Asymptotic of some Selberg-like Integrals

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Outline

The physical problem

- Quantum transport in chaotic cavities: the unitary scattering matrix relates the wave functions of incoming and outgoing electrons.
- Many interesting properties are well represented by linear statistics on a random matrix belonging to a suitable ensemble.

The physical problem

- Quantum transport in chaotic cavities: the unitary scattering matrix relates the wave functions of incoming and outgoing electrons.
- Many interesting properties are well represented by linear statistics on a random matrix belonging to a suitable ensemble.
- Unitary constraint: the joint probability density for the eigenvalues T_i of the matrix is of the following form:

$$P(T_1, \dots, T_N) = \frac{1}{\mathfrak{N}} \prod_{i<j} |T_i - T_j|^2 \prod_{i=1}^{N} T_i^{\alpha-1}.$$

- Interest in non-linear statistics on the eigenvalues and in asymptotic behavior (great number of electrons).

Selberg integrals

We are therefore interested in a fast computation of the following integrals:

$$\langle f(x_1,\dots,x_N)\rangle_{a,b,c}$$

$$= \frac{1}{N!}\int_{[0,1]^N} f(x_1,\dots,x_N)\prod_{i<j}(x_i-x_j)^{2c}\prod_i x_i^{a-1}(1-x_i)^{b-1}dx_i$$

Known result (Kaneko)

$$\frac{1}{N!}\int_{[0,1]^N} P_\lambda^{1/c}(x)\prod_{i<j}|x_i-x_j|^{2c}(x)\prod_{i=1}^{N}x_i^{a-1}\,(1-x_i)^{b-1}\,dx_i$$

$$=\prod_{i<j}\frac{\Gamma\left[\lambda_i-\lambda_j+c(j-i+1)\right]}{\Gamma\left[\lambda_i-\lambda_j+c(j-i)\right]}\prod_{i=1}^{N}\frac{\Gamma\left[\lambda_i+a+c(N-i)\right]\Gamma\left[b+c(N-i)\right]}{\Gamma\left[\lambda_i+a+b+c(2N-i-1)\right]}$$

The algorithm

1. Expand f in terms of Jack polynomials;
2. Replace each occurrence of a Jack polynomial $P_\lambda^{1/c}$ by $\langle P_\lambda^{1/c} \rangle_{a,b,c}$ which can be computed with the previous formula.

Advantages:

- the number of terms do not depend on the number of variables N;
- simplifications occur such that we get rational fractions in N (the number of factors do not depend on N);
- possibility to study the asymptotic behavior of this integral for $N \to \infty$.

Simplest case: Schur functions

Using the identity

$$\prod_{i=a+1}^{N} \frac{b+i}{b+c+i} = \prod_{i=1}^{c} \frac{a+b+i}{b+N+i} \text{ for } (a, c \in \mathbb{N}),$$

one has

$$\frac{\langle s_\lambda \rangle_{a,b}}{\langle 1 \rangle_{a,b}} = \prod_{i=1}^{\ell(\lambda)} \left[\frac{\lambda_i - \lambda_j + j - i}{j - i} \times \right.$$
$$\left. \times \prod_{j=0}^{\lambda_i - 1} \frac{(j + N - i + 1)(a + N - i + j)}{(\ell(\lambda) + j - i + 1)(a + b + 2N - i + j - 1)} \right]$$

Rational fraction of N, the number of terms does not depend on N, asymptotic behaviour in $N^{|\lambda|}$.

Integral for power sums

$$I_k = \frac{\langle p_k \rangle_{a,b}}{\langle 1 \rangle_{a,b}} \text{ where } p_k(\mathbb{X}) = x_1^k + \ldots x_N^k$$

Using the expansion of p_k in terms of hook-Schur functions:

$$p_k(\mathbb{X}) = \sum_{i=0}^{k} (-1)^i s_{[(k-i)1^i]},$$

one has

$$\frac{I_k}{N} = \frac{1}{N} \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} \prod_{j=-i}^{k-i-1} \frac{(N+j)(a+N+j-1)}{a+b+2N+j-2}.$$

Convergence of the integral? Limit if convergence?

Rewriting the integral for power sums

$$\frac{I_k}{N} = \frac{1}{Nk!}\frac{\mathfrak{N}_k(N)}{\mathfrak{D}_k(N)} \text{ (rational fraction of } N\text{)}$$

with

$$\mathfrak{D}_k(N) = \prod_{j=-k+1}^{k-1} [a(N) + b(N) + 2N + j - 2] \quad (\deg_N [\mathfrak{D}_k(N)] = 2k - 1)$$

and

$$\mathfrak{N}_k(N) = \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} \mathfrak{R}(N, i)$$

Inverse binomial transform of $(\mathfrak{R}(x, i))_i|_{x=N}$.

Binomial and inverse binomial transforms

Let $\mathbb{P}(x) = (P_i(x))_{i\in\mathbb{N}}$ be a sequence of polynomials. Then its binomial transform $\mathfrak{B}_k\left[\mathbb{P}(x)\right]$ is defined as follows:

$$\mathfrak{B}_k\left[\mathbb{P}(x)\right] = \sum_{i=0}^{k} \binom{k}{i} P_i(x)$$

This transformation is invertible:

$$\mathfrak{B}_k^{-1}\left[\mathbb{P}(x)\right] = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} P_i(x)$$

Definitions: Divided Differences, Newton interpolation

Divided differences

The divided differences $\partial_{y_i y_j}$ acting on a function f of the variables y_i is defined by:

$$f\partial_{y_i y_j} = \frac{f^{\sigma_{y_i y_j}} - f}{y_i - y_j}$$

where $\sigma_{y_i y_j}$ permutes y_i and y_j in f.

Newton interpolation

Let f be a function of y. Then there exists a unique polynomial of degree n, $\mathtt{N}_n = \sum_{k=0}^{n} \alpha_k \prod_{j=0}^{k-1} (y - j)$, such that $\mathtt{N}_n(i) = f(i)$. The coefficients α_k are given by

$$f(y_0)\partial_{y_0 y_1} \ldots \partial_{y_{k-1} y_k} |_{y_i = i, i=0,1..} .$$

Link between the divided differences and $\mathfrak{B}^{-1}$

Proposition

If f is a polynomial,

$$\begin{aligned}\mathfrak{B}_k^{-1}\left[(f(x,i))_{i\in\mathbb{N}}\right] &= k!f(x,y_0)\partial_{y_0y_1}\dots\partial_{y_{k-1}y_k}|_{y_i=i,i=0,1..} \\ &:= k!f(x,y_0)\partial_{0\dots k}\end{aligned}$$

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$$:= k!f(x,y_0)\partial_{0\ldots k}$$

Proof: By linearity, on the monomials y^p:

$$\mathfrak{B}_k^{-1}\left[(i^p)_{i\in\mathbb{N}}\right] = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} i^p = k!S_{p,k}$$

($S_{p,k}$ denotes the Stirling numbers of the 2^{nd} kind).
But the Newton interpolation of y^p reads, with $(y)_k$ the falling factorial,

$$y^p = \sum_{k=0}^{p} S_{p,k}(y)_k \text{ or } S_{p,k} = y^p\partial_{0\ldots k+1}.$$

Properties

Degree of the binomial transform

- Assuming that $k \leq p$, the degree of

$$y_0^p \partial_{y_0 y_1} \dots \partial_{y_{k-1} y_k}$$

is equal to $p - k$.
- Therefore, if $g(x, y)$ is a polynomial of degree p in x and y, the degree of $\mathfrak{B}_k^{-1}[(g(x, i))_i]$ equals $p - k$.

Shift-like property

$$y^p \partial_{0 \dots k+1} = (y+1)^{p-1} \partial_{0 \dots k}$$

An example of binomial transform

Definition

$$P_i^k(x; a, b) = P_i^k(x) = \prod_{j=0}^{k-i-1} (x + j + a) \prod_{j=0}^{i-1} (x - j + b)$$

Using the properties of the divided differences, it is possible to prove by induction the following property:

$$\mathfrak{B}_{k-p}^{-1} \left[P_p^k, \ldots P_k^k \right] = \prod_{i=0}^{p-1} (x + b + i) \prod_{i=0}^{k-p-1} (b - a - p - i)$$

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With $p = 0$,

$$\mathfrak{B}_k^{-1}[P_0^k, \dots P_k^k] = \prod_{i=0}^{k-1} (b - a + i)$$

Leading coefficient

For **f** of degree m,

$$\mathfrak{L}_{k,p}(\mathbf{f}) = [x^{p+m-k}]\mathbf{f}(x, y_0)y_0^p\partial_0 \ldots \partial_{k-1}\Big|_{y_i=i}$$

is the coefficient of the leading term in the binomial transform of $\mathbf{f}(x, y_0)y^p$.

Proposition

$$\mathfrak{L}_{k,p}(\mathbf{f}) = \begin{cases} \mathfrak{L}_{k-p,0}(\mathbf{f}_p) & \text{if } p \leq k \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathbf{f}_p(x, y) = \mathbf{f}(x, y + p)$.

Let $\mathbf{P}^{k-p}$ be the unique polynomial of degree $k - p$ such that $\mathbf{P}^{k-p}(i) = P^k_{p+i}$ for each $i = 0 \ldots k - p$. Then

$$(k - p)!\mathfrak{L}_{k,p}(\mathbf{P^k}) = [x^p]\mathfrak{B}^{-1}_{k-p}[P^k_p, \ldots P^k_k].$$

Binomial transform and the P_i^k's

Definition

For any sequence of polynomials $\mathbb{T}(x) = (T_i(x))_{i\in\mathbb{N}}$, we define

$$\mathfrak{T}_k^{a,b}\left[\mathbb{T}(x)\right] = (-1)^k \mathfrak{B}_k^{-1}\left[\left(P_i^k(x;a,b)\,T_i(x)\right)_{i\in\mathbb{N}}\right].$$

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Let $f(x,y)$ be a bivariate polynomial of degree m. Then $\mathfrak{T}_k^{a,b}\left[(f(x,i))_{i\in\mathbb{N}}\right]$ is polynomial in x of degree m.

Indeed, $\deg_x\left[P_i^k(x;a,b)f(x,i)\right] = k+m$ and

$$\deg_x\left[\mathfrak{B}^{-1}\left[P_i^k(x;a,b)f(x,i)\right]\right] = k+m-k.$$

Leading coefficient

The iteration of a short computation shows that

Proposition

$$[x^p]\,\mathfrak{T}_k^{a,b}\,[(i^p)_i] = \begin{cases} (-1)^p \dfrac{k!}{(k-p)!} \displaystyle\prod_{i=0}^{k-p-1} (b-a-p-i) & \text{if } p \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

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Proposition

$$[x^p]\,\mathfrak{T}_k^{a,b}\,[(i^p)_i] = \begin{cases} (-1)^p \dfrac{k!}{(k-p)!} \displaystyle\prod_{i=0}^{k-p-1} (b-a-p-i) & \text{if } p \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover,

$$[x^p]\,\mathfrak{T}_k^{a,b}\left[\left(\prod_{j=0}^{p-1} (c_j x + d_j i + e_j)\right)_i\right]$$

is independent of the e_j's.

Expression of $\frac{I_k}{N}$ in terms of $\mathfrak{T}$ for a, b linear in N

Let $a = a_0 + a_1 N$ and $b = b_0 + b_1 N$. Then

$$\mathfrak{N}_k(N) = \mathfrak{T}^{a_0+b_0-1-k, a_0+b_0+k-3}\left[(\mathbf{Q}_k(x,i))_{i\in\mathbb{N}}\right]\Big|_{x=(a_1+b_1+2)N}$$

where $\mathbf{Q}_k(x,y)$ is a polynomial of degree $2k$ in x and y:

$$\mathbf{Q}_k(x,y) := \prod_{j=0}^{k-1}\left(\frac{x}{2+a_1+b_1}+j-y\right)\left(\frac{1+a_1}{2+a_1+b_1}x+a_0+j-1-y\right).$$

Expression of $\frac{I_k}{N}$ in terms of $\mathfrak{T}$ for a, b linear in N

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where $\mathbf{Q}_k(x,y)$ is a polynomial of degree $2k$ in x and y:

$$\mathbf{Q}_k(x,y) := \prod_{j=0}^{k-1} \left(\frac{x}{2+a_1+b_1} + j - y\right)\left(\frac{1+a_1}{2+a_1+b_1}x + a_0 + j - 1 - y\right).$$

The results presented for the y^p still hold for the $\mathbf{Q}_k$'s by linearity. Therefore,

$$\deg\left[\mathfrak{N}_k(N)\right] = \deg\left[N\mathfrak{D}_k(N)\right] = 2k$$

Thus $\lim_{N\to\infty} \frac{I_k}{N}$ exists.

Value of the limit

Finally, using the property presented about leading coefficient, one has

$$\lim_{N\to\infty} \frac{I_k}{N} =$$

$$\frac{1+a_1}{k(2+a_1+b_1)^k} \sum_{j=0}^{k-1} (-1)^j \left(\frac{1+a_1}{2+a_1+b_1}\right)^j \binom{j+k-1}{j} \times$$

$$\times \sum_{i=0}^{k-1-j} (1+a_1)^i \binom{k}{i+j+1} \binom{k}{i}$$

Catalan triangle

Let $a_1 = 0$ and $b_1 = \ell - 1$. Then

$$\lim_{N \to \infty} \frac{I_k}{N} = \frac{\displaystyle\sum_{i=0}^{k-1} \frac{k-i}{k} \binom{2k}{i} \ell^i}{(1+\ell)^{2k-1}}$$

The triangle $\left(\frac{k-i}{k} \binom{2k}{i} \right)_{k,i \in \mathbb{N}}$ is called Catalan triangle.

Number of Dyck paths

Let $b_1 = 0$ and write $a_1 = \ell - 1$. Then

$$\lim_{N\to\infty} \frac{I_k}{N} = \frac{\ell}{(1+\ell)^{2k-1}} \sum_{i=0}^{2(k-1)} \binom{k-1}{\lceil \frac{i}{2} \rceil} \binom{k-1}{\lfloor \frac{i}{2} \rfloor} \ell^i$$

where

$$\binom{k-1}{\lceil \frac{i}{2} \rceil} \binom{k-1}{\lfloor \frac{i}{2} \rfloor}$$

is the number of Dyck paths of odd semi-length $2k-1$ with i peaks.

Central binomial coefficients

Let $a_1 = b_1 = 0$. Then

$$\lim_{N\to\infty} \frac{I_k}{N} = \frac{1}{2^k k} \sum_{j=0}^{k-1} \left(\frac{1}{2}\right)^i \binom{j+k-1}{j} \sum_{i=0}^{k-1-j} \binom{k}{i+j+1} \binom{k}{i}$$

Simplifications yield the following equality:

$$\lim_{N\to\infty} \frac{I_k}{N} = \frac{1}{2^k k} \binom{2k}{k}.$$

Conclusion

Next steps:

- Computation of the integral for power sums for any value of c:

  $$\text{power sums} \xrightarrow{\text{known}} \text{(hook)-Schur functions} \xrightarrow{?} \text{Jack polynomials;}$$

- Factorization property?

$$\lim_{N\to\infty} \langle \frac{1}{N^{\ell(\lambda)}} p_\lambda(x_1, \ldots, x_N) \rangle^\sharp = \prod_{i=1}^{\ell(\lambda)} \lim_{N\to\infty} \langle \frac{1}{N} p_{\lambda_i}(x_1, \ldots, x_N) \rangle^\sharp$$

  where $\langle f \rangle^\sharp = \dfrac{\langle f \rangle_{a,b,c}}{\langle 1 \rangle_{a,b,c}}$ and $p_\lambda = \prod_i p_{\lambda_i}$.

- Find a combinatorial interpretation: Dyck paths...