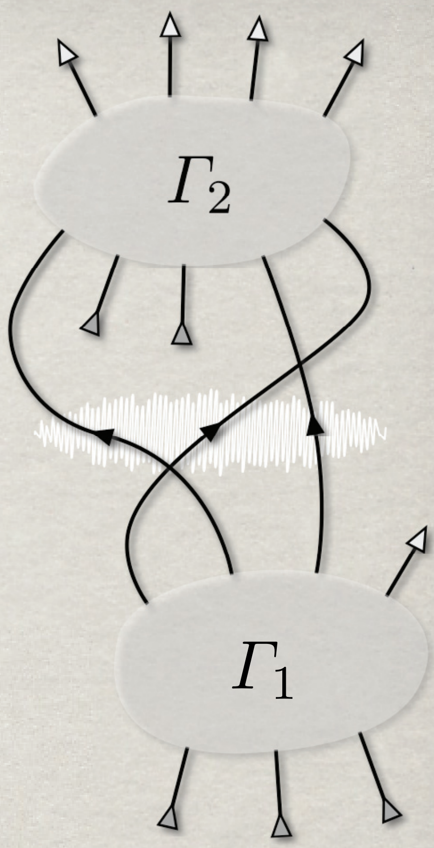

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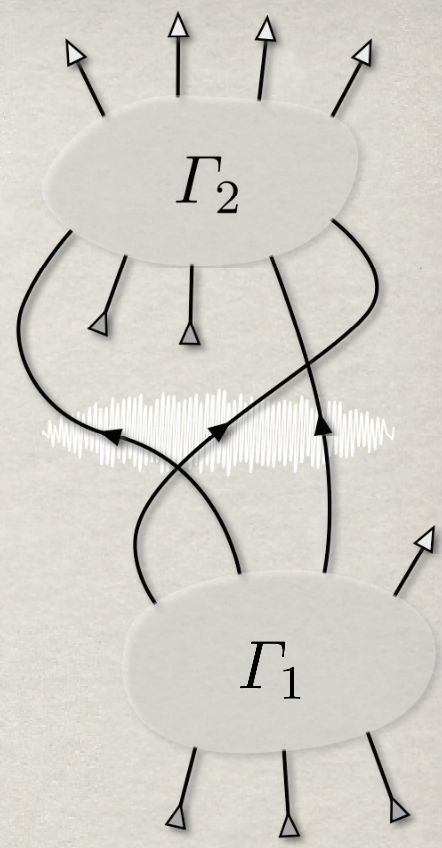

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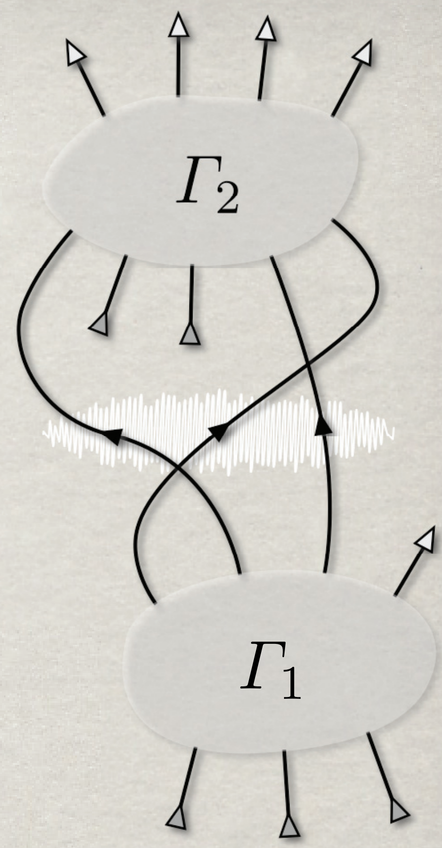

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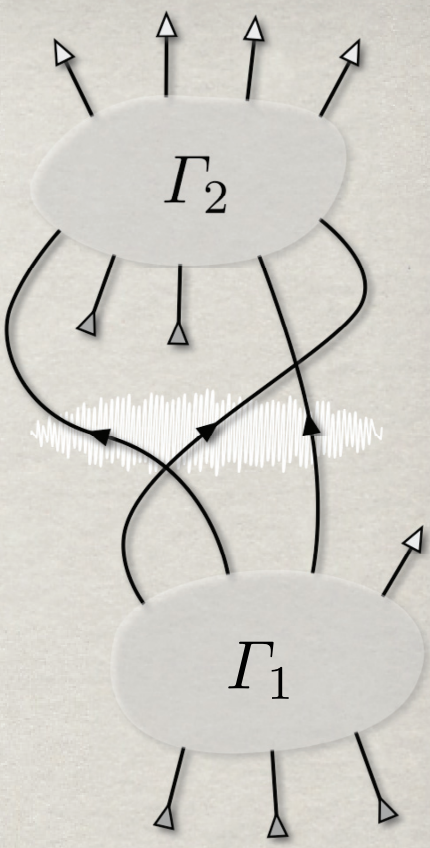

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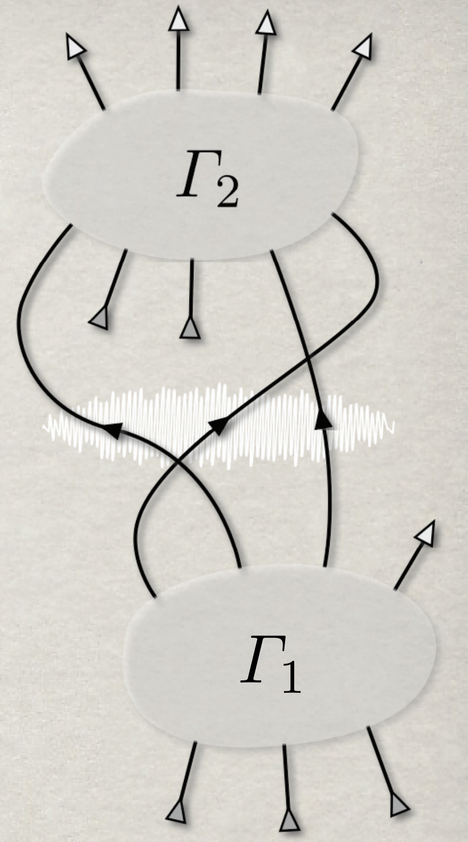

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- Compatible with size $\quad d\left(\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1\right) = d\left(\Gamma_2\right) + d\left(\Gamma_1\right)$

Graph decomposition

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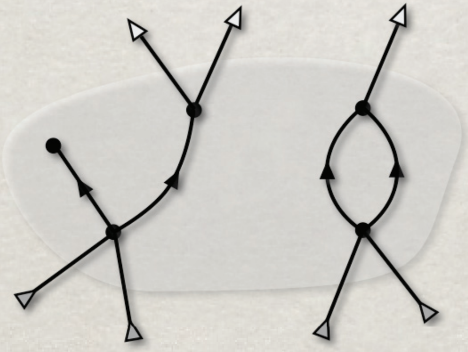
Decomposition of a graph Γ is a splitting $\Gamma \rightsquigarrow (\Gamma|_L\,, \Gamma|_R)$ induced by an ordered partition of its edges $L + R = E_\Gamma$.



- *A sub-graph $\Gamma|_L$ is a restriction of the head and tail mappings to the subset $L \subset E_\Gamma$*
- *Enumeration of all decompositions according to the number of lines in the left component:*

$$\# \left\{ (\Gamma|_L\,, \Gamma|_R) \in \langle \Gamma \rangle : \begin{smallmatrix} |\Gamma|^+_L| = i \\ |\Gamma|^-_L| = j \\ |\Gamma|^0_L| = k \end{smallmatrix} \right\} = \binom{|\Gamma^+|}{i}\binom{|\Gamma^-|}{j}\binom{|\Gamma^0|}{k}$$

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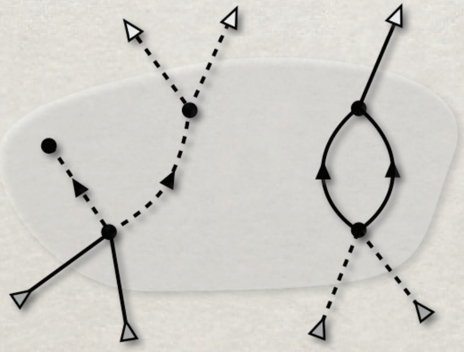

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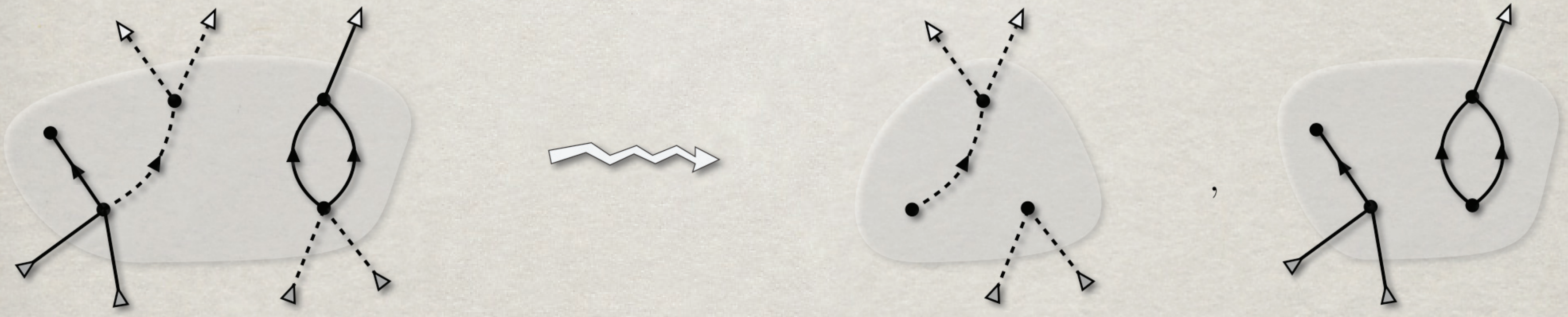

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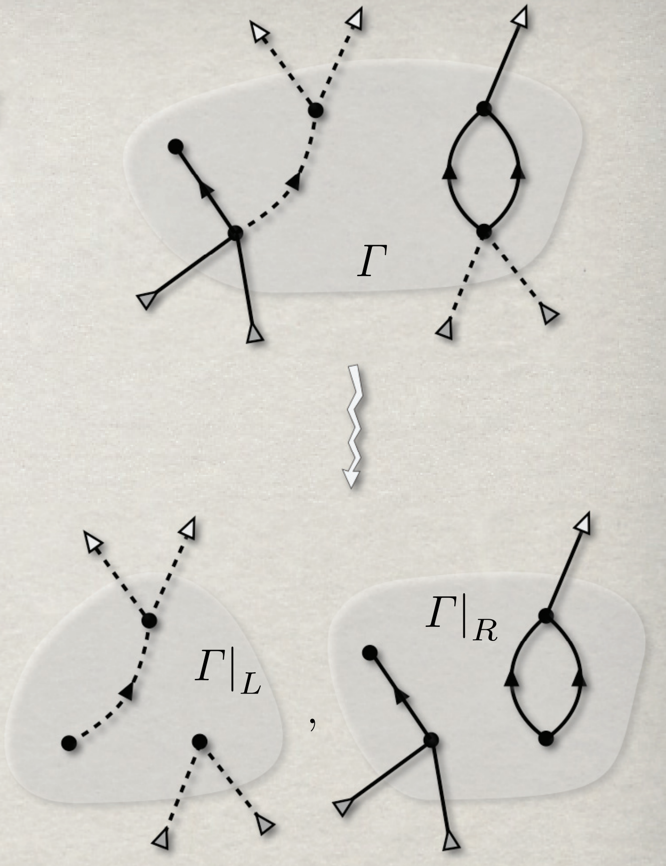
Graph decomposition - Properties



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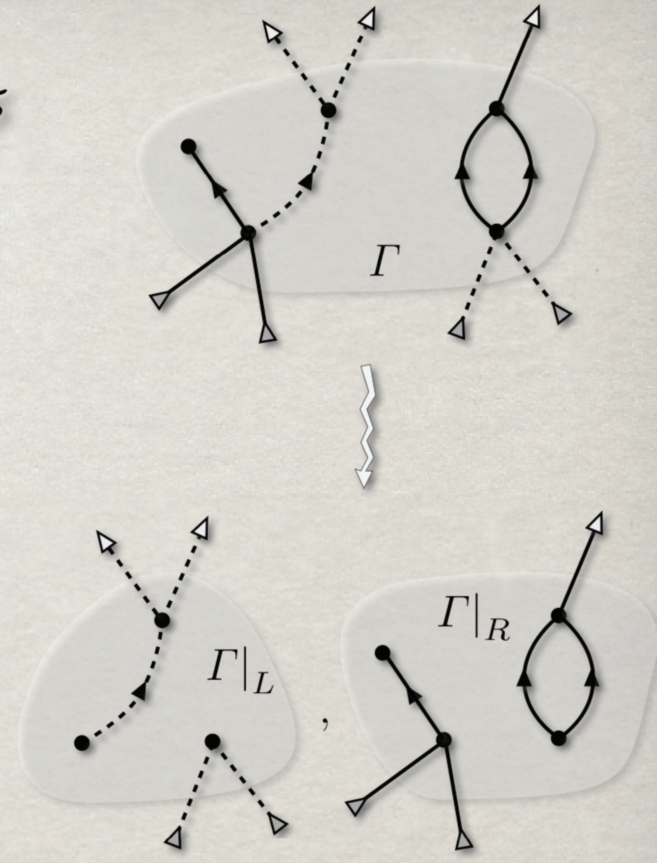

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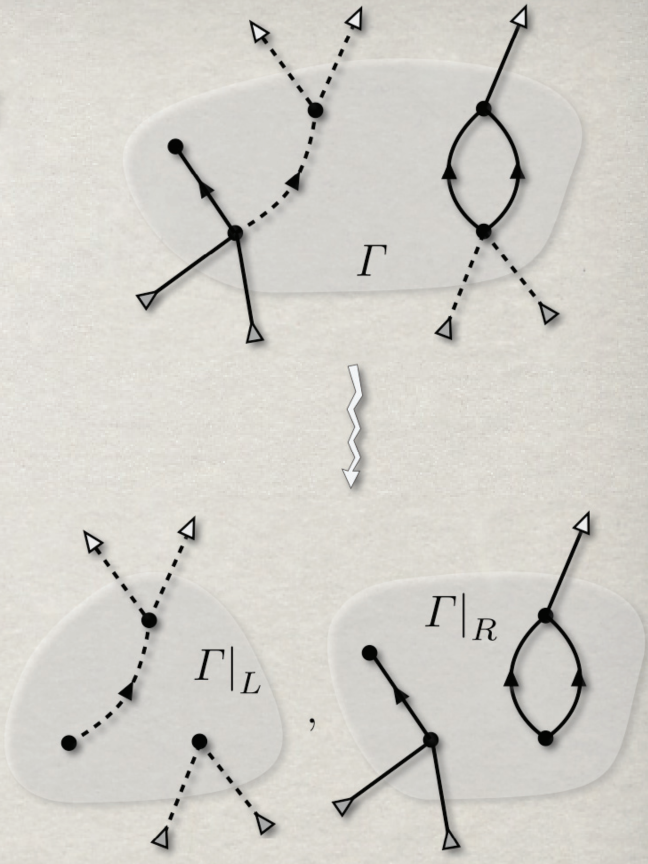

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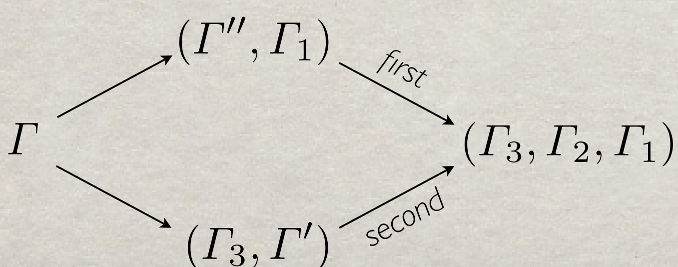

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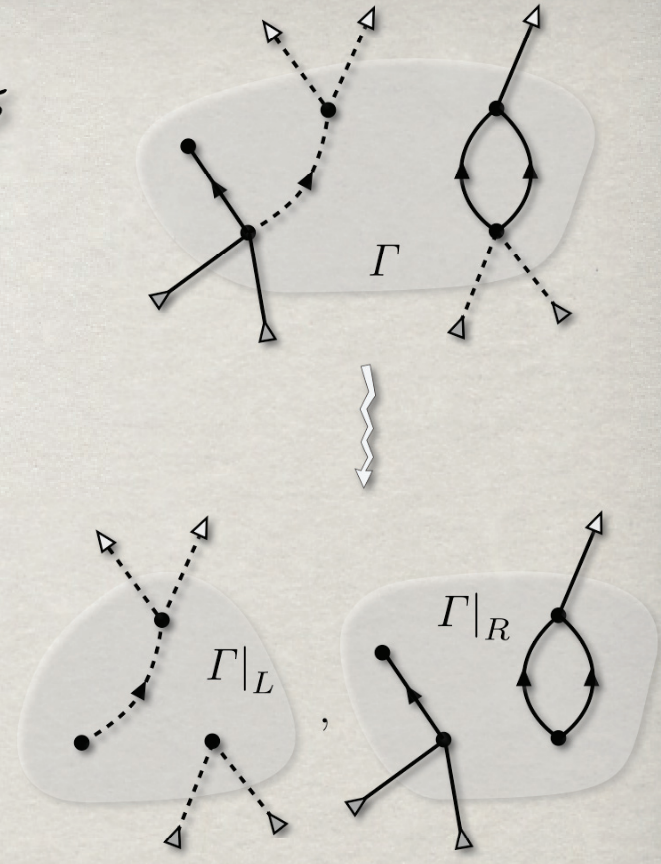

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- *Triple decomposition*

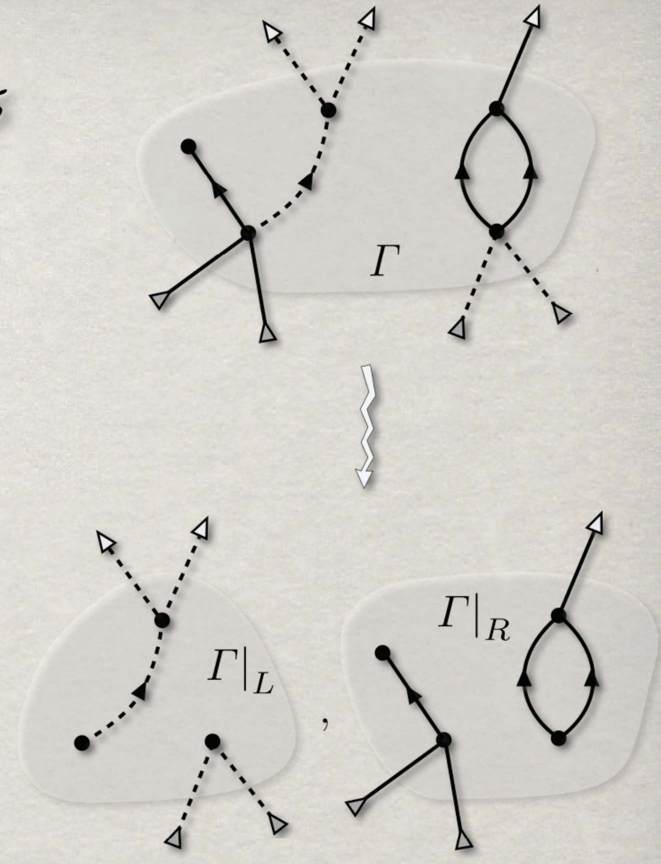
$$\Gamma \longrightarrow (\Gamma'', \Gamma_1) \xrightarrow{\text{first}} (\Gamma_3, \Gamma_2, \Gamma_1)$$

$$\Gamma \longrightarrow (\Gamma_3, \Gamma') \xrightarrow{\text{second}} (\Gamma_3, \Gamma_2, \Gamma_1)$$

- *Void graph*

$$\underbrace{\Gamma \longrightarrow (\emptyset, \Gamma) \quad \& \quad \Gamma \longrightarrow (\Gamma, \emptyset)}_{\text{unique decomposition}}$$

Graph decomposition - Properties



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- *Symmetry*

$$(\Gamma', \Gamma'') \in \langle \Gamma \rangle \implies (\Gamma'', \Gamma') \in \langle \Gamma \rangle$$

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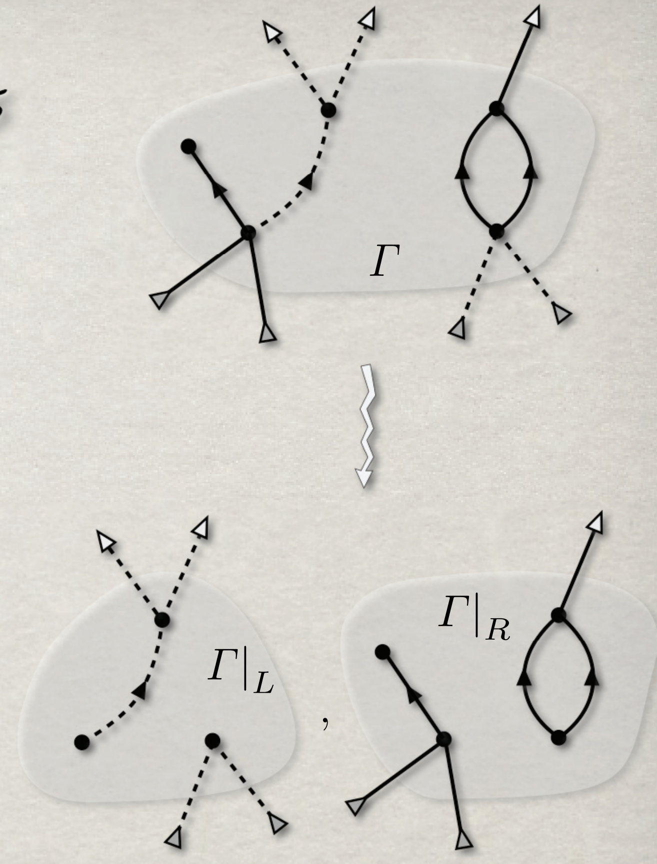

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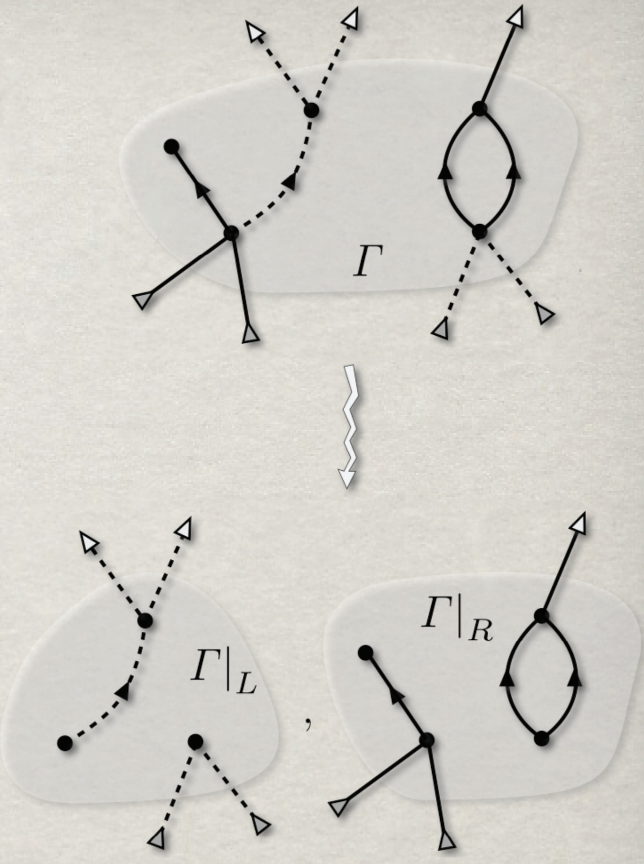

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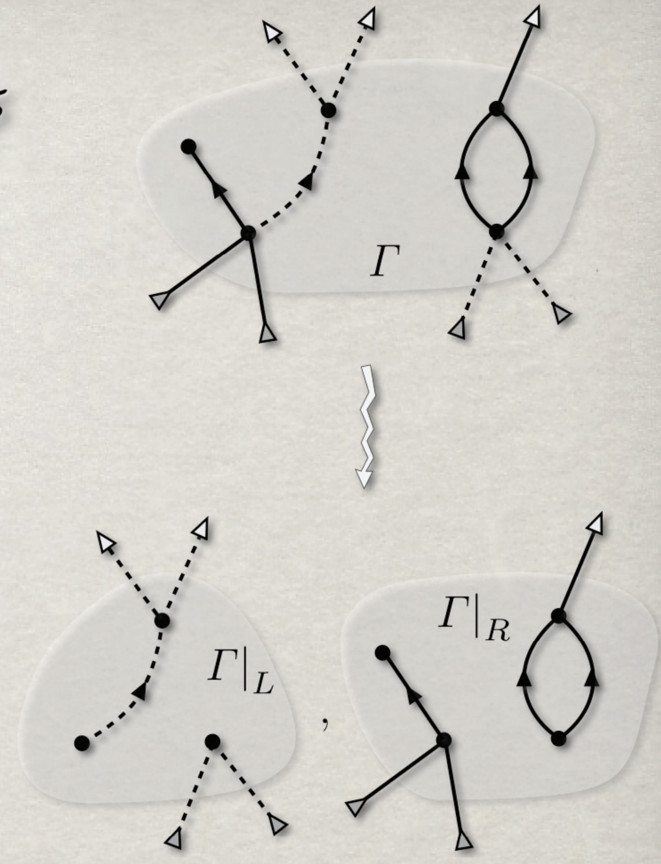
- Composition-decomposition compatibility

$(\Gamma_2, \Gamma_1) \xrightarrow{\text{comp.}} \Gamma_2 \blacktriangleleft \Gamma_1 \xrightarrow{\text{decomp.}} \langle (\Gamma_2, \Gamma_1) \rangle$

$(\Gamma_2, \Gamma_1) \xrightarrow{\text{decomp.}} \langle \Gamma_2 \rangle \times \langle \Gamma_1 \rangle \xrightarrow{\text{comp.}} \langle (\Gamma_2, \Gamma_1) \rangle$

Graph decomposition - Properties



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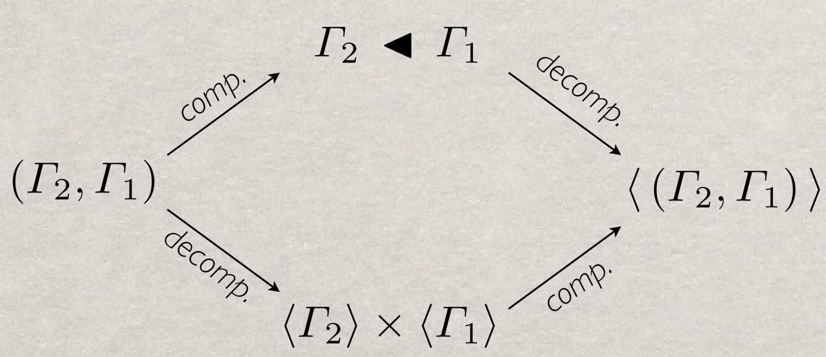
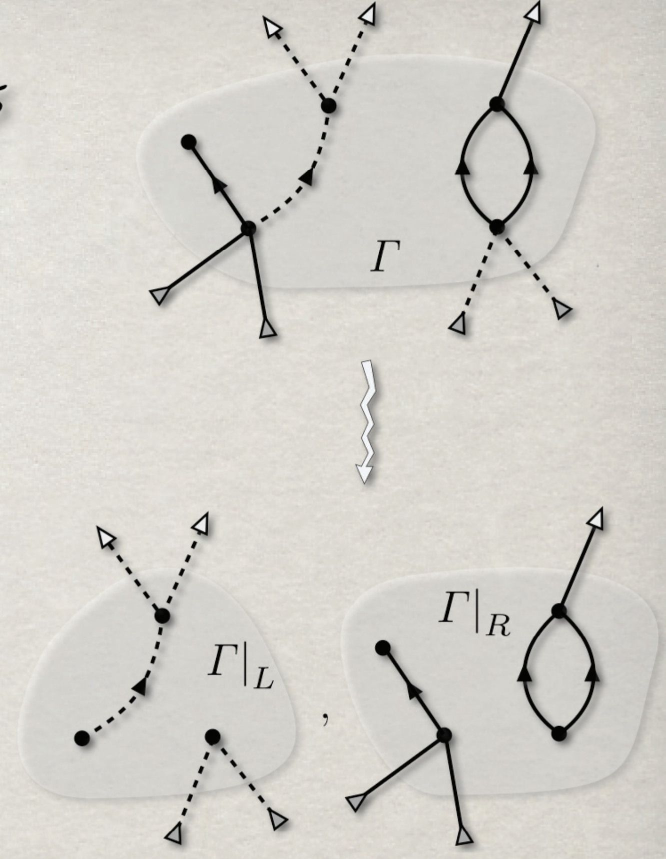


- *Compatible with size*

$$d\,(\Gamma) = d\,(\Gamma|_L) + d\,(\Gamma|_R)$$

Graph decomposition - Properties



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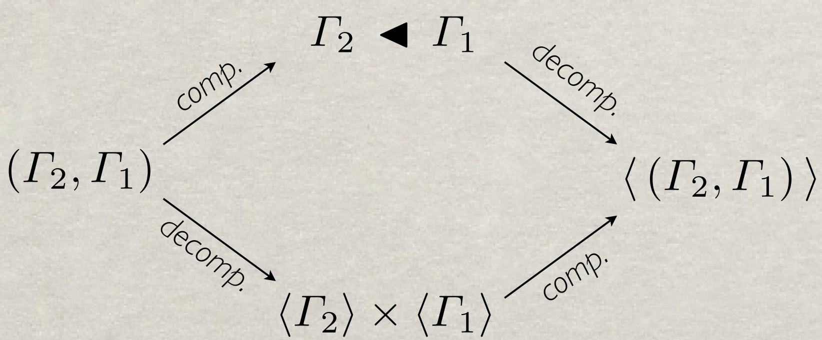


- Compatible with size

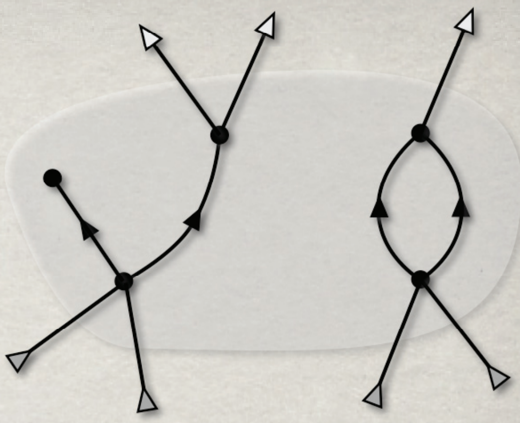
$$d(\Gamma) = d(\Gamma|_L) + d(\Gamma|_R)$$

- Finiteness of multiple decompositions

$$\{\Gamma \rightsquigarrow (\Gamma_n, \ldots \Gamma_1) : \Gamma_n, \ldots, \Gamma_1 \neq \varnothing\} = \emptyset$$

for $n \geq N(\Gamma)$

Vector space of graphs



We define $\mathcal{G}$ as a vector space over $\mathbb{K}$ spanned by the basis set consisting of all Heisenberg - Weyl diagrams, i.e.

$$\mathcal{G} = \left\{ \sum_i \alpha_i \,\mathit{\Gamma}_i : \ \alpha_i \in \mathbb{K}, \ \mathit{\Gamma}_i \text{ - Heisenberg–Weyl graph} \right\}$$

Addition in $\mathcal{G}$ has the usual form:

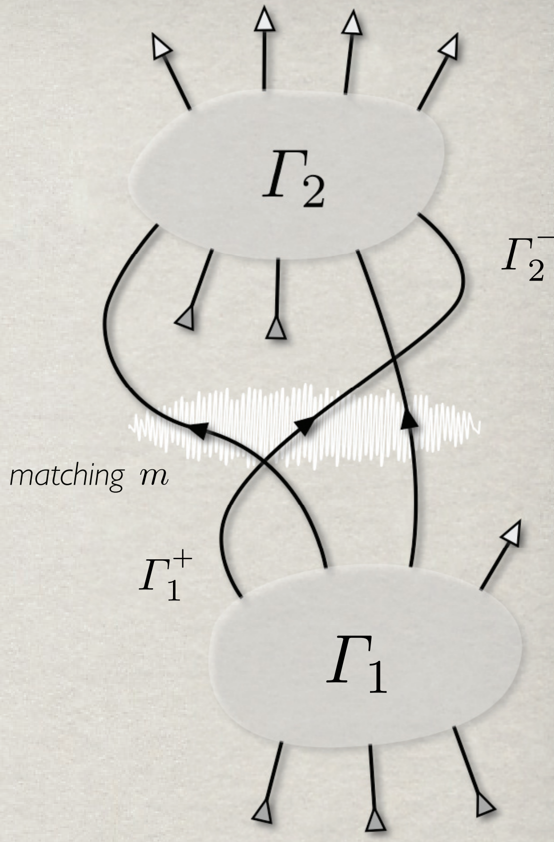
$$\sum_i \alpha_i \,\mathit{\Gamma}_i + \sum_i \beta_i \,\mathit{\Gamma}_i = \sum_i (\alpha_i + \beta_i) \,\mathit{\Gamma}_i$$

What about multiplication ?

$$\sum_i \alpha_i \,\mathit{\Gamma}_i * \sum_j \beta_j \,\mathit{\Gamma}_j = \sum_{i,j} \alpha_i \beta_j \,\mathit{\Gamma}_i * \mathit{\Gamma}_j$$

What about co-product, co-unit and antipode ?

Multiplication of graphs



Definition

Multiplication of two graphs Γ_2 and Γ_1 in $\mathcal{G}$

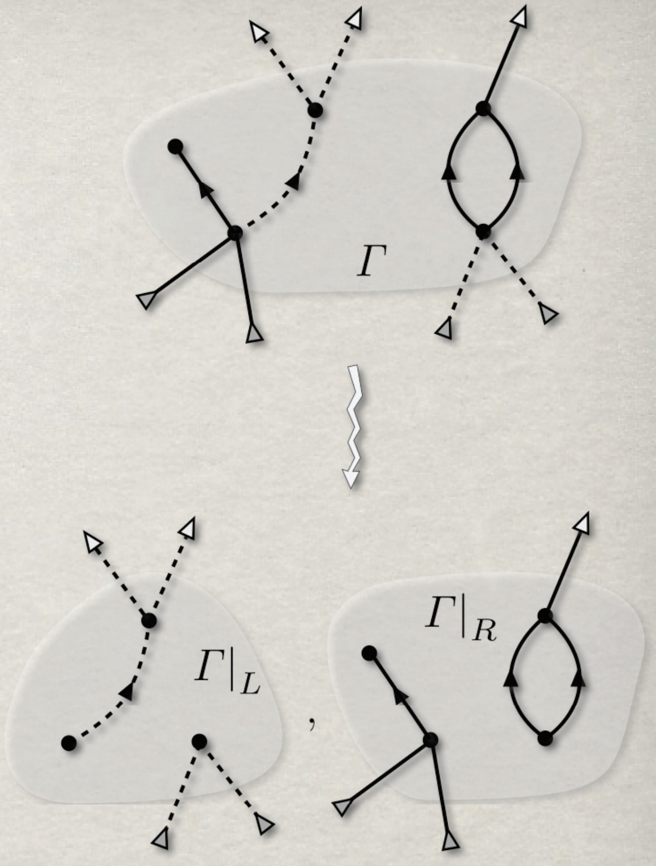
is just a sum over all possible compositions:

$$\Gamma_2 * \Gamma_1 = \sum_{m \in \Gamma_2^- \lll \Gamma_1^+} \Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1$$

Proposition

Heisenberg - Weyl graphs form an associative algebra with unit $(\mathcal{G}, +, *, \emptyset)$.
It is non-commutative !!

Co-product of graphs



Definition

Co-product $\Delta : \mathcal{G} \longrightarrow \mathcal{G} \otimes \mathcal{G}$ *is defined on the basis as a sum over all possible decompositions:*

$$\Delta(\Gamma) = \sum_{L+R=E_\Gamma} \Gamma|_L \otimes \Gamma|_R$$

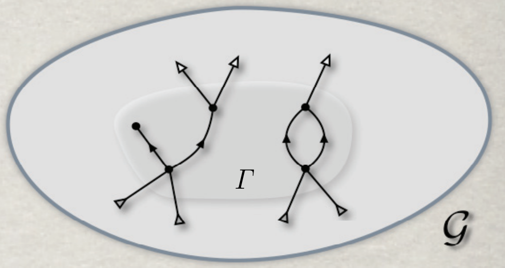
Co-unit $\varepsilon : \mathcal{G} \longrightarrow \mathbb{K}$ *simply extracts the expansion coefficient standing at the void:*

$$\varepsilon(\Gamma) = \begin{cases} 1 & \text{if} \quad \Gamma = \emptyset \,, \\ 0 & \text{otherwise} \,. \end{cases}$$

Proposition

Heisenberg - Weyl graphs form a bi-algebra $(\mathcal{G}, +, *, \emptyset, \Delta, \varepsilon)$. It is co-commutative !!

Algebra of Heisenberg - Weyl graphs



Combinatorial algebra
of Heisenberg-Weyl graphs
(Hopf algebra & AAU)

Theorem

- Heisenberg - Weyl graphs form a bi-algebra.
  It is non-commutative and co-commutative.

- Even more, it has a genuine Hopf algebra structure $(\mathcal{G}, +, *, \emptyset, \Delta, \varepsilon, S)$, with an antipode given by:

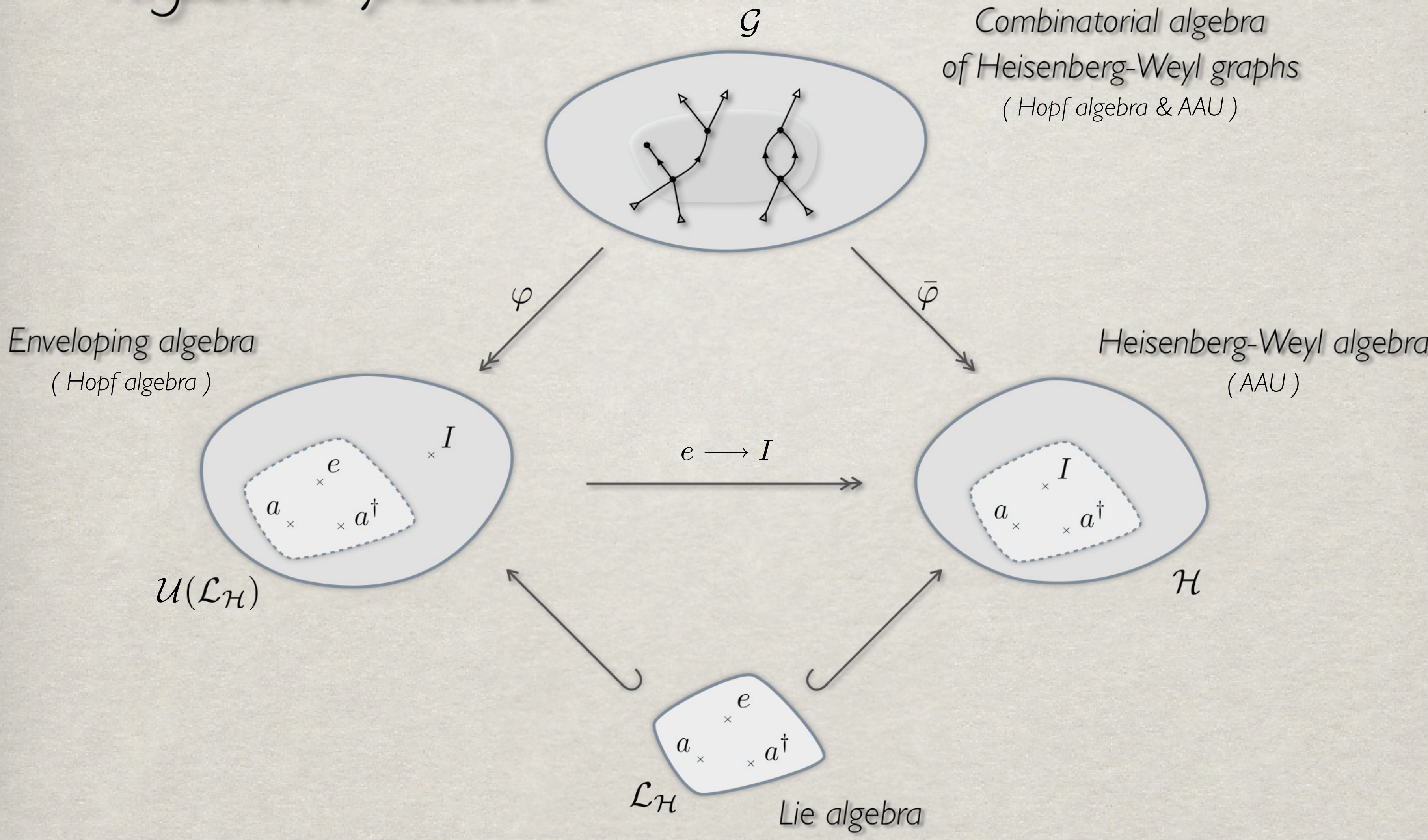
$$S(\Gamma) = \sum_{\substack{A_n + \ldots + A_1 = E_\Gamma \\ A_n, \ldots, A_1 \neq \emptyset}} (-1)^n \;\; \Gamma|_{A_n} * \ldots * \Gamma|_{A_1}$$

and $S(\emptyset) = \emptyset$.

- It is graded $\quad \mathcal{G} = \bigoplus_{n \in \mathbb{N}} \mathcal{G}_n \;, \qquad \mathcal{G}_n = \mathrm{Span}\,\{\Gamma : d(\Gamma) = n\}$

$$* : \mathcal{G}_i \times \mathcal{G}_j \longrightarrow \mathcal{G}_{i+j} \;, \qquad \Delta : \mathcal{G}_k \longrightarrow \bigoplus_{i+j=k} \mathcal{G}_i \otimes \mathcal{G}_j$$

Algebraic "picture"



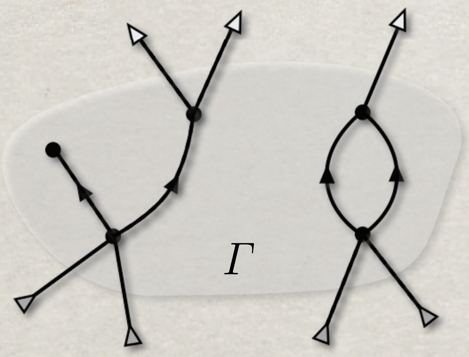
We still need to need to provide mappings $\varphi : \mathcal{G} \longrightarrow \mathcal{U}(\mathcal{L}_\mathcal{H})$ and $\bar{\varphi} : \mathcal{G} \longrightarrow \mathcal{H}$ preserving (Hopf) algebraic structure of the Heisenberg - Weyl graphs $\mathcal{G}$.

Model of the Heisenberg-Weyl algebra

Definition

We define a linear mapping $\varphi : \mathcal{G} \longrightarrow \mathcal{U}(\mathcal{L}_{\mathcal{H}})$ which erases inner structure of a graph, given on the basis elements as:

$$\varphi\,(\Gamma) = a^{\dagger\,|\Gamma^{+}|}\;a^{\,|\Gamma^{-}|}\;e^{\,|\Gamma^{0}|}$$



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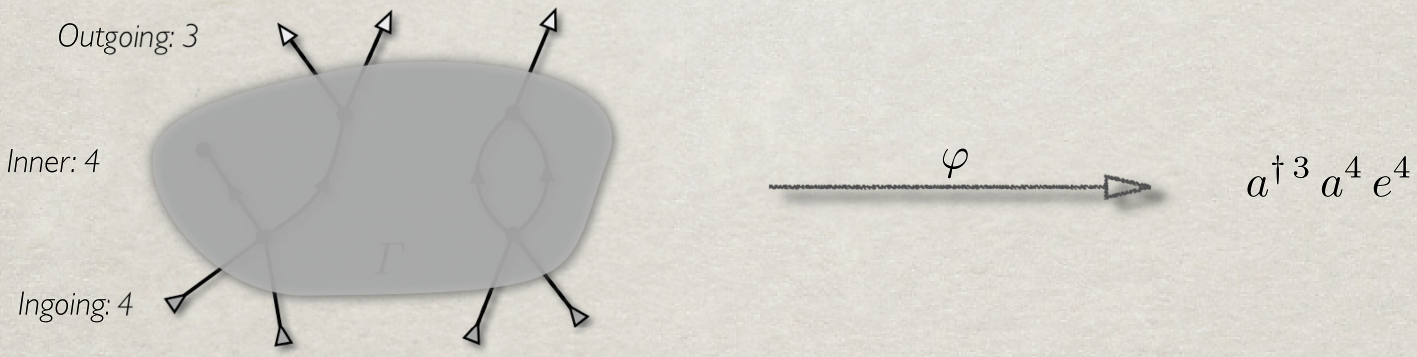

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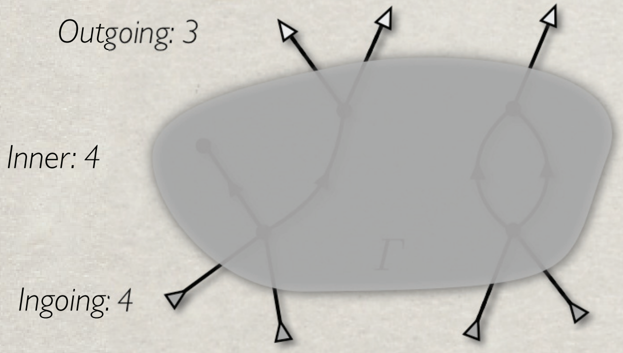

$$\xrightarrow{\;\varphi\;} \quad a^{\dagger\,3}\, a^4\, e^4$$

Theorem

Forgetful mapping $\varphi : \mathcal{G} \longrightarrow \mathcal{U}(\mathcal{L}_{\mathcal{H}})$ is a Hopf algebra morphism.

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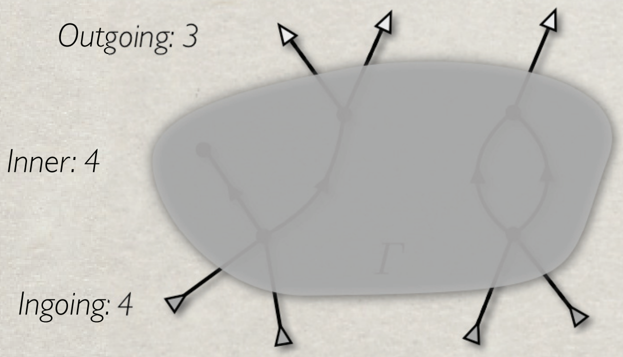


$$\xrightarrow{\ \varphi\ } \quad a^{\dagger\,3}\, a^4\, e^4$$

Theorem

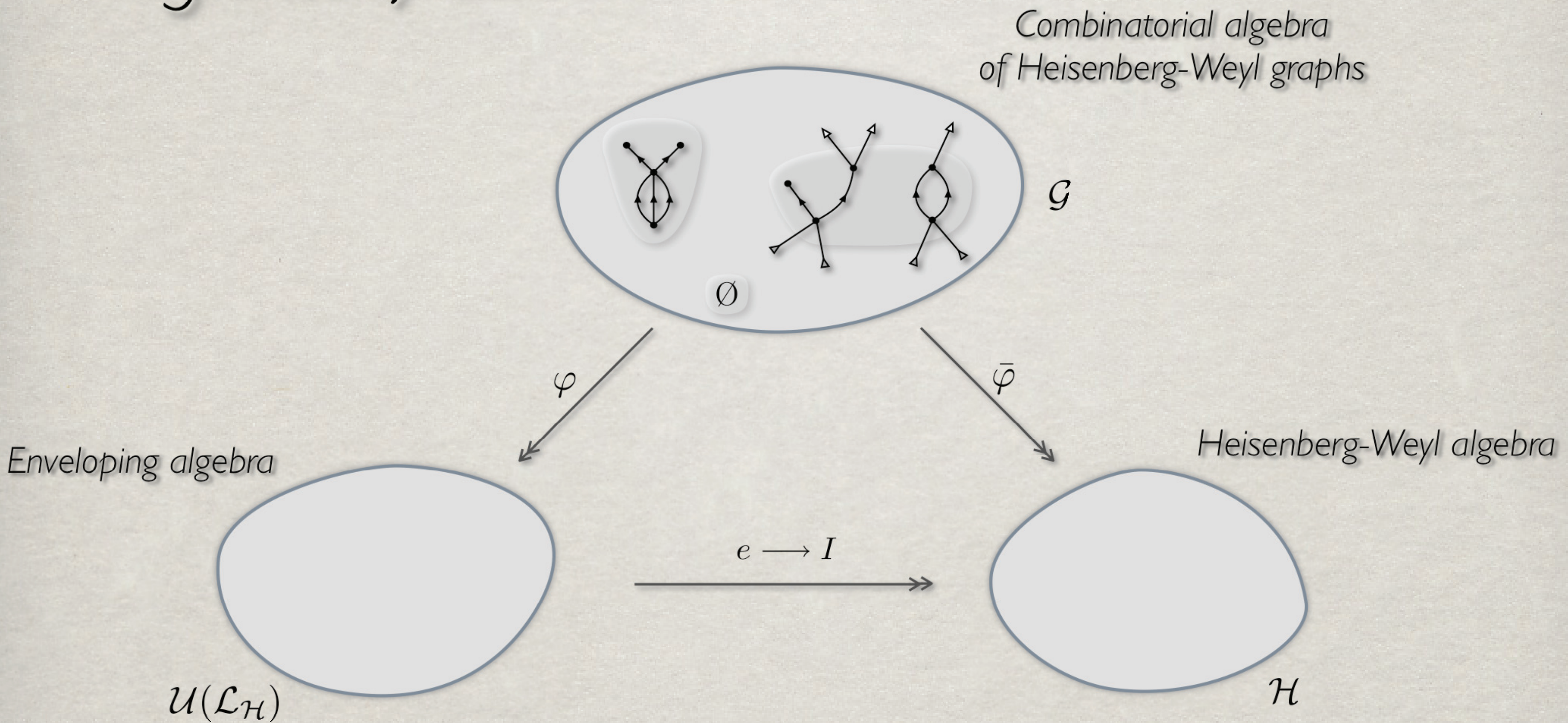
Forgetful mapping $\varphi : \mathcal{G} \longrightarrow \mathcal{U}(\mathcal{L}_{\mathcal{H}})$ is a Hopf algebra morphism.

Note

By additionally neglecting number of the inner edges $\bar{\varphi}\,(\Gamma) = a^{\dagger\,|\Gamma^{+}|}\; a^{|\Gamma^{-}|}$,

we get an (AAU) algebra morphism $\bar{\varphi} : \mathcal{G} \longrightarrow \mathcal{H}$.

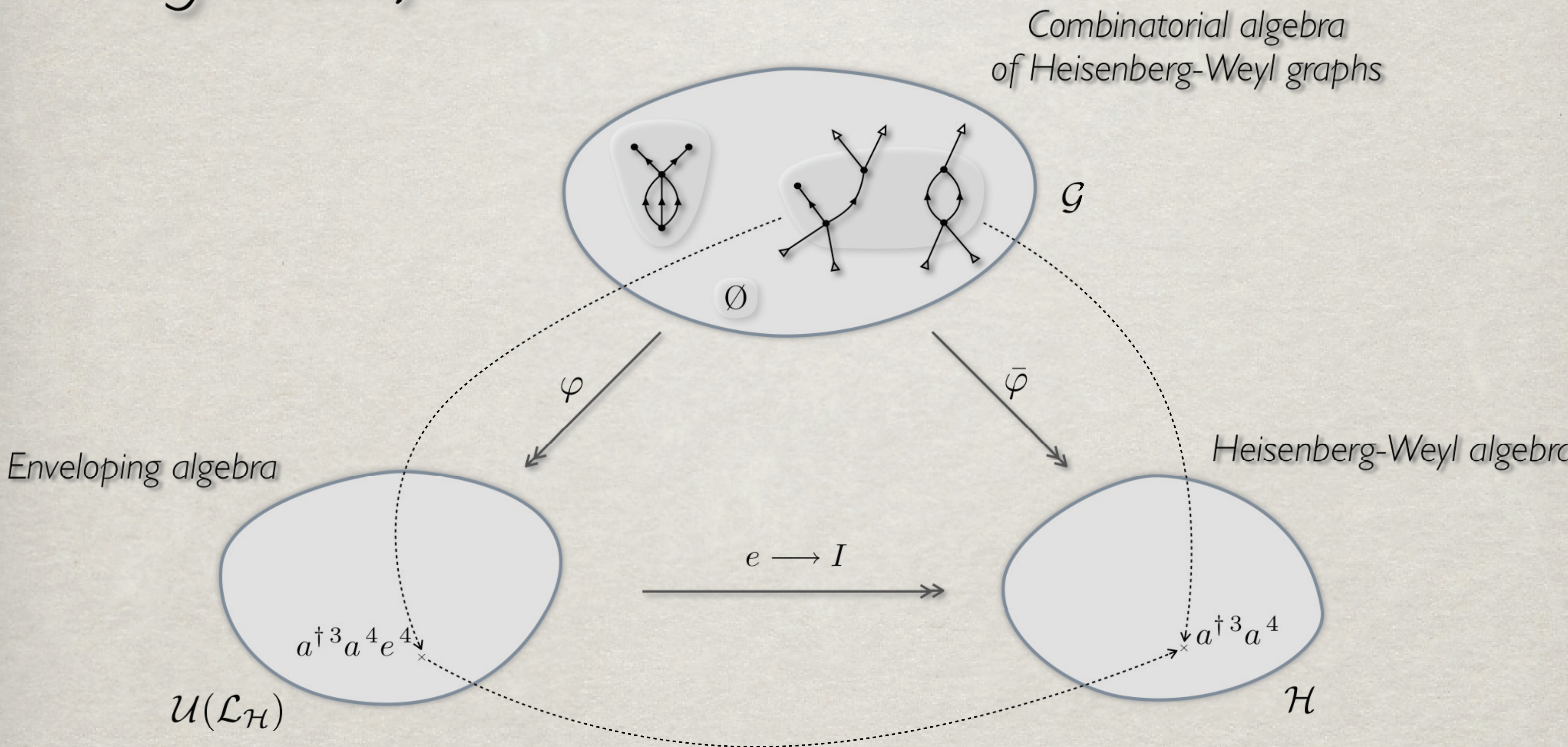
Algebraic "picture"



Morphism $\varphi : \mathcal{G} \longrightarrow \mathcal{U}(\mathcal{L}_{\mathcal{H}})$ erases inner structure of a graph ,

and $\bar{\varphi} : \mathcal{G} \longrightarrow \mathcal{H}$ erases inner structure of a graph & forgets number of its inner edges .

Algebraic "picture"



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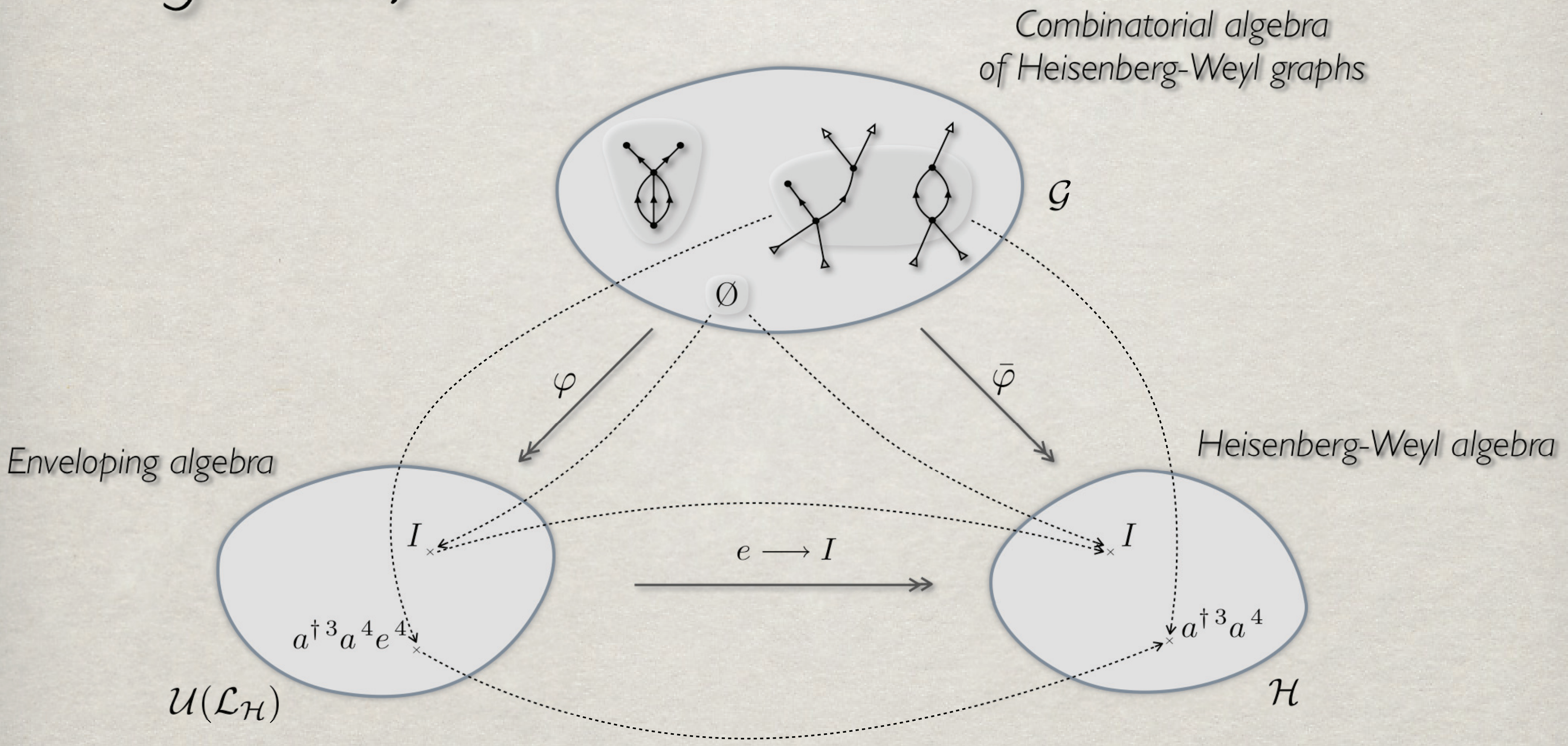

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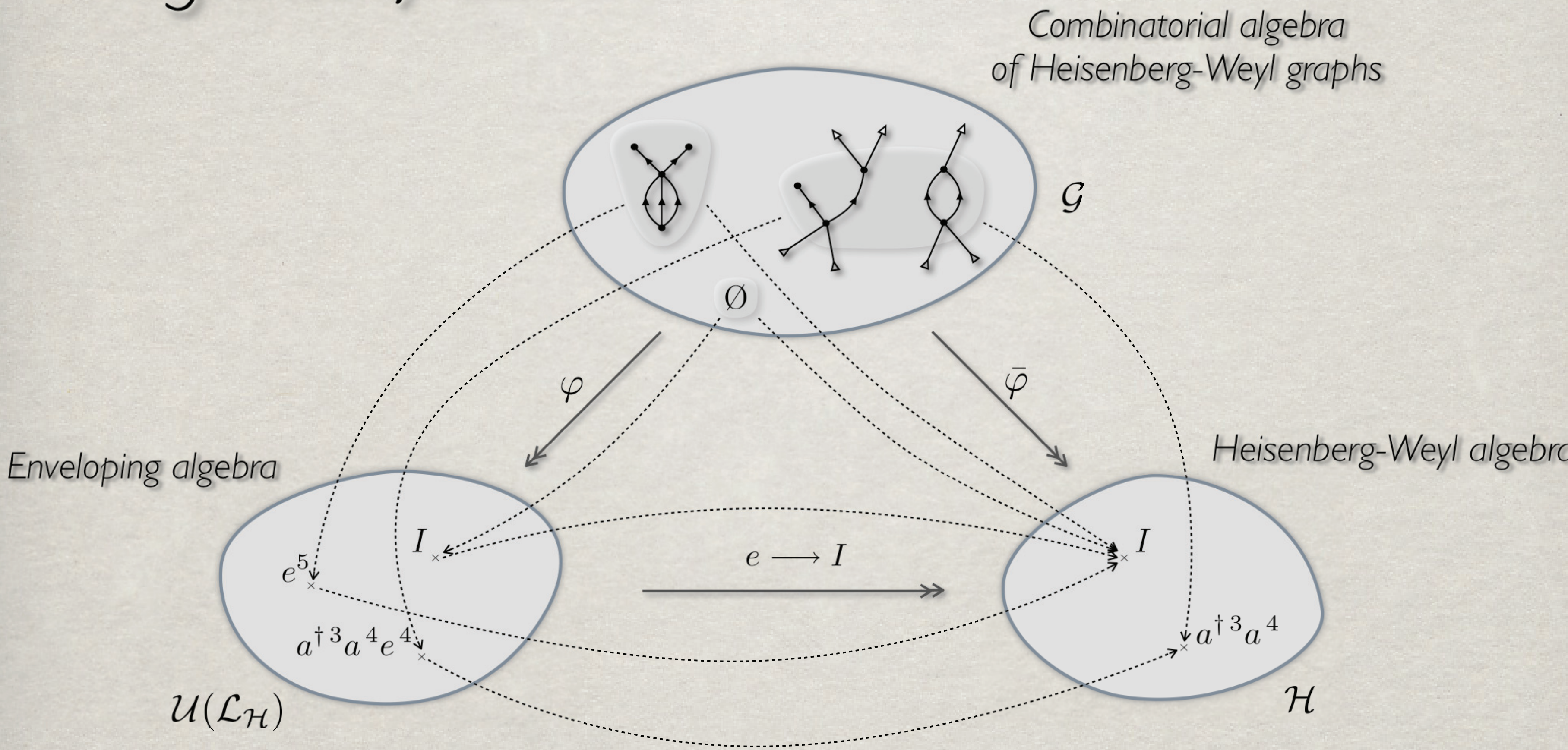

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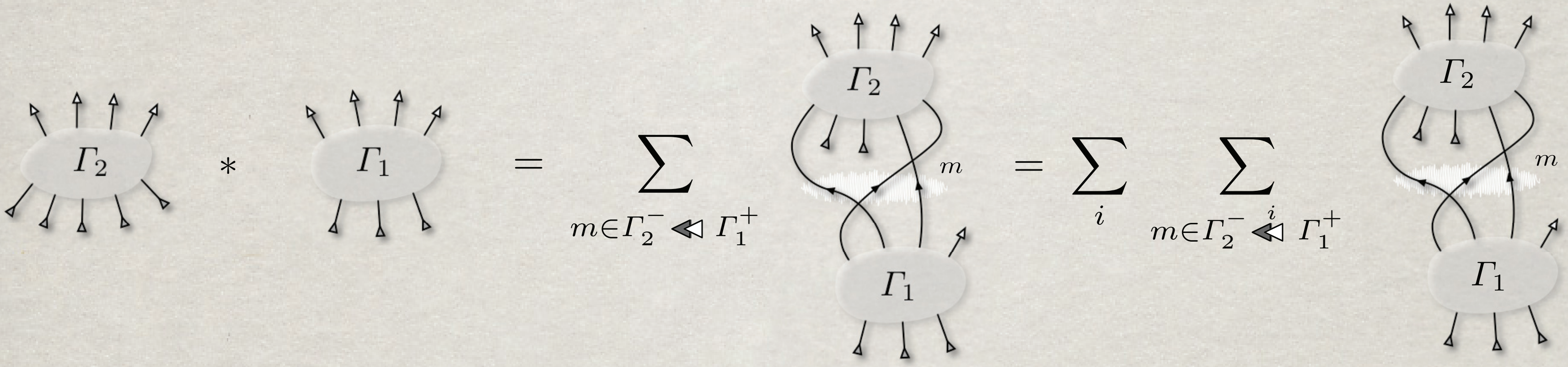
Sketch of Proof: Product

We need to prove that $\varphi : \mathcal{G} \longrightarrow \mathcal{U}(\mathcal{L}_{\mathcal{H}})$ preserves product, i.e. $\varphi(\Gamma_2 * \Gamma_1) = \varphi(\Gamma_2)\,\varphi(\Gamma_1)$



$$\Gamma_2 * \Gamma_1 = \sum_i \sum_{m \in \Gamma_2^- \overset{i}{\lll} \Gamma_1^+} \Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1$$

$$\Gamma_2^- \lll \Gamma_1^+ = \bigcup_i \Gamma_2^- \overset{i}{\lll} \Gamma_1^+$$

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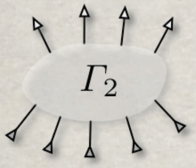
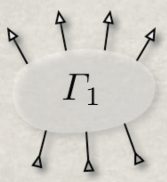
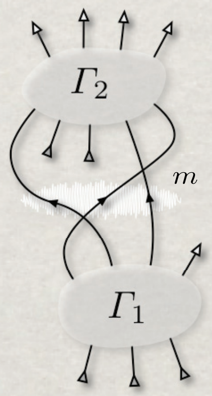


$*$



$= \displaystyle\sum_{m \in \Gamma_2^- \lll \Gamma_1^+}$



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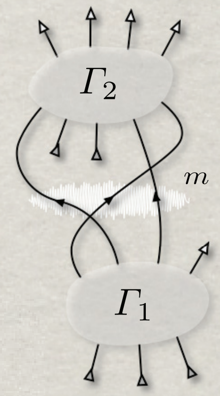


$$\Gamma_2^- \lll \Gamma_1^+ = \bigcup_i \Gamma_2^- \overset{i}{\lll} \Gamma_1^+$$

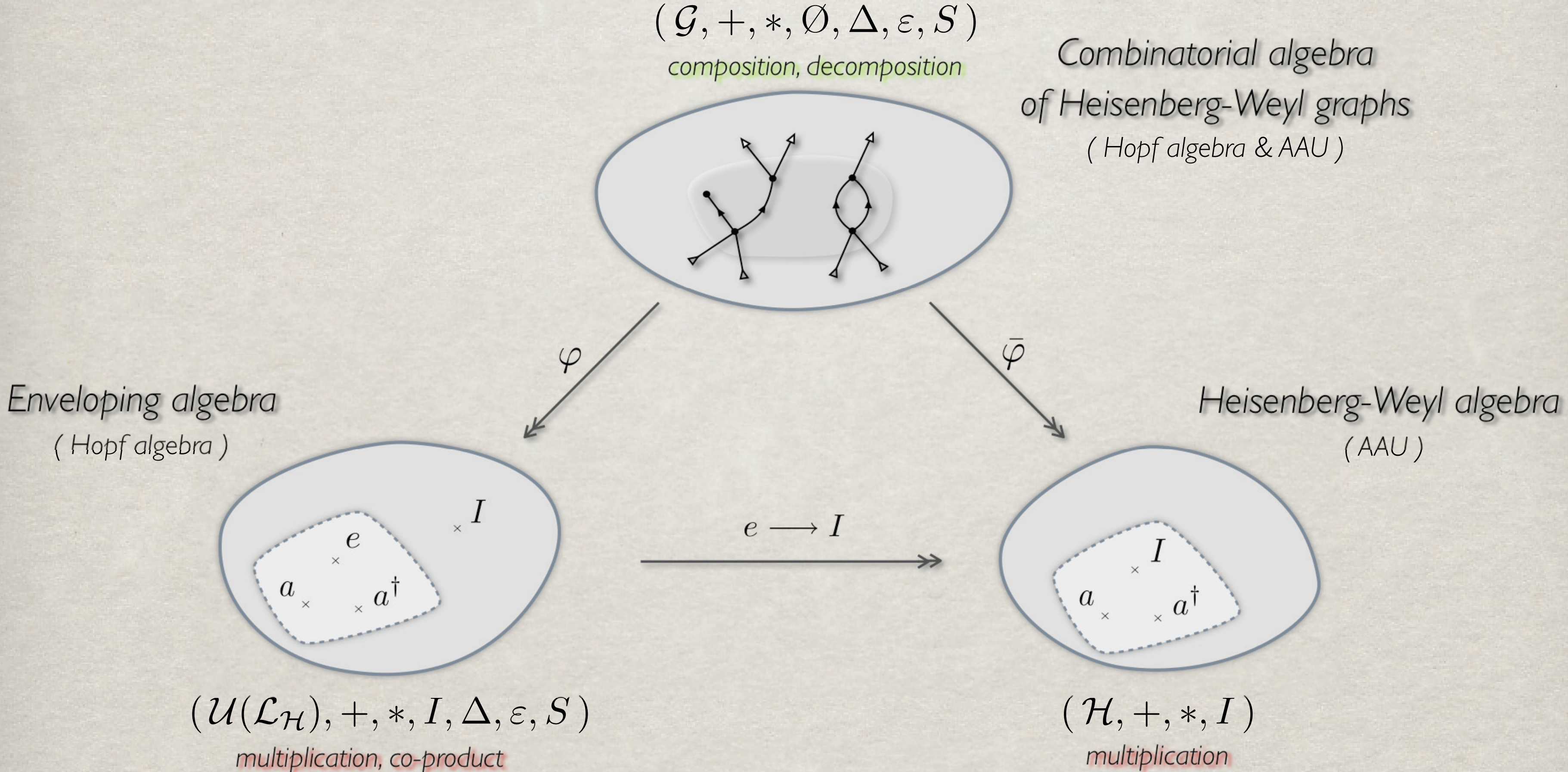
$$\varphi(\Gamma_2 * \Gamma_1) = \sum_i \sum_{m \in \Gamma_2^- \overset{i}{\lll} \Gamma_1^+} \varphi\left(\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1\right)$$

$$= \sum_i \sum_{m \in \Gamma_2^- \overset{i}{\lll} \Gamma_1^+} (a^\dagger)^{|\Gamma_2^+|+|\Gamma_1^+|-i}\; a^{|\Gamma_2^-|+|\Gamma_1^-|-i}\; e^{|\Gamma_2^0|+|\Gamma_1^0|+i}$$

$$= \sum_i \binom{|\Gamma_2^-|}{i}\binom{|\Gamma_1^+|}{i}\, i!\; (a^\dagger)^{|\Gamma_2^+|+|\Gamma_1^+|-i}\; a^{|\Gamma_2^-|+|\Gamma_1^-|-i}\; e^{|\Gamma_2^0|+|\Gamma_1^0|+i}$$

$$= \left((a^\dagger)^{|\Gamma_2^+|}\, a^{|\Gamma_2^-|}\, e^{|\Gamma_2^0|}\right)\left((a^\dagger)^{|\Gamma_1^+|}\, a^{|\Gamma_1^-|}\, e^{|\Gamma_1^0|}\right) = \varphi(\Gamma_2)\,\varphi(\Gamma_1)$$

Algebraic "picture"



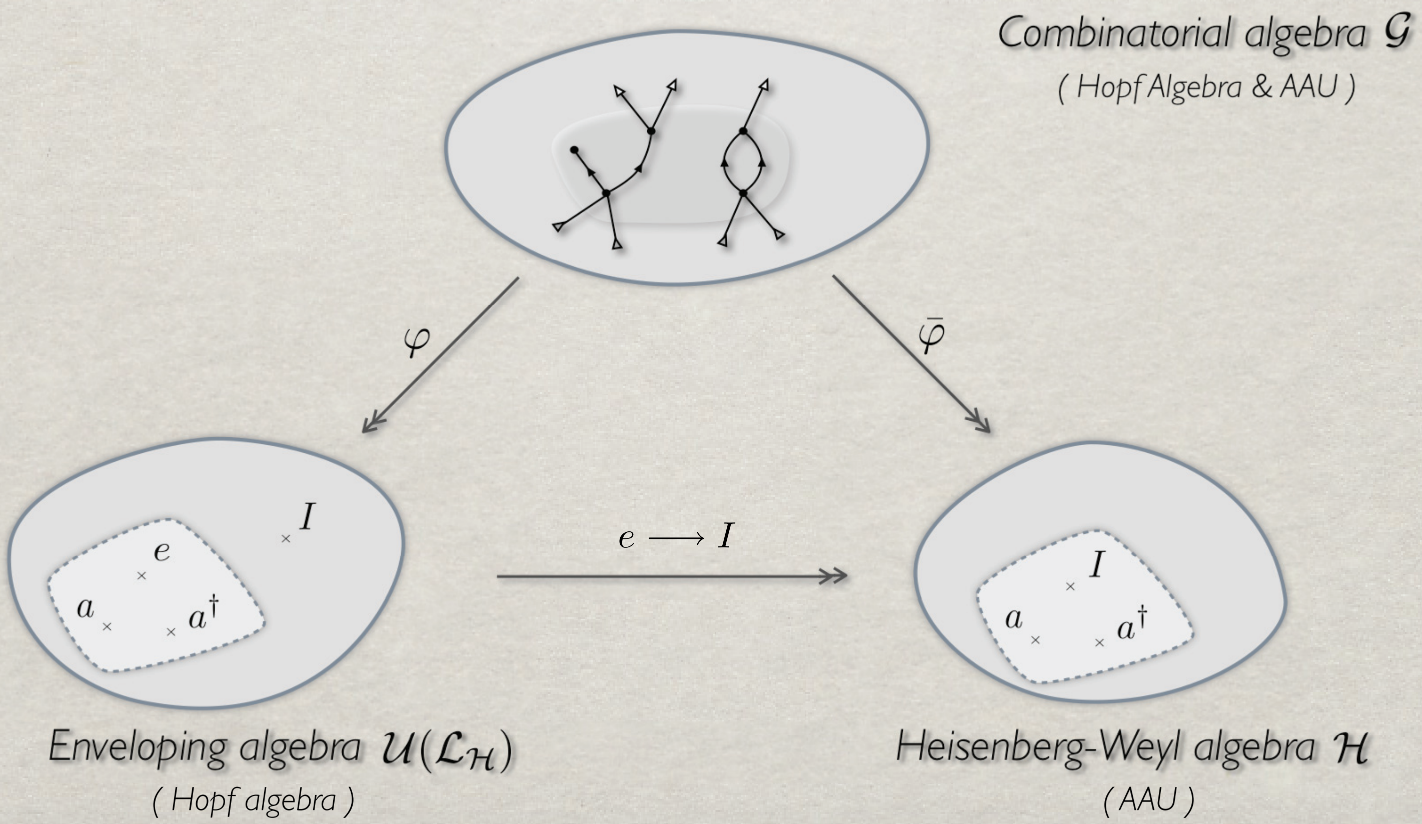
Structures preserved by morphisms $\varphi : \mathcal{G} \longrightarrow \mathcal{U}(\mathcal{L}_\mathcal{H})$ (Hopf algebra)

and $\bar{\varphi} : \mathcal{G} \longrightarrow \mathcal{H}$ (AAU)

Conclusions

More structured algebra of graphs can be seen as a combinatorial model of the Heisenberg-Weyl algebra.

In this way, abstract algebraic structures $\mathcal{H}$ and $\mathcal{U}(\mathcal{L}_{\mathcal{H}})$ gain intuitive interpretation as a shadow of natural constructions on graphs in $\mathcal{G}$.



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