Generalized Stretched Littlewood-Richardson Coefficients

Christian Gutschwager

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Outline

Partitions

Diagram

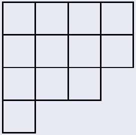
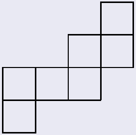
$$\lambda = (4^2, 3, 1) \qquad \mu = (3, 2)$$

$$\lambda = \ydiagram{4,4,3,1} \Longrightarrow \lambda/\mu = \ydiagram{3+1,2+2,3,1}$$

Partitions

Skew-diagram

$$\lambda = (4^2, 3, 1) \qquad \mu = (3, 2)$$

$\lambda =$  $\implies \lambda/\mu =$ 

Partitions

Multiplication and Addition

$$n\lambda = (n\lambda_1, n\lambda_2, n\lambda_3, \ldots)$$

$$10(4,4,3,1) = (40,40,30,10)$$

$$\lambda + \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \lambda_3 + \mu_3, \ldots)$$

$$(30,30,20,10) + (5,4,3,2,1) = (35,34,23,12,1)$$

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Littlewood-Richardson Coefficients

- Semistandard (weakly increasing among rows from left to right, strictly increasing among columns from top to bottom)
- Tableauword w is a lattice permutations.

Semistandard

semistandard:

			1	1	1
2	2	3	3		
4	4	4	4		

not semistandard:

			1	1	1
2	2	1	1		
2	2				

Littlewood-Richardson Coefficients

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- Tableauword w is a lattice permutation.

Lattice permutation

$$\begin{array}{cccccc} & & & \boxed{1} & \boxed{1} \\ \boxed{3} & \boxed{3} & \boxed{2} & \boxed{2} & \end{array}$$

$w = (112233)$

lattice permutation

$$\begin{array}{cccccc} & & & \boxed{1} & \boxed{1} \\ \boxed{1} & \boxed{2} & \boxed{2} & \boxed{2} & \end{array}$$

$w = (112221)$

no lattice permutation

Littlewood-Richardson Coefficients

- Semistandard (weakly increasing among rows from left to right, strictly increasing among columns from top to bottom)
- Tableauword w is a lattice permutations.

Definition

LR-Coefficient $c(\lambda; \mu, \nu)$ *equals the number of tableaux of shape* λ/μ *with content* ν *satisfying the above conditions.*

Skew characters

Skew characters

$$[\lambda/\mu] = \sum_{\nu} c(\lambda;\mu,\nu)[\nu]$$

Example $\lambda = (3,3,1), \mu = (2,1)$:

$$\begin{array}{ccc} & & 1 \\ & 1 & 2 \\ 1 & & \end{array} \qquad \begin{array}{ccc} & & 1 \\ & 1 & 2 \\ 2 & & \end{array} \qquad \begin{array}{ccc} & & 1 \\ & 1 & 2 \\ 3 & & \end{array}$$

$$[(3,3,1)/(2,1)] = [3,1] + [2,2] + [2,1,1]$$

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Skew Schur functions

$$s_{\lambda/\mu} = \sum_{\nu} c(\lambda;\mu,\nu) s_{\nu}$$

Schur functions

$$s_{\mu} s_{\nu} = \sum_{\lambda} c(\lambda;\mu,\nu) s_{\lambda}$$

Schubert Calculus, ...

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$$f(n) = c(n\lambda; n\mu, n\nu)$$

is a polynomial in n for $n \geq 0$.

Example

$$c(n(8,5,3,1); n(4,2,1), n(5,3,2)) = \frac{1}{6}(n+1)(n+2)(n+3)$$

Addition

$$c(\lambda + \lambda'; \mu + \mu', \nu + \nu') \geq c(\lambda; \mu, \nu)$$

if $c(\lambda'; \mu', \nu') \neq 0$

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Generalized Stretched LR-Coefficients

Question:

What can be said about:

$$P(n) = c(n\lambda + \lambda'; n\mu + \mu', n\nu + \nu')?$$

(with $c(\lambda; \mu, \nu), c(\lambda'; \mu', \nu') \neq 0$)

Easier:

What can be said about:

$$Q(n) = \sum_{\nu} c(n\lambda + \lambda'; n\mu + \mu', \nu)?$$

(with λ/μ and λ'/μ' skew diagrams)

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Results for $Q(n) = \sum_\nu c(n\lambda + \lambda'; n\mu + \mu', \nu)$

Results

- If λ/μ is neither a partition nor a rotated partition then $Q(n)$ increases without bound.

$$\text{(Because of } \sum_\nu c(n(2,1), n(1), \nu) = n + 1)$$

- If λ/μ is a partition then
  - there exists an m with $Q(n) = Q(m)$ for $n \geq m$.
  - $Q(n)$ is strictly increasing before it gets constant.
  - we have a formula to get the smallest m with $Q(n) = Q(m)$ for $n \geq m$.

$$m = \left\lceil \max_{\substack{1 \leq j \leq k \\ \alpha_j > \alpha_{j+1}}} \left(\frac{\lambda'_1 - \lambda'_{a_j} + \lambda'_{a_j+1} + \mu'_{a_1} - \mu'_{a_1-1}}{\alpha_j - \alpha_{j+1}} \right) \right\rceil$$

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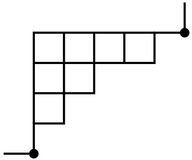
Example for $Q(n) = \sum_\nu c(n\lambda + \lambda'; n\mu + \mu', \nu)$

Parameter

$\lambda = (6,5,3,2,1), \mu = (6,1^4), \lambda' = (8^2,5,3^2,2,1), \mu' = (4,3,2,1^2)$ and $\mathcal{A}(n) = (n\lambda + \lambda')/(n\mu + \mu')$

$\lambda/\mu =$ 

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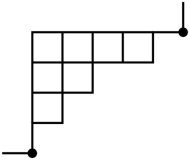

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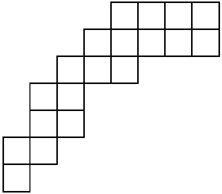

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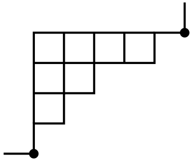
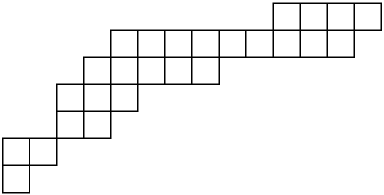
$\lambda/\mu =$ 

$\mathcal{A}(1) =$ 

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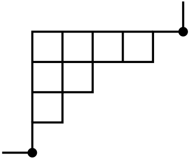
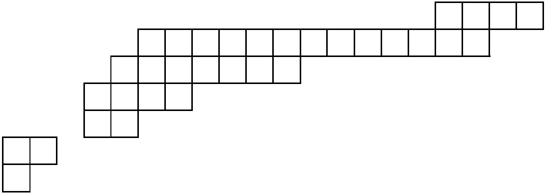


$\mathcal{A}(2) =$



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$Q(n)$

$Q(0)$	$Q(1)$	$Q(2)$	$Q(3)$	$Q(4)$	$Q(5)$	$Q(6)$	$Q(7)$
910	18.271	38.016	49.635	54.176	55.480	55.826	55.889

$Q(n \geq 8)$
55.895

Results for $P(n) = c(n\lambda + \lambda'; n\mu + \mu', n\nu + \nu')$

Results

- If $\lambda/\mu, \lambda/\nu$ or $\left((\lambda_1)^{l(\lambda)}/\mu\right)^{\circ}/\nu$ is a partition or rotated partition then it follows from $Q(n)$ that there is an m with $P(n) = P(m)$ for $n \geq m$.
- We get an upper bound for m.
- If $c(\lambda; \mu, \nu) \neq 1$ then $P(n)$ increases without bound (because then $c(n\lambda; n\mu, n\nu)$ increases without bound)

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Conjecture for $P(n) = c(n\lambda + \lambda'; n\mu + \mu', n\nu + \nu')$

Known:

$c(n\lambda; n\mu, n\nu)$ is a polynomial in n.

Conjecture

There exists a polynomial $g(n)$ of the same degree as $c(n\lambda; n\mu, n\nu)$ and an m such that $P(n) = g(n)$ for $n \geq m$.
In particular for $c(\lambda; \mu, \nu) = 1$ there exists an integer m with $P(n) = P(m)$ for $n \geq m$.

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Example for $P(n) = c(n\lambda + \lambda'; n\mu + \mu', n\nu + \nu')$

Parameter

For $\lambda = (6, 5, 4, 3^2, 1), \mu = (5, 3, 2, 1), \nu = (5, 3, 2, 1)$

$$c(n\lambda; n\mu, n\nu) = \frac{(n+1)(n+2)(n+3)(n+4)(n+5)(2n^2+5n+7)}{840}$$

is of degree 7.
Let $\lambda' = (9^3, 7, 3^4, 2, 1), \mu' = (7^2, 3, 2^3, 1^2), \nu' = (8, 5, 3^2, 2^2, 1)$.

Example for $P(n) = c(n\lambda + \lambda'; n\mu + \mu', n\nu + \nu')$

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$P(n)$

n :	0	1	2	3	4	$n \geq 5$
$P(n)$:	39	30.920	509.202	3.101.626	12.098.348	$g(n)$
$g(n)$:	55.407	50.333	513.782	3.102.223	12.098.382	$g(n)$

with

$$g(n) = \frac{1}{360}\big(8490n^7 + 214.525n^6 + 1.664.232n^5 + 5.835.910n^4 + 904.140n^3 + 8.621.725n^2 - 19.075.662n + 19.946.520\big).$$

Thank You

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On the arXiv you can find a paper with the same title containing proofs and so on.