The reduced Kronecker coefficients

Mercedes Rosas (Universidad de Sevilla)

joint work with

Emmanuel Briand (Universidad de Sevilla)
and
Rosa Orellana (Dartmouth College).

The Kronecker coefficients

The Kronecker coefficients $g^{\lambda}_{\mu,\nu}$: cf

- are the multiplicities in the Clebsch–Gordan decompositions for the symmetric group:

$$S_{\mu}(\mathfrak{S}_n) \otimes S_{\nu}(\mathfrak{S}_n) \cong \bigoplus_{\lambda} g^{\lambda}_{\mu,\nu} S_{\lambda}(\mathfrak{S}_n)$$

- describe the decompositions in $GL_n \times GL_m$ irreps of the irreducible representations of GL_{mn}.

$$W_{\lambda}(GL_{mn}) \cong \bigoplus_{\mu,\nu} g^{\lambda}_{\mu,\nu} W_{\mu}(GL_m) \otimes W_{\nu}(GL_n)$$

Problems raised by the the Kronecker coefficients

1. **Combinatorial interpretation:** Find a combinatorial interpretation for the Kronecker coeffs, akin to the Littlewood–Richardson rule for the LR coeffs (= multiplicities in the Clebsch–Gordan decompositions for GL_n).

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"One of the main open problems in the combinatorial representation theory of $\mathfrak{S}_n$ is to obtain a combinatorial interpretation of $g^{\lambda}_{\mu\nu}$ in general." (R. Stanley, 1998)

"An outstanding problem in the theory of symmetric functions is that of finding an explicit formula or combinatorial interpretation for the Kronecker coefficients." (I. Gessel, 2008)

Problems raised by the the Kronecker coefficients

1. **Combinatorial interpretation:** Find a combinatorial interpretation for the Kronecker coeffs, akin to the Littlewood–Richardson rule for the LR coeffs (= multiplicities in the Clebsch–Gordan decompositions for GL_n or U_n).

2. **Computation:** Given λ, μ, ν, compute (quickly) $g^{\lambda}_{\mu,\nu}$.

3. **Decision:** Given λ, μ, ν, decide (quickly) whether or not $g^{\lambda}_{\mu,\nu}$ is zero.

Summary of the talk

I study a related family of coefficients, the

REDUCED KRONECKER COEFFICIENTS

that seems to be simpler than the Kronecker coefficients. Still, it carries enough information about the Kronecker coefficients to allow us to reconstruct them.

The reduced Kronecker coefficients where discovered by *Murhaghan* and *Littlewood* in the 1930's. I will introduce them by an example.

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The *reduced Kronecker coefficients* $\bar{g}_{\mu,\nu}^{\lambda}$ are the stable values of the sequence of Kronecker coefficients, after we disregard the first part.

So, for instance the reduced Kronecker coefficient :

$$\bar{g}_{(2),(2)}^{()} = 1 \qquad \bar{g}_{(2),(2)}^{(1)} = 1$$
$$\bar{g}_{(2),(2)}^{(2)} = 2 \qquad \bar{g}_{(2),(2)}^{(2,1)} = 2$$
$$\bar{g}_{(2),(2)}^{(5)} = 0 \qquad \cdots$$

Since,

$$s_{\bullet,2} * s_{\bullet,2} = s_{\bullet} + s_{\bullet,1} + 2s_{\bullet,2} + s_{\bullet,1,1} + s_{\bullet,3} + 2s_{\bullet,2,1} + s_{\bullet,1,1,1} + s_{\bullet,4} + s_{\bullet,3,1} + s_{\bullet,2,2}$$

MURNAGHAN'S THEOREM, *1938,1955*

There exists a family of non-negative integers $(\bar{g}_{\alpha\beta}^{\gamma})$ indexed by triples of partitions (α, β, γ) such that, given α and β, only finitely many terms $\bar{g}_{\alpha\beta}^{\gamma}$ are nonzero, and for all $n \geq 0$,

$$s_{\alpha[n]} * s_{\beta[n]} = \sum_{\gamma} \bar{g}_{\alpha\beta}^{\gamma} s_{\gamma[n]}$$

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$$s_{\alpha[n]} * s_{\beta[n]} = \sum_{\gamma} \bar{g}_{\alpha\beta}^{\gamma} s_{\gamma[n]}$$

Moreover, the coefficient $\bar{g}_{\alpha\beta}^{\gamma}$ vanishes unless the weights of the three partitions fulfill *Murnaghan's inequalities:*

$$|\alpha| \leq |\beta| + |\gamma|, \qquad |\beta| \leq |\alpha| + |\gamma|, \qquad |\gamma| \leq |\alpha| + |\beta|.$$

MURNAGHAN'S THEOREM HOLDS EVEN FOR SMALL VALUES OF n.

For example, for $n = 4$,

$$s_{\bullet,2} * s_{\bullet,2} = s_{\bullet} + s_{\bullet,1} + 2s_{\bullet,2} + s_{\bullet,1,1} + s_{\bullet,3} + 2s_{\bullet,2,1} + s_{\bullet,1,1,1} + s_{\bullet,4} + s_{\bullet,3,1} + s_{\bullet,2,2}$$

becomes

$$s_{2,2} * s_{2,2} = s_{4} + s_{3,1} + 2s_{2,2} + s_{2,1,1} + s_{1,3} + 2s_{1,2,1} + s_{1,1,1,1} + s_{0,4} + s_{0,3,1} + s_{0,2,2}$$

But in the Jacobi-Trudi expansion $s_{1,2,1}$ and $s_{0,3,1}$ have a repeated column, hence are zero. $s_{1,3} = -s_{2,2}$, $s_{0,3,1} = -s_{2,1,1}$, and

$s_{0,4} = -s_{3,1}$. Hence,

$$s_{2,2} * s_{2,2} = s_4 + s_{2,2} + s_{1,1,1,1}$$

MONOTICITY OF THE KRONECKER COEFFICIENTS, *Brion 1993*

The sequence $g^{\gamma[n]}_{\alpha[n],\beta[n]}$ with indices $n \geq \max(|\alpha| + \alpha_1, |\beta| + \beta_1, |\gamma| + \gamma_1)$ is weakly increasing.

$$g^{(4,2,2,1,1)}_{(3,3,2,1,1),(3,3,2,2)} = 17 \qquad g^{(5,2,2,1,1)}_{(4,3,2,1,1),(4,3,2,2)} = 119$$
$$g^{(6,2,2,1,1)}_{(5,3,2,1,1),(5,3,2,2)} = 256 \qquad g^{(7,2,2,1,1)}_{(6,3,2,1,1),(6,3,2,2)} = 306$$
$$g^{(8,2,2,1,1)}_{(7,3,2,1,1),(7,3,2,2)} = 308 \qquad g^{(9,2,2,1,1)}_{(8,3,2,1,1),(8,3,2,2)} = 308$$
$$g^{(10,2,2,1,1)}_{(9,3,2,1,1),(9,3,2,2)} = 308 \qquad \cdots$$

A BEAUTIFUL RESULT OF MURNAGHAN AND LITTLEWOOD

Note that, in general, there is not relation between the weight of λ, μ and ν. But, what happens when they satisfy

$$|\lambda| = |\mu| + |\nu|$$

?

Surprise !!

when $|\lambda| = |\mu| + |\nu|$, we recover the *Littlewood-Richardson coefficients*:

Theorem [Murnaghan *1955*, Littlewood *1958*] If $|\lambda| = |\mu| + |\nu|$, then

$$\bar{g}^{\lambda}_{\mu,\nu} = c^{\lambda}_{\mu,\nu}$$

Recovering the Kronecker coefficients from reduced Kronecker coefficients.

Theorem [B.O.R]

$$g_{\mu\nu}^{\lambda} = \sum_{i=1}^{\ell(\mu)\ell(\nu)} (-1)^{i+1} \overline{g}_{\overline{\mu},\overline{\nu}}^{\lambda^{\dagger i}}$$

where $\lambda^{\dagger i}$ is the partition obtained from λ by incrementing the $i-1$ first terms by 1 and removing the i–th term, that is:

$$\lambda^{\dagger i} = \left(1+\lambda_1, 1+\lambda_2, \ldots, 1+\lambda_{i-1}, \lambda_{i+1}, \lambda_{i+2}, \ldots\right)$$

and $\overline{\lambda} = \lambda^{\dagger 1}$.

Brion's formula for the reduced Kronecker coefficients

$$\bar{g}^{\gamma}_{\alpha,\beta} = \langle s_\alpha[X] s_\beta[Y], s_\gamma[X+Y+XY]\sigma_1[XY]\rangle$$

where $\sigma_1 = \sum_{n\geq 0} h_n$.

Moreover, the sequence $g^{\gamma[n]}_{\alpha[n],\beta[n]}$ is constant for

$$n \geq |\alpha| + |\beta| + \gamma_1.$$

Stability bounds

- In 1993 *Brion* provided nonsharp bounds for the stabilization of a Kronecker coefficient.

- In 1999 *Vallejo* provided nonsharp bounds for the stabilization of a Kronecker coefficient.

The stabilization of Kronecker products

Theorem [B.O.R]

Let α and β be two partitions. Then

$$\textit{stab}(\alpha,\beta) = |\alpha| + |\beta| + \alpha_1 + \beta_1$$

$$s_{2,2} * s_{2,2} = s_4 + s_{2,2} + s_{1,1,1,1}$$

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Some optimization problems

$$\max\{\gamma_1 \,|\, \gamma \text{ st } \overline{g}^{\gamma}_{\alpha,\beta} > 0\} = |\alpha \cap \beta| + \max(\alpha_1, \beta_1)$$

More generally,

$$\gamma_{i+j-1} \leq |E_i \cap E_j\beta| + \alpha_i + \beta_j$$

where $E_k\gamma$ is the partition obtained erasing the kth part of γ.

Also,

$$\gamma_k \leq \left\lfloor \frac{|\alpha| + |\beta|}{k} \right\rfloor$$

Example, for $\alpha = (3, 1)$ and $\beta = (2, 2)$ we get

k	1	2	3	4	5	6
	6	3	2	1	1	1
	6	4	2	2	1	1

The stabilization of Kronecker coefficients

Lemma [B.O.R] Let f be a function on triples of partitions such that:

$$\overline{g}_{\alpha,\beta}^{\gamma} > 0 \Rightarrow \gamma_1 \leq f(\alpha, \beta, \bar{\gamma})$$

Set

$$M_f(\alpha, \beta, \gamma) = |\gamma| + \text{sup}\{f(\alpha, \beta, \gamma^{\dagger i}) \mid i \geq 1\}$$

Then

$$stab(\alpha, \beta, \gamma) \leq M_f(\alpha, \beta, \gamma)$$

Moreover, if $\overline{g}^{\gamma}_{\alpha,\beta} > 0$ then $M_f(\alpha, \beta, \gamma) \geq |\gamma| + \gamma_1$.

Murnaghan's triangle inequality shows that the previous condition holds for

$$f(\alpha, \beta, \tau) = |\alpha| + |\beta| - |\tau|$$

Indeed, in this situation we recover Brion's bound.

Theorem [B.O.R]

Let

$$N_1(\alpha, \beta, \gamma) = |\gamma| + |\bar{\alpha} \cap \bar{\beta}| + \alpha_1 + \beta_1$$

Then,

$$stab(\alpha, \beta, \gamma) \leq N_1$$

Theorem [B.O.R]

Let

$$N_2(\alpha,\beta,\gamma) = ceil\left(\frac{|\alpha| + |\beta| + |\gamma| + \alpha_1 + \beta_1 + \gamma_1 - 1}{2}\right)$$

where $ceil(x)$ is the smallest integer m fulfilling $m \geq x$. Then,

$$stab(\alpha,\beta,\gamma) \leq N_2$$

The two bounds N_1 and N_2 are no comparable. Sometimes, $N_1 \leq N_2$, others $N_2 \leq N_1$.

They improve both Vallejo's and Brion's bounds.

THE KRONECKER COEFFICIENTS INDEXED BY THREE HOOKS

After deleting the first part of a hook we always obtain a one column shape.

Let $\mu = (1^e)$, $\nu = (1^f)$ and $\lambda = (1^d)$ be the reduced partitions.

In 2001 I showed that Murnaghan's inequalities describe the stable value of the Kronecker coefficient $g_{\mu[n],\nu[n]}^{\lambda[n]}$. That is,

$$\overline{g}_{\mu,\nu}^{\lambda} = ((e \leq d + f))((d \leq e + f))((f \leq e + d))$$

where $((P))$ equals 1 if the proposition is true, and 0 if not.

Moreover, it was shown that the Kronecker coefficient equals 1 only when Murnaghan's inequalities hold, as well as the additional inequality N_2.