Action of the differential Galois group of polylogarithms on their asymptotic expansions

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Summary

FROM DRINDFEL'D EQUATION TO POLYZETAS

Dindfel'd equation and polylogarithms

$(DE)\ dG = (x_0\omega_0 + x_1\omega_1)G,$ with $\omega_0(z) = \frac{dz}{z}, \omega_1(z) = \frac{dz}{1-z}.$
The iterated integral over ω_0 and ω_1 along the path $z_0 \rightsquigarrow z$ and associated to the word $x_{i_1} \dots x_{i_r} \in \{x_0, x_1\}^*$ is denoted by

$$\alpha_{z_0}^{z}(x_{i_1} \dots x_{i_r}) = \int_{z_0}^{z} \omega_{i_1}(t_1) \int_{z_0}^{t_1} \omega_{i_2}(t_2) \dots \int_{z_0}^{t_{r-2}} \omega_{i_r}(t_{r-1}) \int_{z_0}^{t_{r-1}} \omega_{i_r}(t_r).$$

Let $\mathbf{s} = (s_1, \dots, s_r)$. Then,

$$\alpha_0^z(x_0^{s_1-1}x_1 \dots x_0^{s_r-1}x_1) = \mathrm{Li}_{\mathbf{s}}(z) = \sum_{n_1 > \dots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \dots n_r^{s_r}},$$

$$\mathrm{P}_{\mathbf{s}}(z) = \frac{\mathrm{Li}_{\mathbf{s}}(z)}{1-z} = \sum_{N \geq 0} \mathrm{H}_{\mathbf{s}}(N)\, z^N, \text{ where } \mathrm{H}_{\mathbf{s}}(N) = \sum_{n_1 > \dots > n_r = 1}^{N} \frac{1}{n_1^{s_1} \dots n_r^{s_r}}.$$

If $s_1 > 1$, by a theorem of Abel, then

$$\lim_{z \to 1} \mathrm{Li}_{\mathbf{s}}(z) = \lim_{N \to \infty} \mathrm{H}_{\mathbf{s}}(N) = \zeta(\mathbf{s}) = \sum_{n_1 > \dots > n_r > 0} \frac{1}{n_1^{s_1} \dots n_r^{s_r}}$$

else ?

Encoding the multi-indices by words

$Y = \{y_k | k \in \mathbb{N}_+\}$ $(y_1 < y_2 < \ldots)$ and $X = \{x_0, x_1\}$ $(x_0 < x_1)$.
Y^* (resp. X^*) : monoid generated by Y (resp. X).

$$\mathbf{s} = (s_1, \ldots, s_r) \leftrightarrow w = y_{s_1} \ldots y_{s_r} \leftrightarrow w = x_0^{s_1 - 1} x_1 \ldots x_0^{s_r - 1} x_1.$$

u and v are convergent if $s_1 > 1$. A word divergent is of the form

$$(\{1\}^k, s_{k+1}, \ldots, s_r) \leftrightarrow y_1^k y_{s_{k+1}} \ldots y_{s_r} \leftrightarrow x_1^k x_0^{s_{k+1}-1} x_1 \ldots x_0^{s_r - 1} x_1, \quad \text{for} \quad k \geq 1.$$

$$\begin{aligned}
\mathrm{Li} : w &\mapsto \mathrm{Li}_w(z) = \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_r^{s_r}}, \\
\zeta : w &\mapsto \zeta(w) = \sum_{n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}, \\
\mathrm{H} : w &\mapsto \mathrm{H}_w(N) = \sum_{N \geq n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}, \\
\mathrm{P} : w &\mapsto \mathrm{P}_w(z) = \sum_{N \geq 0} \mathrm{H}_w(N) z^N = \frac{\mathrm{Li}_w(z)}{1 - z}.
\end{aligned}$$

Let $\Pi_X : \mathbb{C}\langle\langle Y \rangle\rangle \longrightarrow \mathbb{C}\langle\langle X \rangle\rangle$ and $\Pi_Y : \mathbb{C}\langle\langle X \rangle\rangle \longrightarrow \mathbb{C}\langle\langle Y \rangle\rangle$ denote the "change" of alphabets over $\mathbb{C}\langle\langle X \rangle\rangle$ and $\mathbb{C}\langle\langle Y \rangle\rangle$ respectively.

Structures of polylogarithms

Let $\mathcal{C} = \mathbb{C}[z, z^{-1}, (1-z)^{-1}]$

Theorem (HNM, van der Hoeven & Petitot, 1998)

Putting $\mathrm{Li}_{x_0}(z) = \log z$, $\mathrm{Li} : w \mapsto \mathrm{Li}_w$ *becomes an isomorphism from* $(\mathbb{C}\langle X \rangle, ш)$ *to* $(\mathbb{C}\{\mathrm{Li}_w\}_{w \in X^*}, .)$.

- $\mathrm{Li}_w, w \in X^*$, are $\mathcal{C}$-linearly independent.
- $\mathrm{Li}_l, l \in \mathcal{L}ynX$, are algebraically independent.
- $\zeta(l), l \in \mathcal{L}ynX \setminus \{x_0, x_1\}$, are generators of the $\mathbb{Q}$-algebra generated by convergent polyzêtas, denoted by $\mathcal{Z}$.

Theorem (HNM, 2003)

$(\mathbb{C}\{\mathrm{P}_w\}_{w \in Y^*}, \odot) \cong (\mathbb{C}\langle Y \rangle, ⧢)$.

- P_w (then H_w), $w \in Y^*$, are $\mathcal{C}$-linearly independent.
- P_l (then H_l), $l \in \mathcal{L}ynY$, are algebraically independent.
- $\zeta(l), l \in \mathcal{L}ynY \setminus \{y_1\}$, are generators of the algebra $\mathcal{Z}$.

Towards the structure of polyzêtas

Corollary

$\forall u, v \in X^*, \mathrm{Li}_u \mathrm{Li}_v = \mathrm{Li}_{u \,ш\, v} \Rightarrow \forall u, v \in x_0 X^* x_1, \zeta(u)\zeta(v) = \zeta(u \,ш\, v).$

Example

$$\begin{aligned} x_0x_1 \,ш\, x_0^2x_1 &= x_0x_1x_0^2x_1 + 3x_0^2x_1x_0x_1 + 6x_0^3x_1^2, \\ \mathrm{Li}_2 \mathrm{Li}_3 &= \mathrm{Li}_{2,3} + 3\,\mathrm{Li}_{3,2} + 6\,\mathrm{Li}_{4,1}, \\ \zeta(2)\zeta(3) &= \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1). \end{aligned}$$

Corollary

$\forall u, v \in Y^*, \mathrm{H}_u \mathrm{H}_v = \mathrm{H}_{u \uplus v} \Rightarrow \forall u, v \in Y^* \setminus y_1 Y^*, \zeta(u)\zeta(v) = \zeta(u \uplus v).$

Example

$$\begin{aligned} y_2 \uplus y_3 &= y_2y_3 + y_3y_2 + y_5, \\ \mathrm{P}_{y_2} \odot \mathrm{P}_{y_3} &= P_{y_2y_3} + \mathrm{P}_{y_3y_2} + \mathrm{P}_{y_5}, \\ \mathrm{H}_2\mathrm{H}_3 &= \mathrm{H}_{2,3} + \mathrm{H}_{3,2} + \mathrm{H}_5, \\ \zeta(2)\zeta(3) &= \zeta(2,3) + \zeta(3,2) + \zeta(5). \end{aligned}$$

$$\left.\begin{aligned} \zeta(2)\zeta(3) &= \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1) \\ \zeta(2)\zeta(3) &= \zeta(2,3) + \zeta(3,2) + \zeta(5) \end{aligned}\right\} \Rightarrow \zeta(5) = 2\zeta(3,2) + 6\zeta(4,1).$$

Polynomial relations among $\{\zeta(l)\}_{l\in\mathcal{L}ynX\setminus\{x_0,x_1\}}$

$$
\begin{aligned}
\zeta(2,1) &= \zeta(3)\\
\zeta(4) &= \frac{2}{5}\zeta(2)^2\\
\zeta(3,1) &= \frac{1}{10}\zeta(2)^2\\
\zeta(2,1,1) &= \frac{2}{5}\zeta(2)^2\\
\zeta(4,1) &= 2\zeta(5)-\zeta(2)\zeta(3)\\
\zeta(3,2) &= -\frac{11}{2}\zeta(5)+3\zeta(2)\zeta(3)\\
\zeta(3,1,1) &= 2\zeta(5)-\zeta(2)\zeta(3)\\
\zeta(2,2,1) &= -\frac{11}{2}\zeta(5)+3\zeta(2)\zeta(3)\\
\zeta(2,1,1,1) &= \zeta(5)\\
\zeta(6) &= \frac{8}{35}\zeta(2)^3\\
\zeta(5,1) &= -\frac{1}{2}\zeta(3)^2+\frac{6}{35}\zeta(2)^3\\
\zeta(4,2) &= \zeta(3)^2-\frac{32}{105}\zeta(2)^3\\
\zeta(4,1,1) &= -\zeta(3)^2+\frac{23}{70}\zeta(2)^3\\
\zeta(3,2,1) &= 3\zeta(3)^2-\frac{29}{30}\zeta(2)^3\\
\zeta(3,1,2) &= -\frac{3}{2}\zeta(3)^2+\frac{53}{105}\zeta(2)^3\\
\zeta(3,1,1,1) &= -\frac{1}{2}\zeta(3)^2+\frac{6}{35}\zeta(2)^3\\
\zeta(2,2,1,1) &= \zeta(3)^2-\frac{32}{105}\zeta(2)^3\\
\zeta(2,1,1,1,1) &= \frac{8}{35}\zeta(2)^3
\end{aligned}
$$

GROUP OF DRINDFEL'D ASSOCIATORS

Φ_{KZ} associator

$$\mathrm{L}(z) := \sum_{w \in X^*} \mathrm{Li}_w(z)\, w \quad \text{and} \quad \mathrm{P}(z) := \frac{\mathrm{L}(z)}{1-z} = \sum_{w \in X^*} \mathrm{P}_w(z)\, w.$$

Theorem (HNM, van der Hoeven & Petitot, 1998)

Let $\mathcal{L}ynX$ be the set of Lyndon words. $\{S_l\}_{l \in \mathcal{L}ynX}$ and $\{\check{S}_l\}_{l \in \mathcal{L}ynX}$ denote the transcendental basis of $(\mathbb{C}\langle X \rangle, ш)$ and its dual basis respectively. Then $\mathrm{L}(z) = e^{x_1 \log \frac{1}{1-z}} \mathrm{L}_{\mathrm{reg}}(z) e^{x_0 \log z}$, where

$$\mathrm{L}_{\mathrm{reg}}(z) = \prod_{l \in \mathcal{L}ynX, l \neq x_0, x_1}^{\searrow} e^{\mathrm{Li}_{\check{S}_l}(z)\, S_l}.$$

$\Phi_{KZ} := \mathrm{L}_{\mathrm{reg}}(1).$

Proposition

Let $\zeta_ш : \mathbb{C}\langle\langle X \rangle\rangle \longrightarrow \mathbb{C}$ be the shuffle algebra morphism defined by

- $\zeta_ш(x_0) = \zeta_ш(x_1) = 0$,
- *for any* $r_1 > 1, \zeta_ш(x_0^{r_1-1}x_1 \ldots x_0^{r_k-1}x_1) = \zeta(r_1, \ldots, r_k)$,
- *for any* $u, v \in X^*, \zeta_ш(u ш v) = \zeta_ш(u)\zeta_ш(v)$.

Then $\displaystyle\sum_{w \in X^*} \zeta_ш(w)\, w = \Phi_{KZ}.$

Noncommutative generating series of harmonic sums

$$\mathrm{H}(N) := \sum_{w \in Y^*} \mathrm{H}_w(N)\, w.$$

Let $\mathcal{L}ynY$ be the set of Lyndon words over Y and let $\{\Sigma_l\}_{l \in \mathcal{L}ynY}$ and $\{\check{\Sigma}_l\}_{l \in \mathcal{L}ynY}$ be respectively the transcendental basis of $(\mathbb{C}\langle Y \rangle, \uplus)$ and its dual basis, defined by putting

$$\begin{cases} \check{\Sigma}_\varepsilon = \varepsilon & \\ \check{\Sigma}_l = x\check{\Sigma}_u, & \text{for } l = xu \in \mathcal{L}ynY, \\ \check{\Sigma}_w = \dfrac{\check{\Sigma}_{l_1}^{\uplus i_1} \uplus \ldots \uplus \check{\Sigma}_{l_k}^{\uplus i_k}}{i_1! \ldots i_k!} & \text{for } w = l_1^{i_1} \ldots l_k^{i_k}, l_1 < \ldots < l_k. \end{cases}$$

Theorem

$$\mathrm{H}(N) = \prod_{l \in \mathcal{L}ynY}^{\nearrow} e^{\mathrm{H}_{\check{\Sigma}_l}(N)\, \Sigma_l}.$$

Theorem (*à la Abel*, HNM, 2005)

L *and* H *are group-like and*

$$\lim_{z \to 1} e^{y_1 \log \frac{1}{1-z}} \Pi_Y \mathrm{L}(z) = \lim_{N \to \infty} \left[\sum_{k \geq 0} \mathrm{H}_{y_1^k}(N)(-y_1)^k \right] \mathrm{H}(N) = \Pi_Y \Phi_{KZ}.$$

Asymptotic expansion of harmonic sums

Proposition

$$\mathrm{H}(N) \underset{N\to\infty}{\widetilde{\quad}} \exp\left[-\sum_{k\geq 1} \mathrm{H}_{y_k}(N)\frac{(-y_1)^k}{k}\right]\Pi_Y \Phi_{KZ}.$$

Theorem (Costermans, Enjalbert & HNM, 2005)

There exists algorithmically computable coefficients $b_i \in \mathcal{Z}'$, *the* $\mathbb{Q}$*-algebra generated by convergent polyzêtas and by* γ, $\kappa_i \in \mathbb{N}$ *and* $\eta_i \in \mathbb{Z}$ *s.t.* $\forall w \in Y^*, \mathrm{H}_w(N) \underset{N\to\infty}{\widetilde{\quad}} \sum_{i=0}^{+\infty} b_i N^{\eta_i} \log^{\kappa_i}(N)$.

Definition

For any $k \geq 0$ and for any $w \in Y^* \setminus \{y_1\}$, let $\zeta_{\sqcup\!\sqcup}(y_1^k w)$ be the constant associated to $\mathrm{H}_{y_1^k w}$. Let $\Psi_{KZ} := \sum_{w\in Y^*} \zeta_{\sqcup\!\sqcup}(w)\, w$.

Theorem (HNM, 2005)

Ψ_{KZ} *is group-like and* $\Psi_{KZ} = B(y_1)\Pi_Y \Phi_{KZ}$, *where*

$$B(y_1) := \exp\left[\gamma y_1 - \sum_{k\geq 2} \zeta(k)\frac{(-y_1)^k}{k}\right].$$

Generalized Euler constants

Let $b_{n,k}(t_1, \ldots, t_k)$ be the Bell polynomials. By specializing at $t_1 = \gamma$ and for $l \geq 2$, $t_l = (-1)^{l-1}(l-1)!\zeta(l)$ and by using the identity, for any $u \in X^*$, $x_1^k x_0 u = \sum_{l=0}^{k} x_1^l ш(x_0[(-x_1)^{k-l} ш u])$, we get

Corollary

$$\zeta_{⊎}(y_1^k w) = \sum_{i=0}^{k} \frac{\zeta(x_0[(-x_1)^{k-i} ш \Pi_X w])}{i!} \left[\sum_{j=1}^{i} b_{i,j}(\gamma, -\zeta(2), 2\zeta(3), \ldots)\right].$$

In particular,

$$\zeta_{⊎}(y_1^k) = \sum_{\substack{s_1, \ldots, s_k > 0 \\ s_1 + \ldots + k s_k = k}} \frac{(-1)^k}{s_1! \ldots s_k!} (-\gamma)^{s_1} \left(-\frac{\zeta(2)}{2}\right)^{s_2} \ldots \left(-\frac{\zeta(k)}{k}\right)^{s_k}.$$

Corollary

$\zeta_{⊎}$ *realizes also a morphism from* $(\mathbb{C}\langle Y\rangle, ⊎)$ *to* $(\mathbb{C}, .)$ *s.t.*

- $\zeta_{⊎}(y_1) = \gamma$,
- *for any* $w \in Y^* \setminus y_1 Y^*, \zeta_{⊎}(w) = \zeta(w)$,
- *for any* $u, v \in Y^*, \zeta_{⊎}(u ⊎ v) = \zeta_{⊎}(u)\zeta_{⊎}(v)$.

Noncommutative generating series of regularized polyzêtas

Theorem

$$\Psi_{KZ} = \prod_{l \in \mathcal{L}ynY}^{\nearrow} e^{\zeta_{\text{⧢}}(\check{\Sigma}_l)\, \Sigma_l} = e^{\gamma y_1} \Psi'_{KZ},$$

where Ψ'_{KZ} *is the noncommutative generating series of regularized polyzêtas* $\{\zeta'_{\text{⧢}}(w)\}_{w \in Y^*}$ *:*

$$\Psi'_{KZ} := \sum_{w \in Y^*} \zeta'_{\text{⧢}}(w)\, w = \prod_{l \in \mathcal{L}ynY, l \neq y_1}^{\nearrow} e^{\zeta(\check{\Sigma}_l)\, \Sigma_l},$$

verifying

- $\zeta'_{\text{⧢}}(y_1) = 0$,
- *for any* $w \in Y^* \setminus y_1 Y^*, \zeta'_{\text{⧢}}(w) = \zeta(w)$,
- *for any* $u, v \in Y^*, \zeta'_{\text{⧢}}(u \text{⧢} v) = \zeta'_{\text{⧢}}(u)\zeta'_{\text{⧢}}(v)$.

The meaning of the double regularization to 0

The constant $\zeta_{\text{⧢}}(y_1) = \gamma$ is obtained as the finite part of the asymptotic expansion of $\mathrm{H}_1(n)$ in the scale $\{n^a \log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}$.

In the same way, since for any $n \in \mathbb{N}$, n and $\mathrm{H}_1(n)$ are algebraically independent then $\{n^a \mathrm{H}_1^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}$ constitutes a new scale for asymptotic expansions.

Let $C_1 = \mathbb{Q} \oplus x_0\mathbb{Q}\langle X\rangle x_1$ and $C_2 = \mathbb{Q} \oplus (Y \setminus \{y_1\})\mathbb{Q}\langle Y\rangle$. By the Radford theorem and its generalization over Y (due to Malvenuto & Reutenauer), one has respectively

$$(\mathbb{Q}\langle X\rangle, \text{ш}) \cong \mathbb{Q}[\mathcal{L}ynX] = C_1[x_0, x_1],$$
$$(\mathbb{Q}\langle Y\rangle, \text{⧢}) \cong \mathbb{Q}[\mathcal{L}ynY] = C_2[y_1].$$

Thus, $\zeta_{\text{ш}}(x_1) = 0$ and $\zeta'_{\text{⧢}}(y_1) = 0$ can be interpreted as the finite part of the asymptotic expansions of Li_1 and H_1 in the scales $\{(1-z)^a \log(1-z)^b\}_{a\in\mathbb{Z},b\in\mathbb{N}}$ and $\{n^a \mathrm{H}_1^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}$ respectively.

Differential Galois group of polylogarithms

$\mathrm{LI}_{\mathcal{C}}$ is the smallest algebra containing $\mathcal{C}$ closed by derivation, by integration w.r.t. ω_0 and ω_1. It is the $\mathcal{C}$-modulus generated by $\{\mathrm{Li}_w\}_{w \in X^*}$.

Let $\sigma \in \mathrm{Gal}(\mathrm{LI}_{\mathcal{C}})$. Then $\displaystyle\sum_{w \in X^*} \sigma \, \mathrm{Li}_w \; w = \prod_{l \in \mathcal{L}yn}^{\searrow} e^{\sigma \, \mathrm{Li}_{\check{S}_l} S_l}$.

Since $d\sigma \, \mathrm{Li}_{x_i} = \sigma d \, \mathrm{Li}_{x_i} = \omega_i$ then $\sigma \, \mathrm{Li}_{x_i} = \mathrm{Li}_{x_i} + c_{x_i}$.

More generally, $\displaystyle\sigma \, \mathrm{Li}_{\check{S}_l} = \int \omega_{x_i} \frac{\sigma \, \mathrm{Li}_{\check{S}_{l_1}}^{i_1}}{i_1!} \cdots \frac{\sigma \, \mathrm{Li}_{\check{S}_{l_k}}^{i_k}}{i_k!} + c_{\check{S}_l}$.

Consequently, $\displaystyle\sum_{w \in X^*} \sigma \, \mathrm{Li}_w \; w = \mathrm{L} \prod_{l \in \mathcal{L}yn}^{\searrow} e^{c_{\check{S}_l} S_l} = \mathrm{L} e^{C_\sigma}$.

The action of $\sigma \in \mathrm{Gal}(\mathrm{LI}_{\mathcal{C}})$ over $\{\mathrm{Li}_w\}_{w \in X^*}$ is equivalent to the action of $e^{C_\sigma} \in \mathrm{Gal}(DE)$ over the exponential solution L. So,

Theorem (HNM, 2003)

$\mathrm{Gal}(\mathrm{LI}_{\mathcal{C}}) \cong \mathrm{Gal}(DE) = \{e^C \mid C \in \mathcal{L}ie_{\mathbb{C}}\langle\langle X \rangle\rangle\}$.

Action of $\mathrm{Gal}(DE)$ on the asymptotic expansions

Theorem (Group of associators theorem)

For any commutative $\mathbb{Q}$-algebra A, let $\Phi \in A\langle\langle X\rangle\rangle$ and $\Psi \in A\langle\langle Y\rangle\rangle$ be group-like elements such that $\Psi = B(y_1)\Pi_Y\Phi$. There exists an unique $C \in \mathcal{L}ie_A\langle\langle X\rangle\rangle$ such that $\Phi = \Phi_{KZ}e^C$ and $\Psi = \Psi_{KZ}\Pi_Y e^C$.

If $C \in \mathcal{L}ie_A\langle\langle X\rangle\rangle$ then $\mathrm{L}' = \mathrm{L}e^C$ is group-like and $e^C \in \mathrm{Gal}(DE)$. Let $\mathrm{H}'(N)$ be the n.c.g.s. of the Taylor coefficients, belonging the harmonic algebra, of $\{(1-z)^{-1}\mathrm{L}'_w(z)\}_{w\in Y^*}$. Then $\mathrm{H}'(N)$ is group-like.

$$\frac{\mathrm{L}'(1-\varepsilon)}{\varepsilon} \underset{\varepsilon\to 0^+}{\widetilde{\quad}} e^{-(1+x_1)\log\varepsilon}\Phi_{KZ}e^C \quad \Rightarrow \quad \mathrm{H}'(N) \underset{N\to\infty}{\widetilde{\quad}} \mathrm{H}(N)\Pi_Y e^C.$$

Let κ_w be the constant part of $\mathrm{H}'_w(N)$. Then,

$$\sum_{w\in Y^*} \kappa_w\, w = \Psi_{KZ}\Pi_Y e^C, \quad \text{or eqivalenty} \quad \Pi_X \sum_{w\in Y^*} \kappa_w\, w = B^{-1}(x_1)\Phi_{KZ}e^C.$$

We put then $\Psi := \Psi_{KZ}\Pi_Y e^C$ and $\Phi := \Phi_{KZ}e^C$ (and $\Psi' := \Psi'_{KZ}\Pi_Y e^C$).

Examples (action of the monodromy group)

For $t \in]0,1[$, the monodromies around $0, 1$ of L are given respectively by ($p = 2i\pi$)

$$\mathcal{M}_0\mathrm{L}(t) = \mathrm{L}(t)e^{p\mathfrak{m}_0} \quad \text{and} \quad \mathcal{M}_1\mathrm{L}(t) = \mathrm{L}(t)\Phi_{KZ}^{-1}e^{-px_1}\Phi_{KZ} = \mathrm{L}(t)e^{p\mathfrak{m}_1},$$

$$\text{where} \quad \mathfrak{m}_0 = x_0 \quad \text{and} \quad \mathfrak{m}_1 = \prod_{l \in \mathcal{L}yn, l \neq x_0, x_1}^{\searrow} e^{-\zeta(\check{S}_l)\,\mathrm{ad}_{S_l}}(-x_1).$$

- If $C = p\mathfrak{m}_0$ then $\Phi = \Phi_{KZ}e^{px_0}$ and
$$\Psi = \exp\left[\gamma y_1 - \sum_{k\geq 2}\zeta(k)\frac{(-y_1)^k}{k}\right]\Pi_Y\Phi_{KZ} = \Psi_{KZ}.$$
- If $C = p\mathfrak{m}_1$ then $\Phi = e^{-px_1}\Phi_{KZ}$ and
$$\Psi = \exp\left[\underbrace{(\gamma - p)}_{=T}y_1 - \sum_{k\geq 2}\zeta(k)\frac{(-y_1)^k}{k}\right]\Pi_Y\Phi_{KZ} = e^{-py_1}\Psi_{KZ}.$$

CONCLUSION

Polynomial relations among generators of polyzêtas

Let $B'(y_1) := e^{-\gamma y_1} B(y_1)$. Then,

$$\Psi_{KZ} = B(y_1)\Pi_Y \Phi_{KZ} \iff \Psi'_{KZ} = B'(y_1)\Pi_Y \Phi_{KZ}.$$

Theorem

$$\prod_{\substack{l\in\mathcal{L}ynX,\\ l\neq x_0,x_1}}^{\searrow} e^{\zeta(\check{S}_l)\, S_l} = e^{\displaystyle\sum_{k\geq 2}\zeta(k)\frac{(-x_1)^k}{k}} \Pi_X \prod_{\substack{l\in\mathcal{L}ynY,\\ l\neq y_1}}^{\nearrow} e^{\zeta(\check{\Sigma}_l)\, \Sigma_l}$$

$$\iff \prod_{\substack{l\in\mathcal{L}ynY,\\ l\neq y_1}}^{\nearrow} e^{\zeta(\check{\Sigma}_l)\, \Sigma_l} = e^{\displaystyle -\sum_{k\geq 2}\zeta(k)\frac{(-y_1)^k}{k}} \Pi_Y \prod_{\substack{l\in\mathcal{L}ynX,\\ l\neq x_0,x_1}}^{\searrow} e^{\zeta(\check{S}_l)\, S_l}.$$

$\{\zeta_{\text{ш}}(\check{S}_l)\}_{l\in\mathcal{L}ynX}$ and $\{\zeta_{⊎}(\check{\Sigma}_l)\}_{l\in\mathcal{L}ynY}$ are respectively generators of the algebras $\mathcal{Z}$ and $\mathcal{Z}'$.

By identifying the local coordinates, in the Lyndon-PBW basis, we get polynomial relations among these generators.

A challenge in computer algebra

How to extract the polynomial relations among $\{\zeta(l)\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}$, or equivalently $\{\zeta(l)\}_{l\in\mathcal{L}ynX\setminus\{x_0,x_1\}}$?

$\{\zeta_{ш}(l)\}_{l\in\mathcal{L}ynX}$ and $\{\zeta_{\text{⧢}}(l)\}_{l\in\mathcal{L}ynY}$ are also generators respectively of the algebras $\mathcal{Z}$ and $\mathcal{Z}'$. Let $\{\hat{l}\}_{l\in\mathcal{L}ynX}$ and $\{\hat{l}\}_{l\in\mathcal{L}ynY}$ be the dual basis of the Lyndon basis over X and Y respectively. One also gets

Theorem

$$\prod_{\substack{l\in\mathcal{L}ynX,\\ l\neq x_0,x_1}}^{\searrow} e^{\zeta(l)\,\hat{l}} = e^{\sum_{k\geq 2}\zeta(k)\frac{(-x_1)^k}{k}}\,\Pi_X \prod_{\substack{l\in\mathcal{L}ynY,\\ l\neq y_1}}^{\nearrow} e^{\zeta(l)\,\hat{l}}.$$

Since $\forall l\in\mathcal{L}ynY \iff \Pi_X l\in\mathcal{L}ynX\setminus\{x_0\}$ then

Corollary

For any $l\in\mathcal{L}ynY\setminus\{y_1\}$, let P_l be the decomposition of $\Pi_X\hat{l}$ in the Lyndon-PBW basis, over X, and let $\check{P}_l$ be its dual. Then $\Pi_X l-\check{P}_l\in\ker\zeta$. *Moreover, if $\Pi_X l\equiv\check{P}_l$ then $\zeta(l)$ is irreductible.*

Towards the transcendence of γ over $\mathcal{Z}$

By considering the commutative indeterminates $t_1, t_2, \ldots$, then let $A = \mathbb{Q}[t_1, t_2, \ldots]$.

Lemma

For any $\Phi \in \{\Phi_{KZ} e^{C} | C \in \mathcal{L}ie_A\langle\langle X \rangle\rangle\}$, *one get*

$$\Psi = B(y_1)\Pi_Y \Phi \quad \Longleftrightarrow \quad \Psi' = B'(y_1)\Pi_Y \Phi.$$

Theorem

For all $\Phi \in \{\Phi_{KZ} e^{C} | C \in \mathcal{L}ie_{\mathbb{Q}}\langle\langle X \rangle\rangle\}$, *the identies* $\Psi = B(y_1)\Pi_Y \Phi$ *yield all polynomial relations among convergent polyzêtas. Moreover, these relations are algebraically independent on* γ*.*

THANK YOU FOR YOUR ATTENTION