Surprising correlations in random orientations of graphs

(or what is special with $n = 27$)

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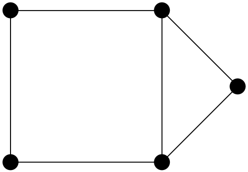
Sept 29, 2009

Edge percolation

- $G = (V, E)$ a graph



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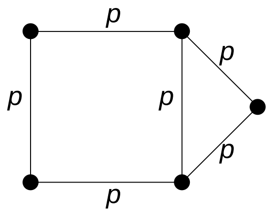

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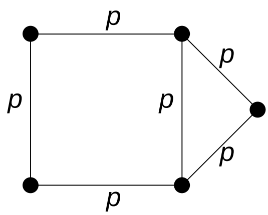

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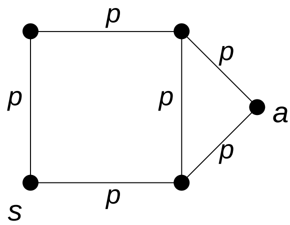

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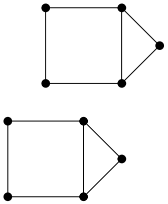

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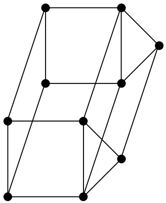

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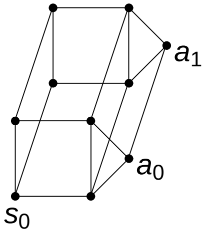

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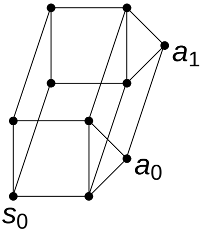

Conjecture (Kasteleyn '85)

For any G and $0 \leq p \leq 1$ *and any vertices* $s, a \in V$ *we have*

$$P(s_0 \leftrightarrow a_0) \geq P(s_0 \leftrightarrow a_1) \quad \text{in } G \times K_2$$

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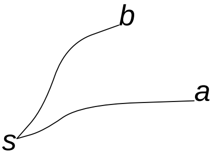
Theorem (Leander '09)

BBC is true for all wheels and subgraphs of wheels.

Correlations

Given any graph $G = (V, E)$ and three vertices $s, a, b \in V$.



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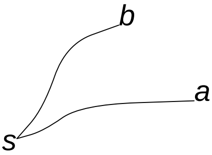

Classical fact:

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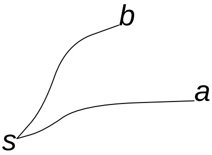

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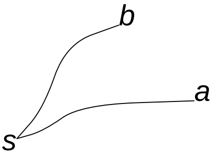

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Note:
$P(s \leftrightarrow a | s \leftrightarrow b) \geq P(s \leftrightarrow a) \Leftrightarrow P(s \leftrightarrow a, s \leftrightarrow b) \geq P(s \leftrightarrow a)P(s \leftrightarrow b)$

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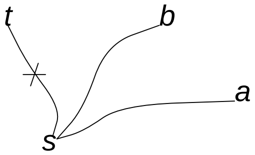

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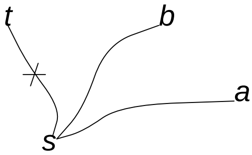

Theorem (van den Berg & Kahn '02)

For any G the events $\{s \leftrightarrow a\}$ and $\{s \leftrightarrow b\}$ are positively correlated in E^p, also when we first condition on $\{s \nleftrightarrow t\}$, i.e.

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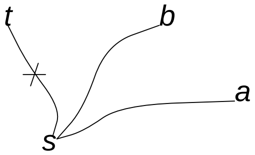

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Clever use of Ahlswede-Daykin's inequality. □

Random Orientations (O)

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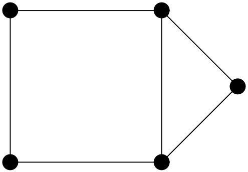

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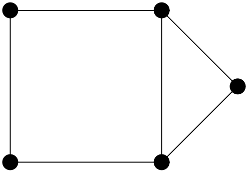

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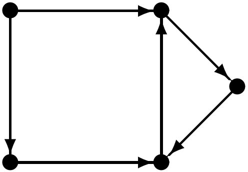

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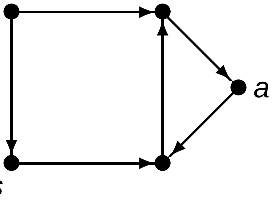

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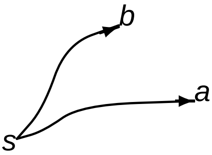

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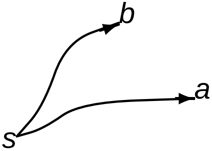

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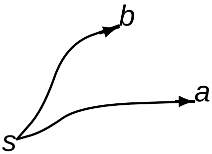

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Follows from:

Lemma (Mc Diarmid '81)

For any graph $G = (V, E)$ and $s, a \in V$ we have

$$P_{E^{1/2}(G)}(s \leftrightarrow a) = P_{O(G)}(s \to a).$$

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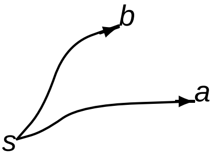

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Surprising?

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Proof.

We have the recursion $P_{E^p}\left(C_s(G) = U\right) =$

$$\sum_{W: s \in W \subseteq U \setminus v} P_{E^p}\left(C_s(G \setminus v) = W\right)(1 - q^r)P_{E^p}\left(C_v(G \setminus W) = U \setminus W\right).$$

□

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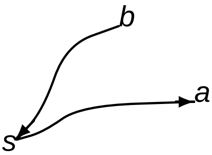
Are the events $\{s \to a\}$ and $\{b \to s\}$ negatively correlated in any graph G?



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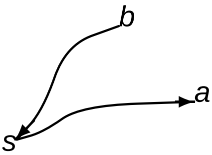

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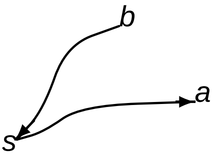

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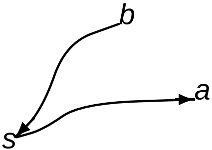

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Theorem (Alm & L. '09)

In model O the events $\{s \to a\}$ and $\{b \to s\}$:
are negatively correlated in K_3,
are independent in K_4,
are positively correlated in $K_n, n \geq 5$,
are negatively correlated in trees and cycles.

Random Orientation on $G(n,p)$

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For fixed p, as $n \to \infty$ we have:
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Proof.

Identify main cases and then long tricky computations. □

We also fixed n and computed $P(s \to a)$ and $P(s \to a, b \to s)$ using exact recursions. With this we computed the value of critical p as in the following table:

n	critical p
4	1
5	0.729
6	0.276
7	0.152
8	0.107
9	0.082
10	0.067
11	0.056
12	0.049
13	0.043
14	0.038
15	0.035
16	0.032

Converges to $1/2$????

Recall:

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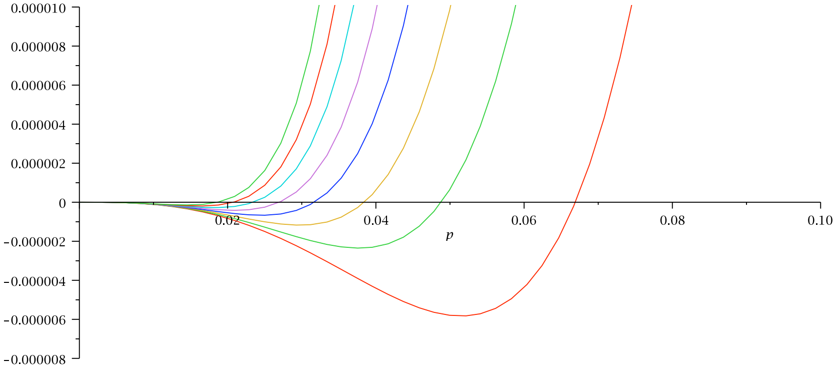
In fact we proved

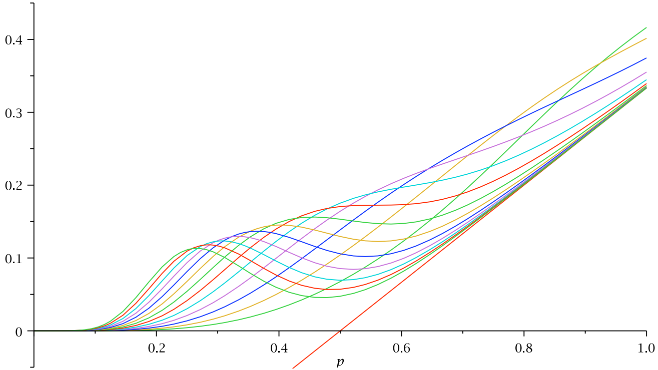
$$1 - \frac{P(b \nrightarrow s)}{P(b \nrightarrow s | s \nrightarrow a)} \to \frac{2p-1}{3}, \quad \text{as } n \to \infty$$

This is a plot of $1 - \frac{P(b \nrightarrow s)}{P(b \nrightarrow s | s \nrightarrow a)}$ for $n = 10..24$



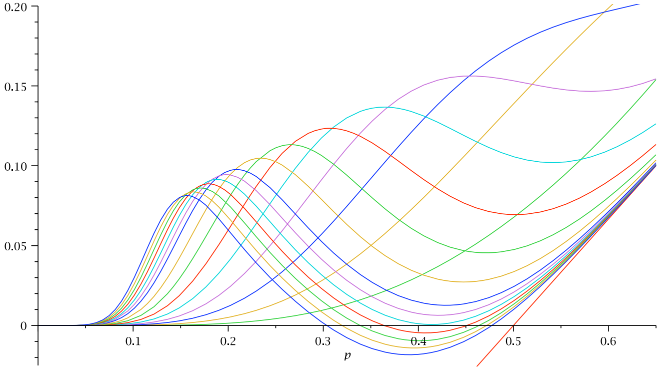
What was wrong? We spent many days looking for an error.

Then I plotted $1 - \frac{P(b \not\to s)}{P(b \not\to s | s \not\to a)}$ for $n = 8..20$ and all p:



What would happen for larger n?

Plot of $1 - \frac{P(b \not\to s)}{P(b \not\to s | s \not\to a)}$ for $n = 12..30$ as a function of p:



Starting from $n = 27$ we get 3 critical values of p.

Some open problems

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- Prove the Bunkbed Conjecture for all graphs!