Tropical Combinatorics

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Overview

1. **Tropical Hypersurfaces**
   - The tropical semi-ring
   - Polyhedral combinatorics
   - Puiseux series
2. **Tropical Convexity**
   - Tropical polytopes
   - Type decomposition
   - Products of simplices
3. **Resolution of Monomial Ideals**
   - The coarse type ideal
   - (Co)-cellular resolutions

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Tropical Arithmetic

tropical semi-ring: $(\mathbb{R} \cup \{+\infty\}, \oplus, \odot)$ where

$$x \oplus y \; := \; \min(x, y) \quad \text{and} \quad x \odot y \; := \; x + y$$

Example

$$(3 \oplus 5) \odot 2 \; = \; 3 + 2 \; = \; 5 \; = \; \min(5, 7) = (3 \odot 2) \oplus (5 \odot 2)$$

History

- can be traced back (at least) to the 1960s
  - e.g., see monography [Cunningham-Green 1979]
- optimization, functional analysis, signal processing, . . .
- recent development (since 2002) initiated by Kapranov, Mikhalkin, Sturmfels, . . .

- read ordinary (Laurent) polynomial with real coefficients as function
- replace operations "$+$" and "$\cdot$" by "$\oplus$" and "$\odot$"

Example

$$F(x) \ := \ (7 \odot x^{\odot 3}) \oplus (3 \odot x) \oplus 4 \ = \ \min(7 + 3x, 3 + x, 4)$$

Definition

tropical polynomial F vanishes at p :$\Leftrightarrow$ there are at least two terms where the minimum $F(p)$ is attained

$$F(1) \ = \ \min(7 + 3 \cdot 1, \boxed{3+1}, \boxed{4}) \ = \ 4$$

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- *tropical semi-module* $(\mathbb{R}^d, \oplus, \odot)$
  - componentwise addition
  - *tropical scalar multiplication*

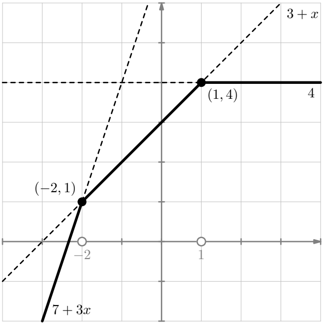
Definition

tropical hypersurface $\mathcal{T}(F)$:= vanishing locus of (multi-variate) tropical polynomial F

Example

$F(x) = (7 \odot x^{\odot 3}) \oplus (3 \odot x) \oplus 4$

$\mathcal{T}(F) = \{-2, 1\} \subset \mathbb{R}^1$



Proposition

For a tropical polynomial $F : \mathbb{R}^d \to \mathbb{R}$ the set

$$\mathcal{P}(F) \ := \ \Big\{(p,s) \in \mathbb{R}^{d+1} : \ p \in \mathbb{R}^d, \ s \in \mathbb{R}, \ s \leq F(p)\Big\}$$

is an unbounded convex polyhedron of dimension $d+1$.

Corollary

The tropical hypersurface $\mathcal{T}(F)$ coincides with the image of the codimension-2-skeleton of the polyhedron $\mathcal{P}(F)$ in $\mathbb{R}^d$ under the orthogonal projection which omits the last coordinate.

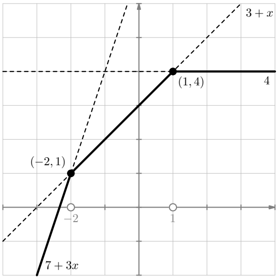
The Newton Polytope of a Tropical Polynomial

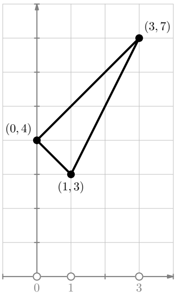
Definition

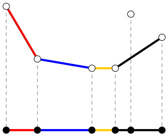
Newton polytope $\mathcal{N}(F)$ = convex hull of the support $\mathrm{supp}(F)$

Theorem

The tropical hypersurface $\mathcal{T}(F)$ of a tropical polynomial F is dual to the 1-coskeleton of the regular subdivision of $\mathcal{N}(F)$ induced by the coefficients of F.







tropical polynomial F *homogeneous of degree* δ if for all $p \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$:

$$F(\lambda \odot p) \; = \; F(\lambda \cdot \mathbf{1} + p) \; = \; \lambda^{\odot \delta} \odot F(p) \; = \; \delta \cdot \lambda + F(p)$$

Definition

tropical $(d-1)$-torus $\mathbb{T}^{d-1} := \mathbb{R}^d / \mathbb{R}\mathbf{1}$

map

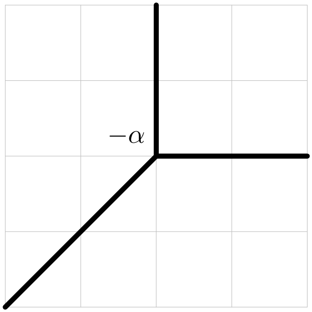
$$\begin{aligned}(x_1, x_2, \ldots, x_d) + \mathbb{R}\mathbf{1} \; &= \; (0, x_2 - x_1, \ldots, x_d - x_1) + \mathbb{R}\mathbf{1} \\ &\mapsto \; (x_2 - x_1, \ldots, x_d - x_1)\end{aligned}$$

defines homeomorphism $\mathbb{T}^{d-1} \approx \mathbb{R}^{d-1}$

Tropical Hyperplanes

$F(x) = (\alpha_1 \odot x_1) \oplus (\alpha_2 \odot x_2) \oplus (\alpha_3 \odot x_3)$ linear homogeneous

$$\begin{aligned}\mathcal{T}(F) &= -(\alpha_1, \alpha_2, \alpha_3) + (\mathbb{R}_{\geq 0} e_1 \cup \mathbb{R}_{\geq 0} e_2 \cup \mathbb{R}_{\geq 0} e_3) + \mathbb{R}\mathbf{1} \\ &= (0, \alpha_1 - \alpha_2, \alpha_1 - \alpha_3) + (\mathbb{R}_{\geq 0}(-e_2 - e_3) \cup \mathbb{R}_{\geq 0} e_2 \cup \mathbb{R}_{\geq 0} e_3)\end{aligned}$$

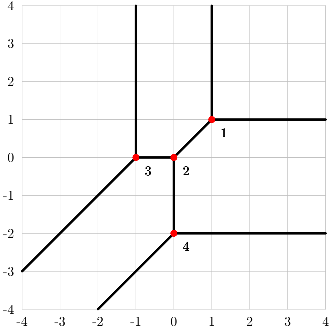


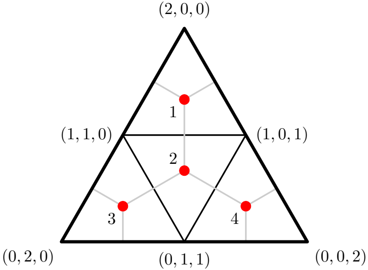
general tropical conic

$$(a_{200} \odot x_1^{\odot 2}) \oplus (a_{110} \odot x_1 \odot x_2) \oplus (a_{101} \odot x_1 \odot x_3) \oplus (a_{020} \odot x_2^{\odot 2}) \oplus (a_{011} \odot x_2 \odot x_3) \oplus (a_{002} \odot x_3^{\odot 2})$$

Example

$$(a_{200}, a_{110}, a_{101}, a_{020}, a_{011}, a_{002}) = (6, 5, 5, 6, 5, 7)$$





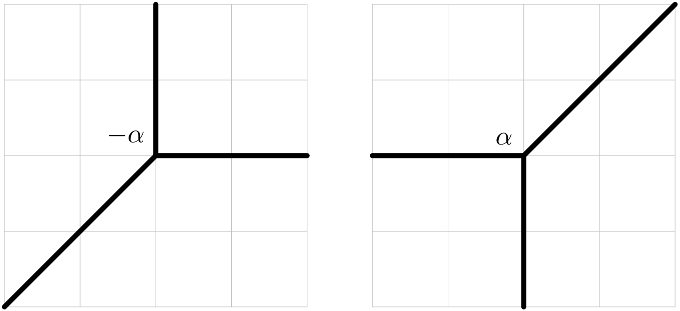
Max-Tropical Hyperplanes

duality between $\min$ and $\max$:

$$\max(-x,-y) \ = \ -\min(x,y)$$

Remark

$\mathcal{T}$ is $\min$-tropical hyperplane $\iff -\mathcal{T}$ is $\max$-tropical hyperplane



min/max

Fields of Puiseux Series

Puiseux series with complex coefficients:

$$\mathbb{C}\{\{z\}\} \;=\; \left\{ \sum_{k=m}^{\infty} a_k \cdot z^{k/N} \,:\; m \in \mathbb{Z}, N \in \mathbb{N}^{\times}, a_k \in \mathbb{C} \right\}$$

- Newton-Puiseux-Theorem: $\mathbb{C}\{\{z\}\}$ isomorphic to algebraic closure of *Laurent series* $\mathbb{C}((z))$
  - isomorphic to $\mathbb{C}$ by [Steinitz 1910]

The Valuation Map

valuation map

$$\mathrm{val}\,:\,\mathbb{C}\{\{z\}\}\to\mathbb{Q}\cup\{\infty\}$$

maps Puiseux series $\gamma(z)=\sum_{k=m}^{\infty}a_k\cdot z^{k/N}$ to lowest degree $\min\{k/N\colon k\in\mathbb{Z},\,a_k\neq 0\}$; setting $\mathrm{val}(0):=\infty$

$$\begin{aligned}\mathrm{val}(\gamma(z)+\delta(z)) &\geq \min\{\mathrm{val}(\gamma(z)),\mathrm{val}(\delta(z))\}\\ \mathrm{val}(\gamma(z)\cdot\delta(z)) &= \mathrm{val}(\gamma(z))+\mathrm{val}(\delta(z))\,.\end{aligned}$$

Remark

inequality becomes equation if no cancellation occurs

Theorem (Einsiedler, Kapranov & Lind 2006)

For $f \in \mathbb{K}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_d^{\pm 1}]$ the tropical hypersurface $\mathcal{T}(\mathrm{trop}(f)) \cap \mathbb{Q}^d$ (over the rationals) equals the set $\mathrm{val}(V(\langle f \rangle))$.

Tropical geometry is a piece-wise linear shadow of classical geometry.

A Lifting Theorem I

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A Lifting Theorem II

Proof of easy inclusion "$\mathcal{T}(\mathrm{trop}(f)) \supseteq \mathrm{val}(V(\langle f \rangle))$".

- let $f = \sum_{i \in I} \gamma_i x^i$ for $I \subset \mathbb{N}^d$ with tropicalization F
- consider zero $u \in (\mathbb{K}^\times)^d$ of f
- for $i \in I$ we have
  $\mathrm{val}(\gamma_i u^i) = \mathrm{val}(\gamma_i) + \langle i, \mathrm{val}(u) \rangle = \mathrm{val}(\gamma_i) \odot \mathrm{val}(u)^{\odot i}$
- minimum

$$F(\mathrm{val}(u)) \;=\; \bigoplus_{i \in I} \mathrm{val}(\gamma_i) \odot \mathrm{val}(u)^{\odot i}$$

attained at least twice since otherwise the terms $\gamma_i u^i$ cannot cancel to yield zero

□

Conclusion I

- tropicalization of (homogeneous) polynomial F
- tropical hypersurface $\mathcal{T}(F)$
  - codimension-2-skeleton of unbounded convex polyhedron
- regular subdivision of Newton polytope $\mathcal{N}(F)$
- tropical hypersurface = image of ordinary hypersurface under valuation map

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Tropical Convexity

for $x, y \in \mathbb{R}^d$ let [Zimmermann 1977] [Develin & Sturmfels 2004]

$$[x,y]_{\mathrm{trop}} \;:=\; \{(\lambda \odot x) \oplus (\mu \odot y) : \; \lambda, \mu \in \mathbb{R}\}$$

- $S \subseteq \mathbb{R}^d$ tropically convex: $[x,y]_{\mathrm{trop}} \subseteq S$ for all $x, y \in S$
- S tropically convex $\Rightarrow \lambda \odot S = \lambda \mathbf{1} + S \subseteq S$ for all $\lambda \in \mathbb{R}$
  - consider tropically convex sets in $\mathbb{T}^{d-1} = \mathbb{R}^d / \mathbb{R}\mathbf{1}$
  - recall: we identify

$$(x_0, x_1, \ldots, x_d) + \mathbb{R}\mathbf{1} \;=\; (0, x_1 - x_0, \ldots, x_d - x_0) + \mathbb{R}\mathbf{1}$$

    with $(x_1 - x_0, \ldots, x_d - x_0)$
- tropical polytope := *tropical convex hull* of finitely many points in $\mathbb{T}^{d-1} \approx \mathbb{R}^{d-1}$

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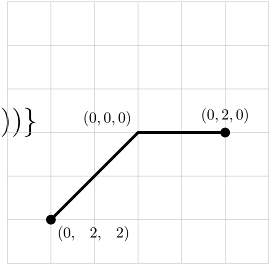
$$[(0,2,0),(0,-2,-2)]_{\text{trop}}$$
$$= \{\lambda \odot (0,2,0) \oplus \mu \odot (0,-2,-2) : \lambda, \mu \in \mathbb{R}\}$$
$$= \{(\min(\lambda,\mu), \min(\lambda+2,\mu-2), \min(\lambda,\mu-2))\}$$
$$= \{(\lambda,\lambda+2,\lambda) : \lambda \le \mu-4\}$$
$$\cup \{(\lambda,\mu-2,\lambda) : \mu-4 \le \lambda \le \mu-2\}$$
$$\cup \{(\lambda,\mu-2,\mu-2) : \mu-2 \le \lambda \le \mu\}$$
$$\cup \{(\mu,\mu-2,\mu-2) : \mu \le \lambda\}$$
$$= \{(0,\mu-\lambda-2,0) : 2 \le \mu-\lambda \le 4\}$$
$$\cup \{(0,\mu-\lambda-2,\mu-\lambda-2) : 0 \le \mu-\lambda \le 2\}$$



Case Distinction

$\lambda \in (-\infty, \mu-4] \cup [\mu-4, \mu-2] \cup [\mu-2, \mu] \cup [\mu, \infty)$

Example: Tropical Line Segment in $\mathbb{T}^2$

$$[(0,2,0),(0,-2,-2)]_{\text{trop}}$$
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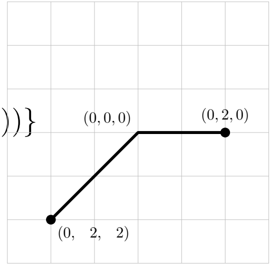

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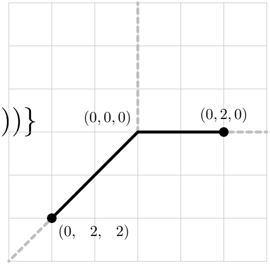

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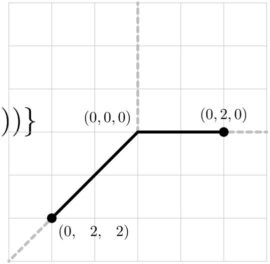

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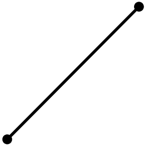
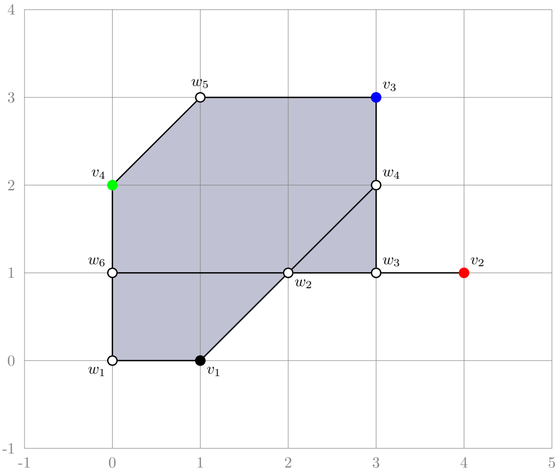


The Running Example

$n = 4$, $d = 3$
$v_1 = (0,1,0)$, $v_2 = (0,4,1)$, $v_3 = (0,3,3)$, $v_4 = (0,0,2)$



Fine Types

consider $V = (v_1, v_2, \ldots, v_n)$ in $\mathbb{T}^{d-1}$

Definition

fine type of $p \in \mathbb{T}^{d-1}$ w.r.t. V given by $T_V(p) \in \{0,1\}^{n \times d}$ with

$$T_V(p)_{ik} = 1 \quad \Leftrightarrow \quad v_{ik} - p_k \leq v_{ij} - p_j \text{ for all } j \in [d]$$

- identify T with $(T_1, T_2, \ldots, T_d)$, where $T_k = \{i \in [n] : T_{ik} = 1\}$

Example

$$V = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{pmatrix} \qquad T_V(0,2,0) = (\{2,3\}, \{1,4\}, \emptyset)$$

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consider $V = (v_1, v_2, \dots, v_n)$ in $\mathbb{T}^{d-1}$

Definition

fine type of $p \in \mathbb{T}^{d-1}$ w.r.t. V given by $T_V(p) \in \{0,1\}^{n \times d}$ with

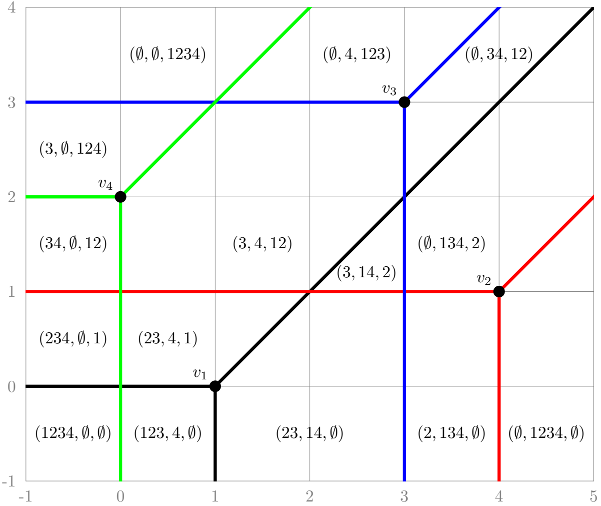
$$T_V(p)_{ik} = 1 \quad \Leftrightarrow \quad v_{ik} - p_k \leq v_{ij} - p_j \text{ for all } j \in [d]$$

- identify T with $(T_1, T_2, \dots, T_d)$, where $T_k = \{i \in [n] \colon T_{ik} = 1\}$

Example

$$V = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{pmatrix} \qquad T_V(0,2,0) = (\{2,3\}, \{1,4\}, \emptyset)$$

Fine Type Decomposition of $\mathbb{T}^{d-1}$



... induced by $\max$-tropical hyperplane arrangement $\mathfrak{A}(V)$

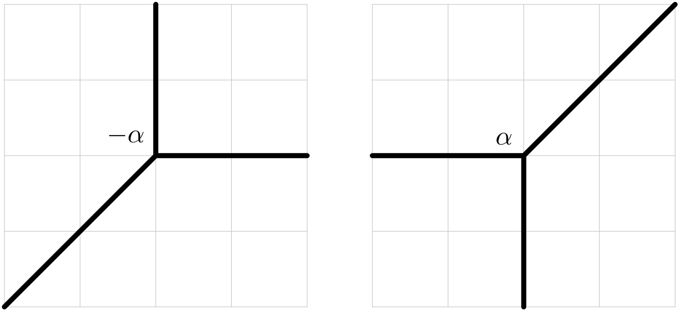
Recall: Max-Tropical Hyperplanes

duality between $\min$ and $\max$:

$$\max(-x,-y) \; = \; -\min(x,y)$$

Remark

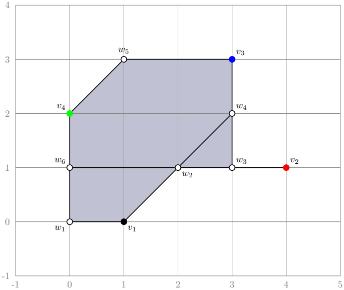
$\mathcal{T}$ is $\min$-tropical hypersurface $\Leftrightarrow$ $-\mathcal{T}$ is $\max$-tropical hypersurface

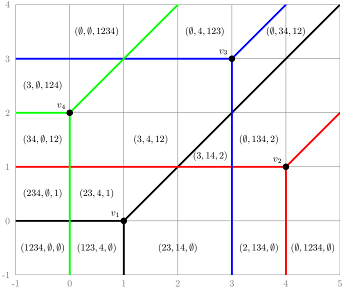


min/max

Theorem (Develin & Sturmfels 2004)

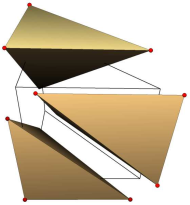
The $\min$*-tropical polytope* $\operatorname{tconv}(V)$ *is the union of the bounded closed cells of the* $\max$*-tropical hyperplane arrangement* $\mathfrak{A}(V)$.



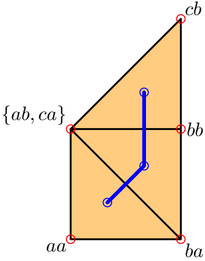


Products of Simplices

- $\mathrm{tconv}\{v_1, \ldots, v_n\} \subset \mathbb{T}^{d-1}$ dual to regular subdivision of $\Delta_{n-1} \times \Delta_{d-1}$ defined by lifting $e_i \times e_j$ to height v_{ij}
  - *general position* $\longleftrightarrow$ triangulation
- extra feature: exchanging the factors $\rightsquigarrow$ $\mathrm{tconv}(\text{rows}) \cong \mathrm{tconv}(\text{columns})$



$\Delta_1 \times \Delta_2$



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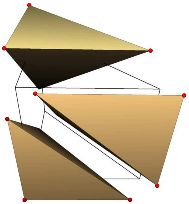

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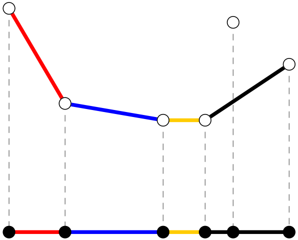
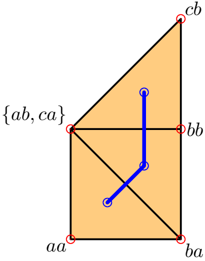


recall: regular subdivision



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Mixed Subdivisions

- P, Q : polytopes in $\mathbb{R}^d$
- $P + Q = \{p + q \colon p \in P,\ q \in Q\}$ *Minkowski sum*
- *Minkowski cell* of $P + Q$ = full-dimensional subpolytope which is Minkowski sum of subpolytopes of P and Q

Definition

Polytopal subdivision of $P + Q$ into Minkowski cells is mixed if for any two of its cells $P' + Q'$ and $P'' + Q''$ the intersections $P' \cap P''$ and $Q' \cap Q''$ both are faces.

- can be generalized to finitely many summands
- fine = cannot be refined (as a mixed subdivision!)

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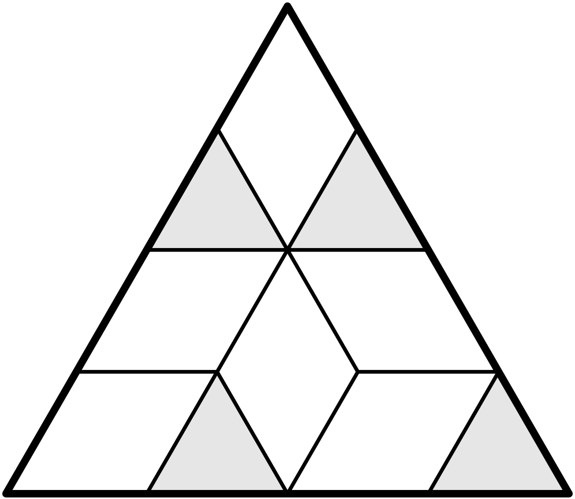
Example With 4 Summands

fine mixed subdivision of *dilated simplex*

$\Delta_2 + \Delta_2 + \Delta_2 + \Delta_2 = 4\Delta_2$



Cayley Trick, General Form

- $e_1, e_2, \ldots, e_n$: affine basis of $\mathbb{R}^{n-1}$
- $\phi_k : \mathbb{R}^d \to \mathbb{R}^{n-1} \times \mathbb{R}^d$ embedding $p \mapsto (e_k, p)$
- *Cayley embedding* of $P_1, P_2, \ldots, P_n$:

$$\mathcal{C}(P_1, P_2, \ldots, P_n) \;=\; \operatorname{conv} \bigcup_{i=1}^{n} \phi_i(P_i)\,.$$

Theorem (Sturmfels 1994; Huber, Rambau & Santos 2000)

1. *For any polyhedral subdivision of* $\mathcal{C}(P_1, P_2, \ldots, P_n)$ *the intersection of its cells with* $\{\frac{1}{n}\sum e_i\} \times \mathbb{R}^d$ *yields a mixed subdivision of* $\frac{1}{n}\sum P_i$*.*
2. *This correspondence is a poset isomorphism from the subdivisions of* $\mathcal{C}(P_1, P_2, \ldots, P_n)$ *to the mixed subdivisions of* $\sum P_i$*. Further, the coherent mixed subdivisions are bijectively mapped to the regular subdivisions.*

Cayley Trick for Products of Simplices

- consider $P_1 = P_2 = \cdots = P_n = \Delta_{d-1} = \text{conv}\{e_1, e_2, \ldots, e_d\}$
- $\mathcal{C}(\underbrace{\Delta_{d-1}, \Delta_{d-1}, \ldots, \Delta_{d-1}}_{n}) \cong \Delta_{n-1} \times \Delta_{d-1}$

Corollary

1. *For any polyhedral subdivision of $\Delta_{n-1} \times \Delta_{d-1}$ the intersection of its cells with $\{\frac{1}{n} \sum e_i\} \times \mathbb{R}^d$ yields a mixed subdivision of $\frac{1}{n} \cdot (n\Delta_{d-1})$.*
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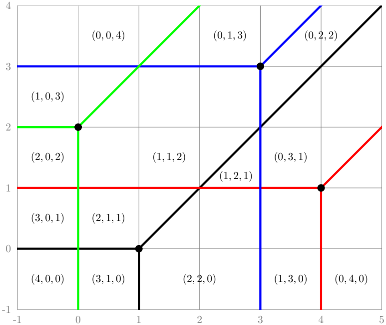

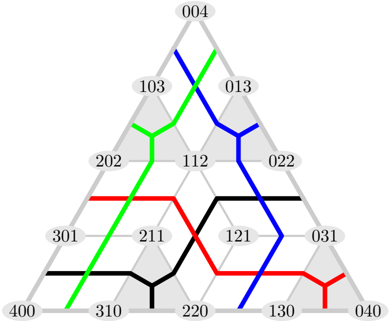

- fine types ⇝ coarse types
  - sum columns of type matrix ∼ replace sets by their cardinality
- coarse types of maximal cells = vertex coordinates of mixed subdivision

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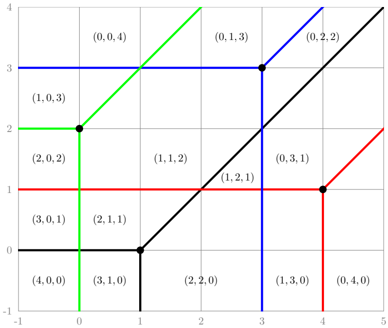

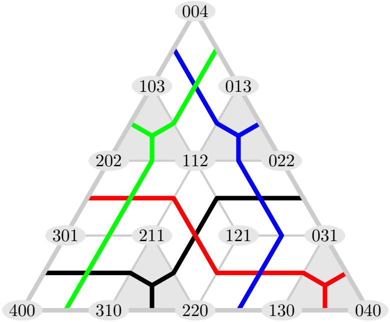

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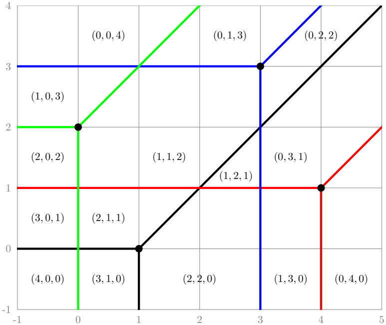

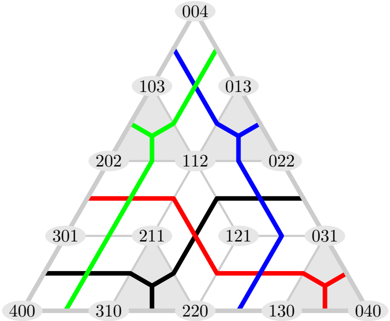

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A Tropical Hypersurface

- point $v_i \in \mathbb{T}^{d-1}$ = apex of unique max-tropical hyperplane $H^{\max}(v_i)$
- homogeneous linear form $h_i \in \mathbb{C}\{\{z\}\}[x_1, x_x, \ldots, x_d]$;

$$h \;:=\; h_1 \cdot h_2 \cdots h_n$$

Proposition

The tropical hypersurface defined by $\operatorname{trop}^{\max}(h)$ *is the union of the* max*-tropical hyperplanes in* $\mathfrak{A}$.

Corollary

Let $p \in \mathbb{T}^{d-1} \setminus \mathfrak{A}$ *be a generic point. Then its coarse type* $\mathfrak{t}_{\mathfrak{A}}(p)$ *equals the exponent of the monomial in* h *which defines the unique facet of* $\mathcal{P}(\operatorname{trop}^{\max}(h))$ *above* p.

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- n points in $\mathbb{T}^{d-1}$ $\leftrightarrow$ arrangement of n tropical hyperplanes in $\mathbb{T}^{d-1}$
  - tropical polytope = union of bounded cells
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The Coarse Type Ideal

Definition

Let $\mathfrak{A} = \mathfrak{A}(V)$ be an arrangement on n tropical hyperplanes in $\mathbb{T}^{d-1}$. The coarse type ideal of $\mathfrak{A}$ is the monomial ideal

$$I_{\mathbf{t}(\mathfrak{A})} \;=\; \langle x^{\mathbf{t}(p)} : p \in \mathbb{T}^{d-1} \rangle \;\subset\; K[x_1, \ldots, x_d]$$

where $x^{\mathbf{t}(p)} = x_1^{\mathbf{t}(p)_1} x_2^{\mathbf{t}(p)_2} \cdots x_d^{\mathbf{t}(p)_d}$.

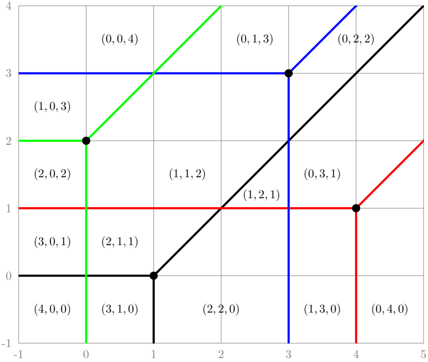
- similar construction for (oriented) matroids
  [Novik, Postnikov & Sturmfels 2002]

Powers of the Maximal Ideal

Proposition

If $\mathfrak{A} = \mathfrak{A}(V)$ is sufficiently generic the coarse cotype ideal is

$$I_{\mathbf{t}(\mathfrak{A})} \;=\; \langle x_1, x_2, \ldots, x_n \rangle^d \, .$$



Resolutions via Coarse Tropical Convexity

Theorem (Dochtermann, Sanyal & J. 2009+)

Let $\mathfrak{A}$ be an arrangement of n tropical hyperplanes in $\mathbb{T}^{d-1}$. The colabeled complex $\mathcal{C}_{\mathfrak{A}}$ supports a minimal cocellular resolution of the coarse type ideal $I_{\mathbf{t}(\mathfrak{A})}$.

- *Eliahou-Kervaire resolution* of $\langle x_1, x_2, \ldots, x_n \rangle^d$ in the sufficiently generic case

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Minimal Free Resolutions

- $S = K[x_1, \ldots, x_d]$ polynomial ring with $\mathbb{Z}^d$-grading $\deg x^a = a$
- free $\mathbb{Z}^d$-graded resolution $\mathcal{F}_\bullet$ of *monomial* ideal $I \subseteq S$ is exact (algebraic) complex of $\mathbb{Z}^d$-graded S-modules:

$$\mathcal{F}_\bullet : \quad \cdots \xrightarrow{\phi_{k+1}} F_k \xrightarrow{\phi_k} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \longrightarrow 0$$

  - $F_i \cong \bigoplus_{a \in \mathbb{Z}^d} S(-a)^{\beta_{i,a}}$ free $\mathbb{Z}^d$-graded S-modules
  - maps ϕ_i homogeneous
  - $F_0 = S$ and $\operatorname{img} \phi_1 = I$
  - i-th syzygy module $\operatorname{img} \phi_{i+1} \subset F_i$
- resolution minimal if $\beta_{i,a} = \dim_K \operatorname{Tor}_i^S(I, K)_a$

  fine graded Betti numbers

(Co-)Cellular Resolutions I

- $\mathcal{P}$ polyhedral complex
- $\mathbb{Z}^d$-labeling of cells with $\mathbf{t}_H = \mathrm{lcm}\,\{\mathbf{t}_G : \text{ for } G \subset H \text{ a face}\}$
- free modules

$$F_i \;=\; \bigoplus_{H\in\mathcal{P},\,\dim H = i+1} S(-\mathbf{t}_H)$$

- differentials $\phi_i : F_i \longrightarrow F_{i-1}$ given on generators by

$$\phi_i(H) \;=\; \sum_{\dim G = \dim H - 1} \epsilon(H,G) x^{\mathbf{t}_H - \mathbf{t}_G} G$$

- $\mathcal{P}_{\leq b}$ = subcomplex of $\mathcal{P}$ given by cells $H \in \mathcal{P}$ with $\mathbf{t}_H \leq b$ for some $b \in \mathbb{Z}^d$
- b-graded component of $\mathcal{F}^{\mathcal{P}}_{\bullet}$ = cellular chain complex of $\mathcal{P}_{\leq b}$

(Co-)Cellular Resolutions II

Proposition

If for every $b \in \mathbb{Z}^d$ the subcomplex $\mathcal{P}_{\leq b}$ is acyclic over K, then $\mathcal{F}_\bullet^\mathcal{P}$ is a free resolution of the ideal I generated by all monomials corresponding to the vertex labels of $\mathcal{P}$. Moreover, if $\mathbf{t}_F \neq \mathbf{t}_G$ for all cells $F \supset G$ then the cellular resolution is minimal.

- cellular: $\mathbf{t}_H = \operatorname{lcm}\{\mathbf{t}_G : \text{ for } G \subset H \text{ a face}\}$
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[Bayer & Sturmfels 1998] [Miller 1998]

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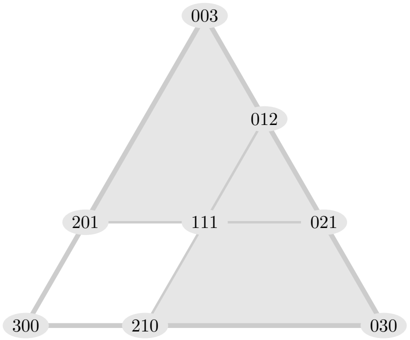

- ideal

$$I = \langle x_1^3, x_1^2x_2, x_2^3, x_1^2x_3, x_1x_2x_3, x_2^2x_3, x_2x_3^2, x_3^3\rangle$$

  - lcm condition $\rightsquigarrow$ labels for all cells
- $f(\mathcal{P}) = (8, 11, 4)$
- check condition for each $b \in \mathbb{Z}^d$
  - e.g., $b = (2, 2, 2)$
- *minimal* resolution

chain complex

$$\mathcal{F}_\bullet^{\mathcal{P}} : \; 0 \to S^4 \xrightarrow{\phi_3} S^{11} \xrightarrow{\phi_2} S^8 \xrightarrow{\phi_1} S \to 0$$

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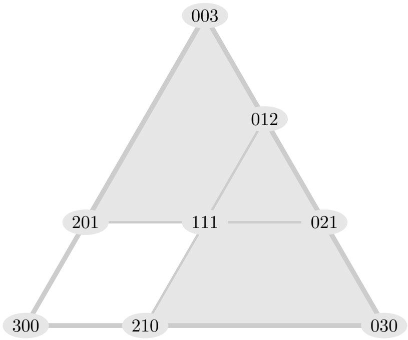

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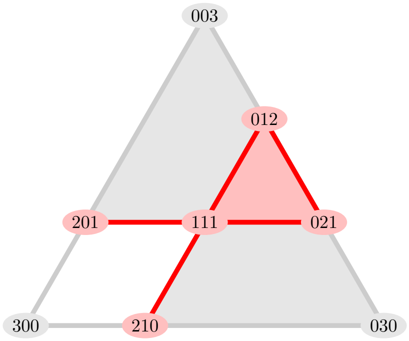

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  - lcm condition $\rightsquigarrow$ labels for all cells
- $f(\mathcal{P}) = (8, 11, 4)$
- check condition for each $b \in \mathbb{Z}^d$
  - e.g., $b = (2,2,2)$
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chain complex

$$\mathcal{F}_\bullet^{\mathcal{P}} : \; 0 \to S^4 \xrightarrow{\phi_3} S^{11} \xrightarrow{\phi_2} S^8 \xrightarrow{\phi_1} S \to 0$$

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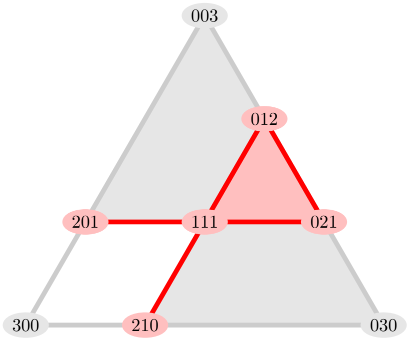
An Example



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Digression: Tropical Convexity

Proposition

Each tropically convex set is contractible.

Proof.

- *distance* of two points $p, q \in \mathbb{T}^{d-1}$:

$$\mathrm{dist}(p,q) \ := \ \max_{1 \leq i < j \leq d} |p_i - p_j + q_j - q_i|$$

  - $(\mathrm{dist}(p,q) \odot p) \oplus q = q$ and $p \oplus (\mathrm{dist}(p,q) \odot q) = p$

- for point p in tropically convex set S define continuous map

$$\eta \ : \ (q,t) \mapsto ((1-t) \cdot \mathrm{dist}(p,q)) \odot p) \oplus ((t \cdot \mathrm{dist}(p,q)) \odot q)$$

- contracts the set S to the point p

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Key Observation

$\mathfrak{A} = \mathfrak{A}(V)$: max-tropical hyperplane arrangement in $\mathbb{T}^{d-1}$

Proposition

For $p, q \in \mathbb{T}^{d-1}$ let $r \in \mathrm{tconv}^{\max}\{p, q\}$ be an arbitrary point on the max-tropical line segment between p and q. Then for arbitrary $k \in [d]$:

$$\mathbf{t}_{\mathfrak{A}}(r)_k \;\leq\; \max\{\mathbf{t}_{\mathfrak{A}}(p)_k, \mathbf{t}_{\mathfrak{A}}(q)_k\}$$

Corollary

For arbitrary $b \in \mathbb{N}^d$:

$$(C_{\mathfrak{A}})_{\leq b} \;:=\; \left\{p \in \mathbb{T}^{d-1} : \; \mathbf{t}_{\mathfrak{A}}(p) \leq b\right\} \;=\; \bigcup \{C \in \mathcal{C}_{\mathfrak{A}} : \; \mathbf{t}_{\mathfrak{A}}(C) \leq b\}$$

$\max$-*tropically convex and hence contractible.*

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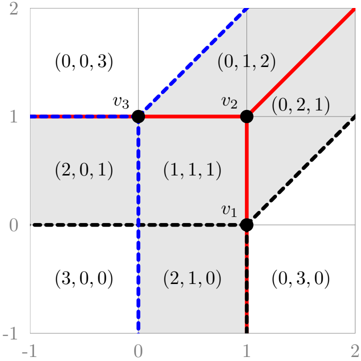
Example

$$n = d = 3\,, \qquad V = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\,, \qquad b = (2,2,2)$$



Main Result, Revisited

$\mathfrak{A}$: arrangement of n tropical hyperplanes in $\mathbb{T}^{d-1}$
$\Sigma_{\mathfrak{A}}$: associated mixed subdivision of $n\Delta_{d-1}$

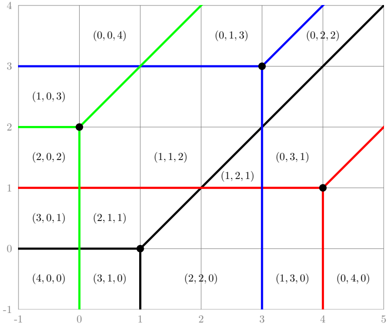
Theorem (Dochtermann, Sanyal & J. 2009+)

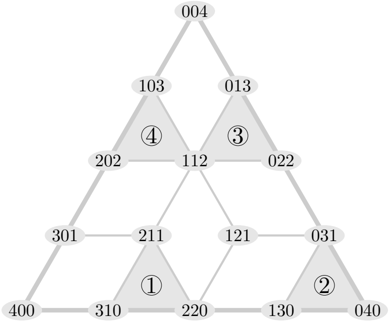
The colabeled complex $\mathcal{C}_{\mathfrak{A}}$ supports a minimal cocellular resolution of the coarse type ideal $I_{\mathbf{t}(\mathfrak{A})}$.

Corollary

The labeled polyhedral complex $\Sigma_{\mathfrak{A}}$ supports a minimal cellular resolution of the coarse type ideal $I_{\mathbf{t}(\mathfrak{A})}$.

A Sufficiently Generic Example

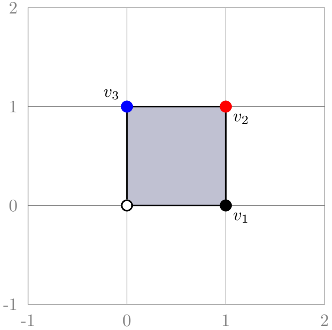


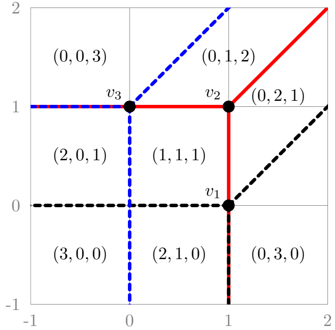


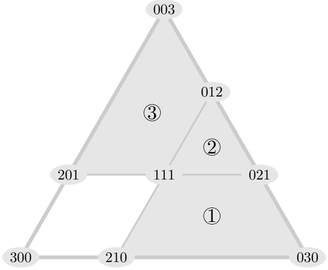
$$I_{\mathbf{t}(\mathfrak{A})} \;=\; \langle x_1, x_2, x_3, x_4 \rangle^3$$

$$0 \to S^{10} \to S^{24} \to S^{15} \to S \to 0$$

Ideal Generated by Non-generic Points







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Conclusion and Remarks III

- tropical hypersurfaces
  - general tropical varieties defined as subfan of Gröbner fan [Sturmfels]
  - tropical curves, coordinate-free approach [Mikhalkin]
- tropical convexity
  - max-plus linear algebra $\rightsquigarrow$ optimization
  - exterior descriptions
  - tropical Grassmannians
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