Symmetry and super-symmetry distribution for partitions

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Outline

1. Introduction
2. Main results
3. Super-Symmetry

Outline

Partitions

- A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of n, write $\lambda \vdash n$, if

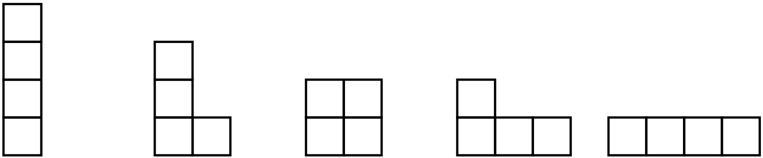
$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_\ell > 0$$

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n.$$

- $p(n)$ = the number of partitions of n.

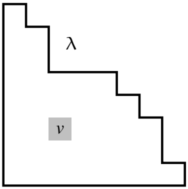
Ferrers diagram

(1,1,1,1) (2,1,1) (2,2) (3,1) (4)



Pointed Partitions

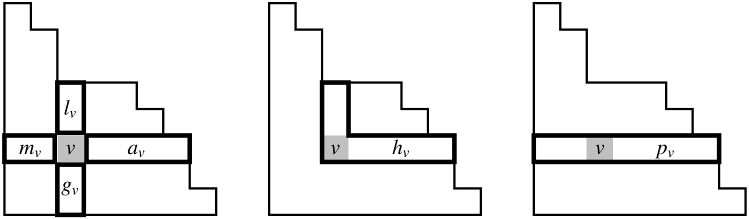
- A pointed partition (λ, v) of n if $\lambda \vdash n$ and v is a cell in the Ferrers diagram of λ.



- $\mathcal{F}_n$ = the set of pointed partitions of n.

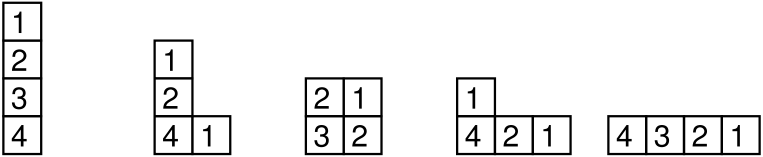
$$|\mathcal{F}_n| = p(n) \times n$$

The arm, leg, coarm, coleg, hook, and part of a pointed partition



$$h_v = l_v + a_v + 1 \quad \text{and} \quad p_v = m_v + a_v + 1$$

The distribution of h_v on $\mathcal{F}_4$ is given below:

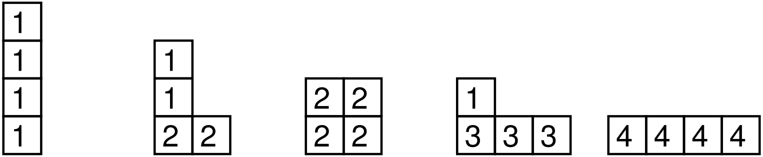


where h_v is written in a cell v.

h_v	1	2	3	4	$\sum$
$\#v$	7	6	3	4	20

The distribution of p_v

The distribution of p_v on $\mathcal{F}_4$ is given below:



where p_v is written in a cell v.

p_v	1	2	3	4	$\sum$
$\#v$	7	6	3	4	20

h_v and p_v are equidistributed

$$\sum_{(\lambda,v)\in\mathcal{F}_4} x^{h_v} = 7x + 6x^2 + 3x^3 + 4x^4.$$

$$\sum_{(\lambda,v)\in\mathcal{F}_4} x^{p_v} = 7x + 6x^2 + 3x^3 + 4x^4.$$

Theorem (Bessenrodt, 1998; Bacher-Manivel, 2001)

The hook length h_v and the part length p_v are equidistributed.

$$\sum_{(\lambda,v)\in\mathcal{F}_n} x^{h_v} = \sum_{(\lambda,v)\in\mathcal{F}_n} x^{p_v}$$

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$$\sum_{(\lambda,v)\in\mathcal{F}_4} x^{h_v} = 7x + 6x^2 + 3x^3 + 4x^4.$$

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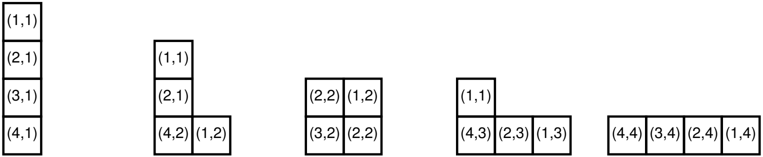
$$\sum_{(\lambda,v)\in\mathcal{F}_n} x^{h_v} = \sum_{(\lambda,v)\in\mathcal{F}_n} x^{p_v}$$

The joint distribution of h_v and p_v

The joint distribution of (h_v, p_v) on $\mathcal{F}_4$ is given below:



where (h_v, p_v) is written in a cell v.

$h_v \backslash p_v$	1	2	3	4	$\sum$
1	3	2	1	1	7
2	2	2	1	1	6
3	1	1	0	1	3
4	1	1	1	1	4
$\sum$	7	6	3	4	20

Theorem (Bessenrodt-Han, 2009)

The hook length h_v and the part length p_v are symmetric.

$$\sum_{(\lambda,v)\in\mathcal{F}_n} x^{h_v} y^{p_v} = \sum_{(\lambda,v)\in\mathcal{F}_n} x^{p_v} y^{h_v}.$$

$h_v \backslash p_v$	1	2	3	4	$\sum$
1	3	2	1	1	7
2	2	2	1	1	6
3	1	1	0	1	3
4	1	1	1	1	4
$\sum$	7	6	3	4	20

Theorem (Bessenrodt-Han, 2009)

The hook length h_v and the part length p_v are symmetric.

$$\sum_{(\lambda,v)\in\mathcal{F}_n} x^{h_v} y^{p_v} = \sum_{(\lambda,v)\in\mathcal{F}_n} x^{p_v} y^{h_v}.$$

Question

How to construct an involution on $\mathcal{F}_n$ exchanging hook length and part length?

$a_v \backslash l_v$	0	1	2	3	$\sum$
0	7	3	1	1	12
1	3	1	1	0	5
2	1	1	0	0	2
3	1	0	0	0	1
$\sum$	12	5	2	1	20

Theorem (Bessenrodt, 1998; Bacher-Manivel, 2001)

The arm length a_v and the leg length l_v are super-symmetric.

$a_v \backslash l_v$	0	1	2	3	$\sum$
0	7	3	1	1	12
1	3	1	1	0	5
2	1	1	0	0	2
3	1	0	0	0	1
$\sum$	12	5	2	1	20

Theorem (Bessenrodt, 1998; Bacher-Manivel, 2001)

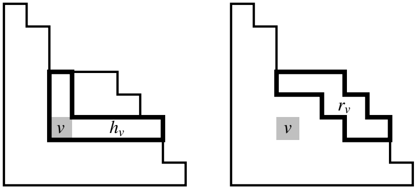
The arm length a_v and the leg length l_v are super-symmetric.

- $\mathcal{F}_n(\alpha, \beta)$ = the set of pointed partitions with arm length α and leg length β.

Question

How to construct an bijection from $\mathcal{F}_n(\alpha, \beta)$ to $\mathcal{F}_n(\alpha', \beta')$ where $\alpha + \beta = \alpha' + \beta'$?

- The rim hook R_v or $R_v(\lambda)$ is the contiguous border strip of λ connecting the rightmost and the uppermost cells of the hook H_v.



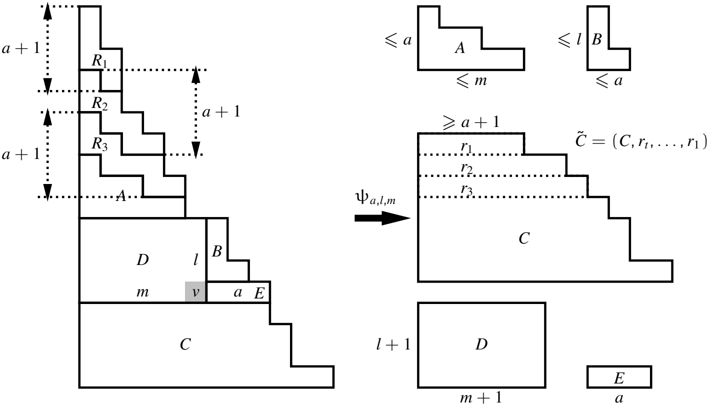
$$h_v = r_v = l_v + a_v + 1$$

- If λ be a partition, denote its conjugate by $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$, that is, λ'_i is the number of parts of λ that are $\geqslant i$.

Outline

Pointed partition to quintuple

We can construct the mapping $\psi_{a,l,m}$ and its inverse as follows:



- $\mathcal{F}_n(a, l, m)$ = the set of pointed partitions (λ, v) of n such that $a_v = a$, $l_v = l$ and $m_v = m$.
- $\mathcal{Q}_n(a, l, m)$ = the set of quintuples $(A, B, \tilde{C}, D, E)$ such that
  $A \subset a \times m$ rectangle,
  $B \subset l \times a$ rectangle,
  $\tilde{C}$ = a partition whose all parts are $\geqslant a + 1$,
  D = $(l + 1) \times (m + 1)$ rectangle,
  E = $1 \times a$ rectangle, and

$$|A| + |B| + |\tilde{C}| + |D| + |E| = n.$$

- $\mathcal{Q}_n$ = the set of such quintuples $(A, B, \tilde{C}, D, E)$.

Define the bijection ψ from $\mathcal{F}_n$ to $\mathcal{Q}_n$ by

$$\psi(\lambda, v) = \psi_{a,l,m}(\lambda, v) \quad \text{if } (\lambda, v) \in \mathcal{F}_n(a, l, m)$$

and the involution ρ on $\mathcal{Q}_n$ by

$$\rho(A, B, \tilde{C}, D, E) = (B', A', \tilde{C}, D', E)$$

where X' is the conjugate of the partition X.

Theorem (S.-Zeng, 2009)

For all $n \geqslant 0$, the mapping

$$\varphi = \psi^{-1} \circ \rho \circ \psi$$

is an involution on $\mathfrak{F}_n$ such that if $\varphi : (\lambda, v) \mapsto (\mu, u)$ then

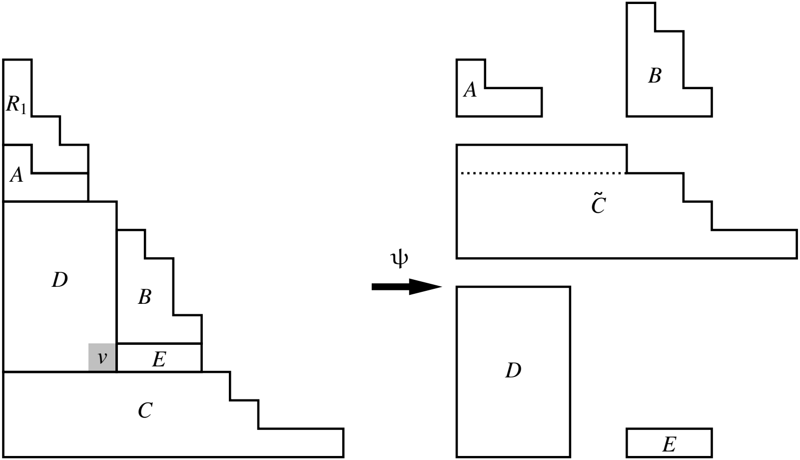
$$(a_v, l_v, m_v)(\lambda) = (a_u, m_u, l_u)(\mu). \quad (1)$$

In particular, the mapping φ also satisfies

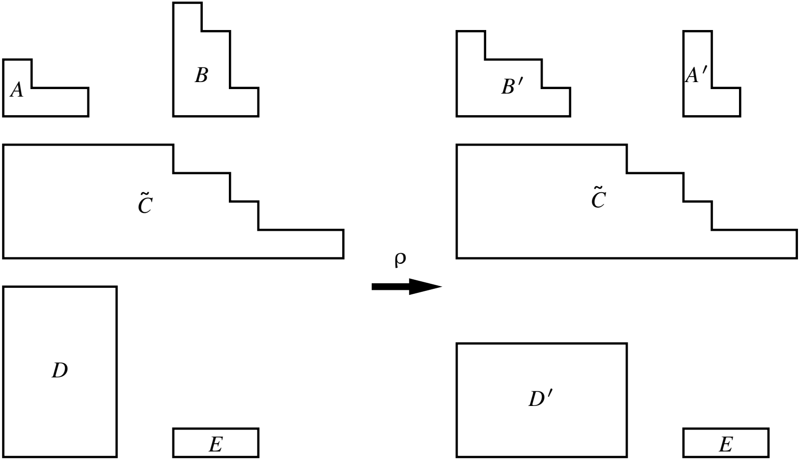
$$(h_v, p_v)(\lambda) = (p_u, h_u)(\mu). \quad (2)$$

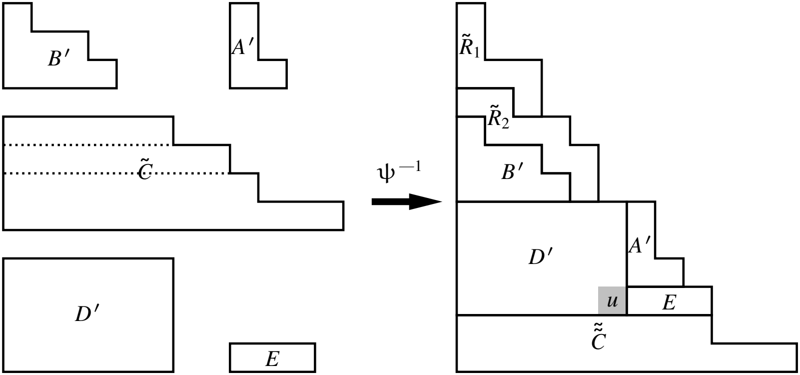
In other words, we have the following diagram:

$$\begin{array}{ccc}
\mathcal{F}_n(a,l,m) & \xrightarrow{\varphi} & \mathcal{F}_n(a,m,l) \\
\psi_{a,l,m}\big\downarrow & & \big\uparrow\psi_{a,m,l}^{-1} \\
\mathcal{Q}_n(a,l,m) & \xrightarrow{\rho} & \mathcal{Q}_n(a,m,l).
\end{array}$$

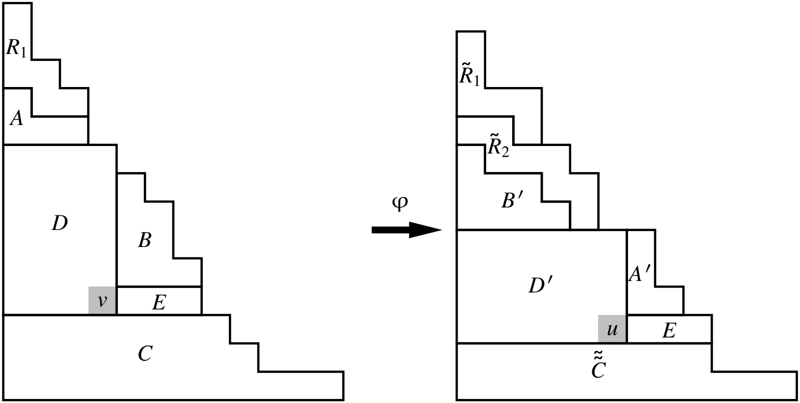


$\lambda = (12, 9, 8, 7, 7, 6, 6, 5, 4, 3, 3, 2, 1, 1)$
$v = (4, 4)$


A
B
$\tilde{C}$
D
E
ρ
B′
A′
$\tilde{C}$
D′
E



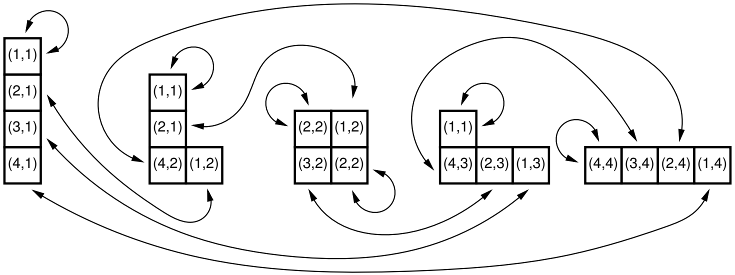
$\mu = (12, 9, 9, 8, 7, 7, 5, 5, 4, 3, 3, 1, 1)$
$u = (6, 3)$



$\lambda = (12, 9, 8, 7, 7, 6, 6, 5, 4, 3, 3, 2, 1, 1)$
$v = (4, 4)$

$\mu = (12, 9, 9, 8, 7, 7, 5, 5, 4, 3, 3, 1, 1)$
$u = (6, 3)$

For example, the bijection φ on $\mathcal{F}_4$ is illustrated below:



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We derive immediately the following result of Bessenrodt and Han [BH09, Theorem 3].

Corollary (Bessenrodt-Han, 2009)

The triple statistic (a_ν, l_ν, m_ν) *has the same distribution as* (a_ν, m_ν, l_ν). *In other words,*

$$Q_n(x,y,z) = Q_n(x,z,y)$$

where

$$Q_n(x,y,z) = \sum_{(\lambda,\nu)\in\mathcal{F}_n} x^{a_\nu} y^{l_\nu} z^{m_\nu}.$$

is the generating function for (a_ν, l_ν, m_ν).

◂ Back

For nonnegative integers m and n,

- q-ascending factorial

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$$

- q-binomial coefficient

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_m (q;q)_{n-m}} \quad \text{for } 0 \leqslant m \leqslant n.$$

It is easy (see [And98, Chapter 3]) to see that

$$A(q) = \begin{bmatrix} m+a \\ a \end{bmatrix}_q,$$

$$B(q) = \begin{bmatrix} l+a \\ a \end{bmatrix}_q,$$

$$\tilde{C}(q) = \frac{1}{(q^{a+1};q)_\infty},$$

$$D(q) = q^{(m+1)(l+1)},$$

$$E(q) = q^a.$$

Let $f_n(a,l,m)$ be the cardinality of $\mathcal{F}_n(a,l,m)$. We can apply the bijection φ to give a different proof of Bessenrodt and Han's formula [BH09, Theorem 2] for $\sum_{n\geqslant 0} f_n(a,l,m)q^n$.

Corollary (Bessenrodt-Han, 2009)

The generating function of $f_n(a,l,m)$ is given by the following formula:

$$\sum_{n\geqslant 0} f_n(a,l,m)q^n = \frac{1}{(q^{a+1};q)_\infty} \begin{bmatrix} m+a \\ a \end{bmatrix}_q \begin{bmatrix} l+a \\ a \end{bmatrix}_q q^{(m+1)(l+1)+a}.$$

Outline

A polynomial $P(x,y)$ in two variables x and y is super-symmetric if

$$[x^{\alpha}y^{\beta}]P(x,y) = [x^{\alpha'}y^{\beta'}]P(x,y)$$

when $\alpha + \beta = \alpha' + \beta'$.

Theorem (Bessenrodt, 1998; Bacher-Manivel, 2001; Bessenrodt-Han, 2009)

The generating function for the pointed partitions of $\mathcal{F}_n$ by the two joint statistics arm length and coarm length (resp. leg length) is super-symmetric. In other words, the polynomial

$$\sum_{(\lambda,v)\in\mathcal{F}_n} x^{a_v} y^{m_v} \quad (resp. \quad \sum_{(\lambda,v)\in\mathcal{F}_n} x^{a_v} y^{l_v})$$

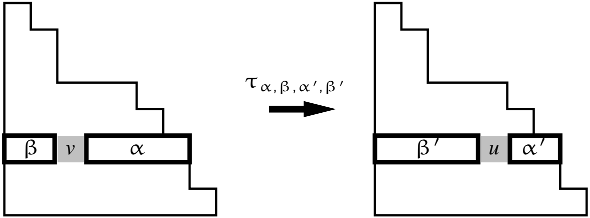
is super-symmetric.

Note that the above two polynomials are actually equal due to the corollary for the polynomial Q_n. Go

- $\mathcal{F}_n(a,*,m)$ = the set of pointed partitions (λ, v) of n such that $a_v = a$ and $m_v = m$.
- $\mathcal{F}_n(a,l,*)$ = the set of pointed partitions (λ, v) of n such that $a_v = a$ and $l_v = l$.

$\tau_{\alpha,\beta,\alpha',\beta'} : \mathcal{F}_n(\alpha, *, \beta) \to \mathcal{F}_n(\alpha', *, \beta')$

It is easy to give a combinatorial proof of the super-symmetry of the first polynomial $\sum_{(\lambda,\nu)\in\mathcal{F}_n} x^{a_\nu} y^{m_\nu}$.



$\zeta_{\alpha,\beta,\alpha',\beta'} : \mathcal{F}_n(\alpha, \beta, *) \to \mathcal{F}_n(\alpha', \beta', *)$

We can prove bijectively the super-symmetry of the polynomial $\sum_{(\lambda,v)\in\mathcal{F}_n} x^{a_v} y^{l_v}$. The bijection $\zeta_{\alpha,\beta,\alpha',\beta'}$ can be defined by

$$\begin{array}{ccc} \mathcal{F}_n(\alpha, \beta, *) & \xrightarrow{\zeta_{\alpha,\beta,\alpha',\beta'}} & \mathcal{F}_n(\alpha', \beta', *) \\ \varphi \Big\downarrow & & \Big\uparrow \varphi \\ \mathcal{F}_n(\alpha, *, \beta) & \xrightarrow{\tau_{\alpha,\beta,\alpha',\beta'}} & \mathcal{F}_n(\alpha', *, \beta'). \end{array}$$

Theorem (S.-Zeng, 2009)

If $\alpha+\beta=\alpha'+\beta'$, *the mapping*

$$\zeta_{\alpha,\beta,\alpha',\beta'}=\varphi\circ\tau_{\alpha,\beta,\alpha',\beta'}\circ\varphi$$

is a bijection from $\mathcal{F}_n(\alpha,\beta,*)$ *to* $\mathcal{F}_n(\alpha',\beta',*)$.

This theorem yields that the generating function of $\mathcal{F}_n$ by the bivariate joint distribution of arm length and leg length is super-symmetric.

Summary

1. h_v and p_v are symmetric. $\leftarrow$ the involution φ.
2. l_v and m_v are symmetric. $\leftarrow$ the involution φ.
3. a_v and m_v are super-symmetric. $\leftarrow$ the bijection $\tau_{\alpha,\beta,\alpha',\beta'}$.
4. a_v and l_v are super-symmetric. $\leftarrow$ the bijection $\zeta_{\alpha,\beta,\alpha',\beta'}$.

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Thank you for your attention.

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