On permutation tableaux of type A and B

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(Joint work with Sylvie Corteel)

University of Paris 7

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Permutation tableaux

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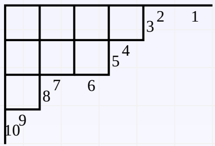
- A connection with partially asymmetric exclusion process (PASEP)
- Type B Permutation tableaux defined by Lam and Williams

Ferrers diagram



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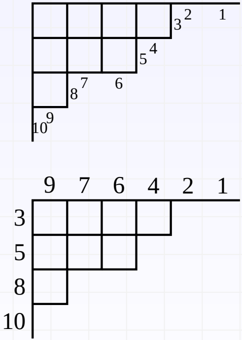

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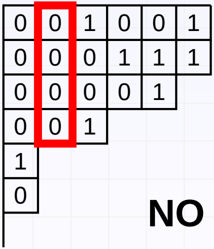

NO

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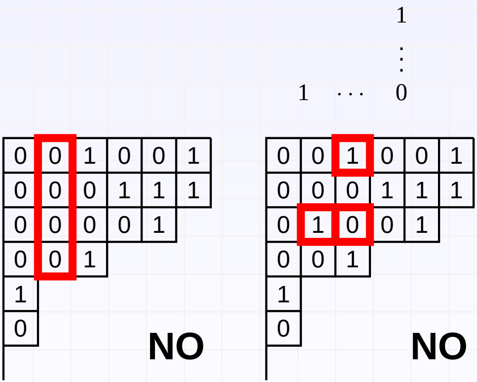

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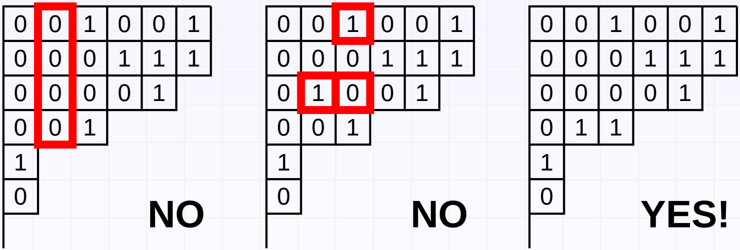

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0	0	1	0	0	1
0	0	0	1	1	1
0	0	0	0	1	
0	0	1			
1					
0					

NO

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0	0	0	1	1	1
0	1	0	0	1	
0	0	1			
1					
0					

NO

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0	0	0	1	1	1
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1					
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YES!

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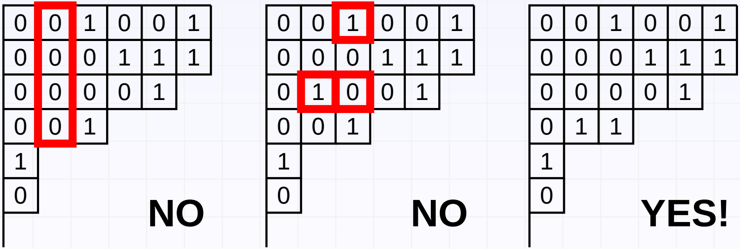

- A **restricted** 0 is

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- An **unrestricted row** has no restricted 0.

The alternative representation

- Topmost 1 is $\uparrow$

The alternative representation

- Topmost 1 is ↑
- Rightmost restricted 0 is ←

	12	9	8	6	5	3
1	0	0	1	0	0	1
2	0	0	0	1	1	1
4	0	0	0	0	1	
7	0	1	1			
10	1					
11	0					
13						

	12	9	8	6	5	3
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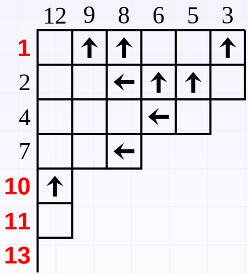
- First introduced by Viennot (alternative tableau) and studied more by Nadeau

The bijection Φ of Corteel and Nadeau



1, 10, 11, 13

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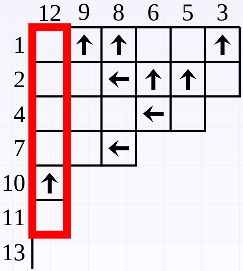

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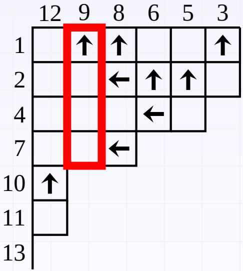

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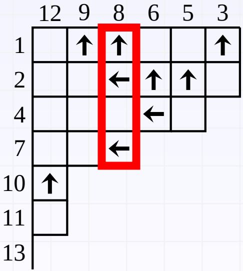

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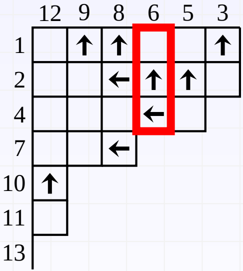

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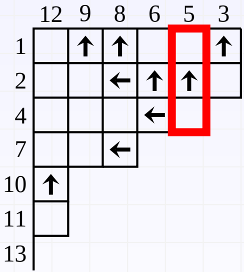

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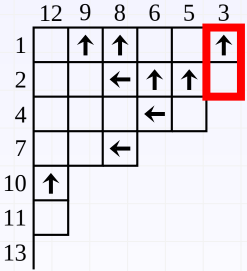

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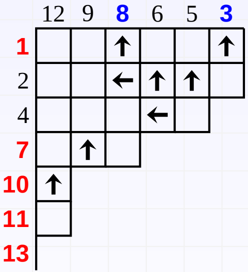
Statistics

- Decompose π as $\sigma 1 \tau$.

$$\pi = \overbrace{4, 6, 5, 2, 8, 3}^{\sigma}, 1, \overbrace{9, 7, 12, 10, 11, 13}^{\tau}$$



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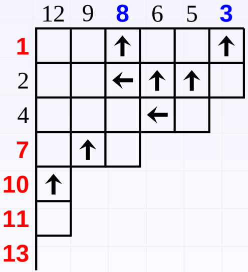

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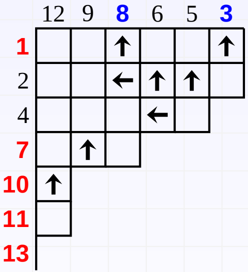

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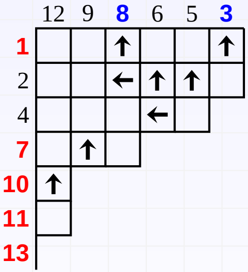

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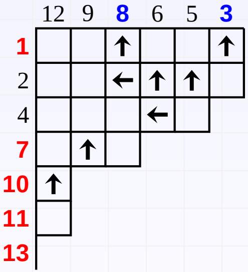

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- *the unrestricted rows of* $T \Leftrightarrow$ *the RL-minima of* π

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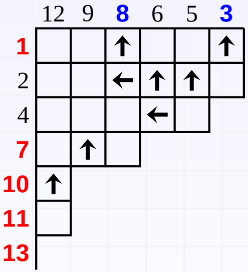

Proposition (Corteel & Nadeau, Nadeau)

Let $\pi = \sigma 1 \tau$ and $\Phi(\pi) = T$. Then

- *the unrestricted rows of $T \Leftrightarrow$ the RL-minima of π*
- *the columns with 1 in the first row of $T \Leftrightarrow$ the RL-maxima of σ*

Nadeau's bijective proof of a theorem of Corteel and Nadeau

Theorem

$$\sum_{T \in \mathcal{PT}(n)} x^{\mathrm{urr}(T)-1} y^{\mathrm{topone}(T)} = (x+y)_{n-1} = (x+y)(x+y+1)\cdots(x+y+n-2).$$

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- $c(n,k)$: the number $\pi \in S_n$ with k cycles

$$(x+y)_{n-1} = \sum_{i,j} c(n-1,i+j)\binom{i+j}{i}x^iy^j$$

$$\#\{T\in\mathcal{PT}(n) : \mathrm{urr}(T)-1=i, \quad \mathrm{topone}(T)=j\} = c(n-1,i+j)\binom{i+j}{i}$$

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- τ is a set of i cycles and σ is a set of j cycles.

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- τ is a set of i cycles and σ is a set of j cycles.
- $\tau\cup\sigma$ is a permutation of $\{2,3,\ldots,n\}$ with $i+j$ cycles.

Another bijective proof

Theorem

We have

$$\sum_{T \in \mathcal{PT}(n)} x^{\mathrm{urr}(T)-1} y^{\mathrm{topone}(T)} = (x+y)_{n-1}.$$

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We have

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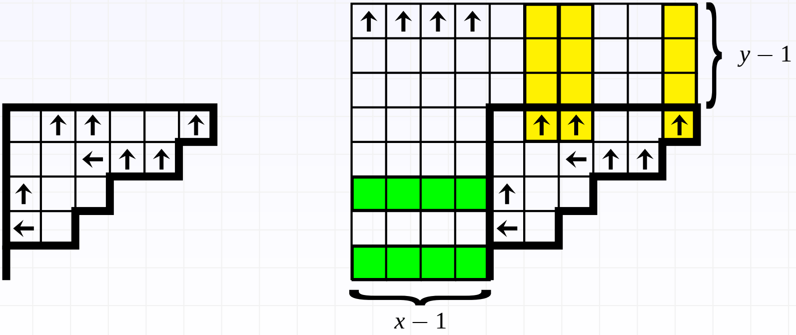
- Let x, y be any positive integers and let $N = n + x + y - 2$.
- Given $T \in \mathcal{PT}(n)$, we construct $T' \in \mathcal{PT}(N)$ as follows.



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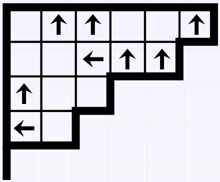
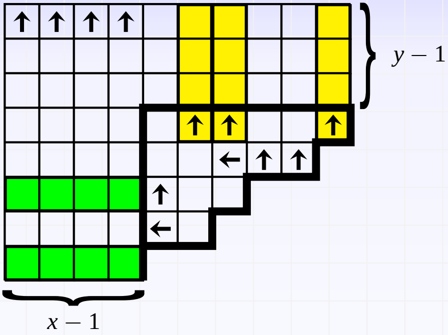
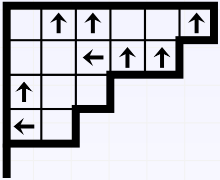
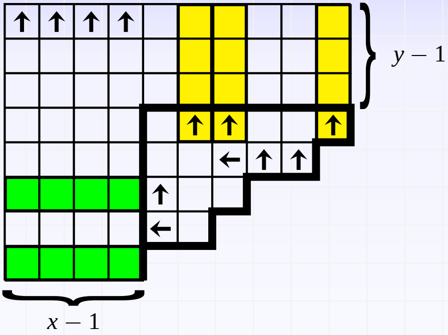




- T' satisfies the following.

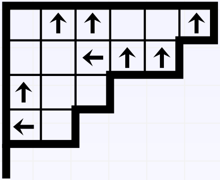
Another bijective proof

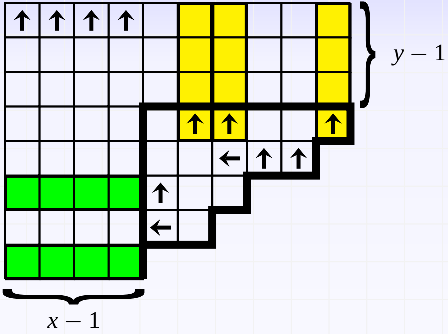




- T' satisfies the following.
  1. The first y steps are south and the first y rows are unrestricted.

Another bijective proof





- T' satisfies the following.
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  2. The last $x-1$ steps are west and the last $x-1$ columns have $\uparrow$'s in the first row.

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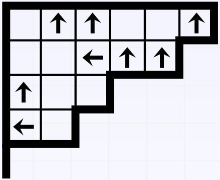

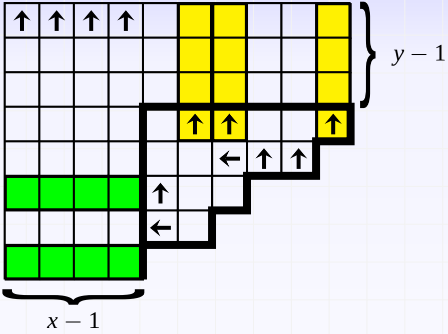

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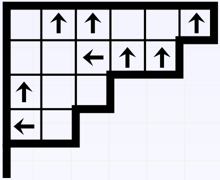
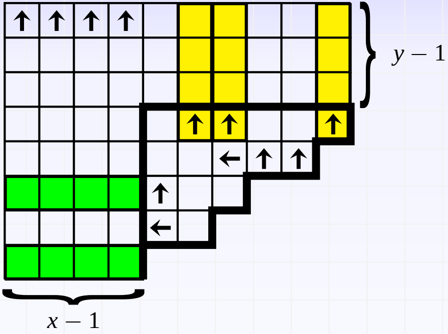
- $\pi' = \Phi(T')$ satisfies the following. ($\pi' = \sigma 1 \tau$)

Another bijective proof





- T' satisfies the following.
  1. The first y steps are south and the first y rows are unrestricted.
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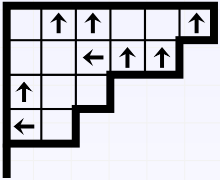

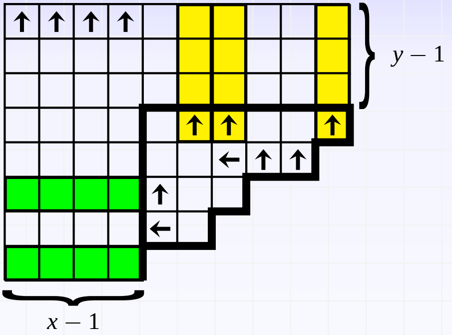

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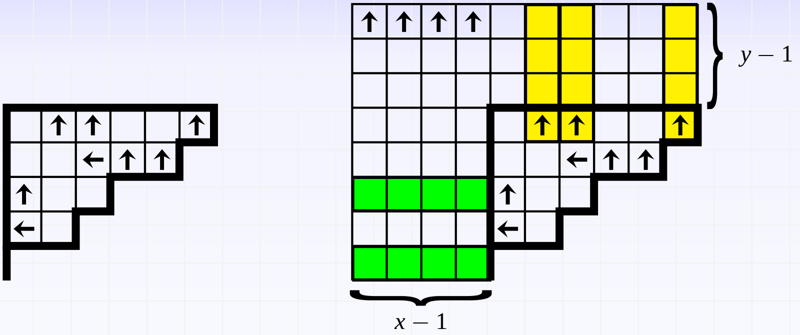

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- $N, N-1, \ldots, N-x+2, 1, 2, \ldots, y$ are arranged in this order in π'.

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Theorem

We have

$$P_t(x) = \frac{1 + E_t(x)}{1 + (t-1)xE_t(x)},$$

where

$$E_t(x) = \sum_{n \geq 1} n(t)_{n-1} x^n.$$

The case $t = 2$: connected permutations

Corollary

$$\sum_{n\geq 0} \sum_{T\in\mathcal{PT}(n)} 2^{\mathrm{urc}(T)} x^n = \frac{1}{x}\left(1 - \frac{1}{\sum_{n\geq 0} n! x^n}\right).$$

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- Combinatorial proof?

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Proposition

$$\#CP(n) = \#SCP(n)$$

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The case $t = -1$: sign-imbalance

Corollary

$$P_{-1}(x) = \sum_{n \geq 0} \sum_{T \in \mathcal{PT}(n)} (-1)^{\mathrm{urc}(T)} x^n = \frac{1-x}{1-2x+2x^2}$$

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Relation between the sign-imbalance of SYT

- The **sign** of a standard Young tableau is defined as follows.

$$\operatorname{sgn}\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}\right) = \operatorname{sgn}(12534) = 1$$

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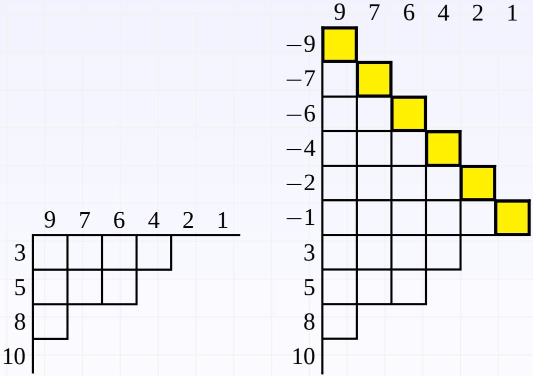
Shifted Ferrers diagram



- The yellow cells are the **diagonal** cells.

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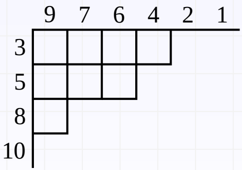

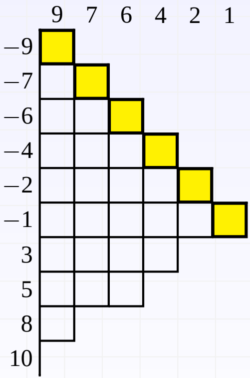

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- The row containing the diagonal cell in Column d is labeled with $-d$.

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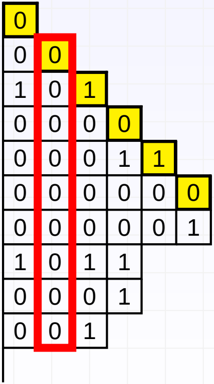

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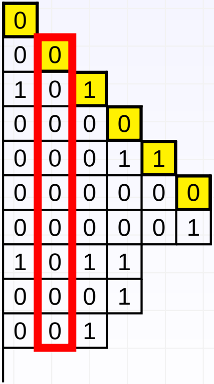

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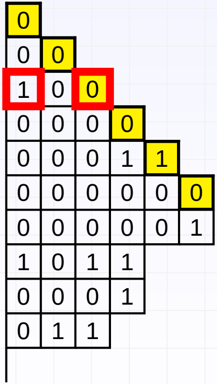

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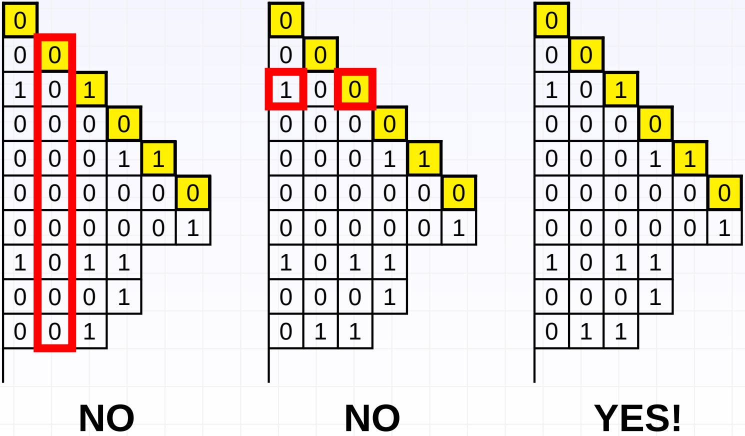

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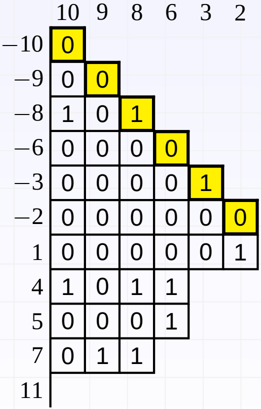

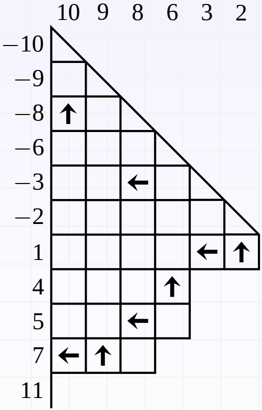

- No arrow points to another.

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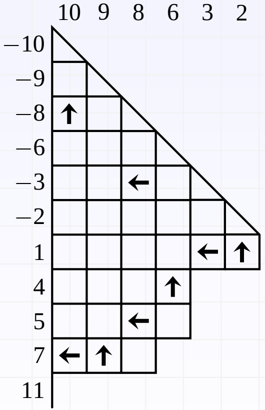
	10	9	8	6	3	2
−10	0					
−9	0	0				
−8	1	0	1			
−6	0	0	0	0		
−3	0	0	0	0	1	
−2	0	0	0	0	0	0
1	0	0	0	0	0	1
4	1	0	1	1		
5	0	0	0	1		
7	0	1	1			
11						



- No arrow points to another.
- The diagonal line acts like a mirror!

A theorem of Lam and Williams

Theorem (Lam and Williams)

$$\sum_{T \in \mathcal{PT}_B(n)} x^{\mathrm{urr}(T)-1} z^{\mathrm{diag}(T)} = (1+z)^n (x+1)_{n-1}$$

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- $\mathrm{urr}(T)$: the number of unrestricted rows, i.e. without

$$\begin{matrix} 1 \\ \vdots \\ 0 \end{matrix} \quad \text{and} \quad \boxed{0}$$

A theorem of Lam and Williams

Theorem (Lam and Williams)

$$\sum_{T\in\mathcal{PT}_B(n)} x^{\mathrm{urr}(T)-1} z^{\mathrm{diag}(T)} = (1+z)^n (x+1)_{n-1}$$

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Theorem

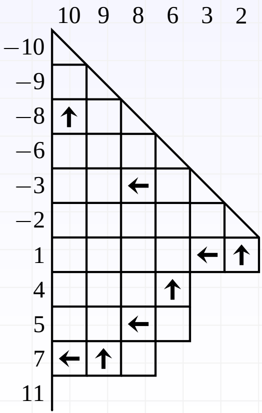
$$\sum_{T\in\mathcal{PT}_B(n)} x^{\mathrm{urr}(T)-1} y^{\mathrm{top}_{0,1}(T)} z^{\mathrm{diag}(T)} = (1+z)^n (x+y)_{n-1}$$

Generalization of a theorem of Lam and Williams

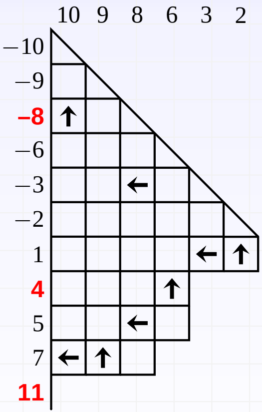
- For $T \in \mathcal{PT}_B(n)$ with the topmost nonzero row labeled m,

$$\begin{aligned}\mathrm{top}_{0,1}(T) &= (\#\text{ 1s in Row } m \text{ except in the diagonal}) \\ &\quad + (\#\text{ rightmost restricted 0s in Column } -m) \\ &= \#\text{ arrows in Row } m \text{ and Column } -m\end{aligned}$$

	10	9	8	6	3	2
−10	0					
−9	0	0				
−8	1	0	1			
−6	0	0	0	0		
−3	0	0	0	0	1	
−2	0	0	0	0	0	0
1	0	0	0	0	0	1
4	1	0	1	1		
5	0	0	0	1		
7	0	1	1			
11						



A type B extension of Corteel and Nadeau's bijection



–8,4,11

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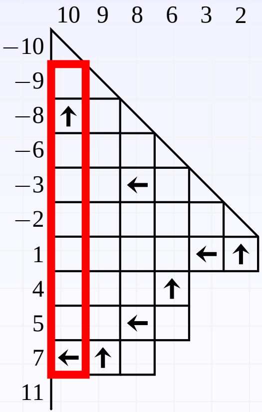

$\mathbf{7}, \mathbf{10}, -8, 4, 11$

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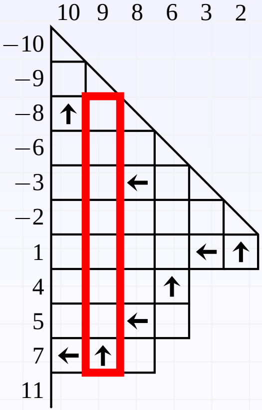

9, 7, 10, −8, 4, 11

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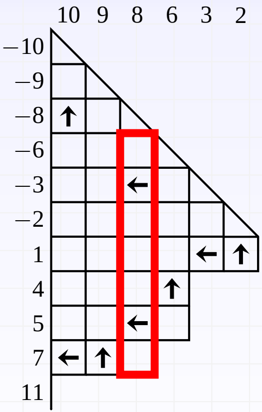

$9, 7, 10, \mathbf{-3}, \mathbf{5} - 8, 4, 11$

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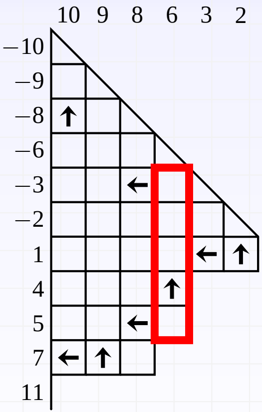

$9, 7, 10, -3, 5 - 8, \mathbf{6}, 4, 11$

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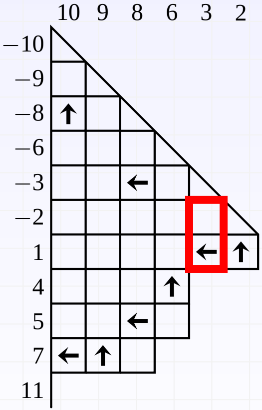

9, 7, 10, **1**, −3, 5 − 8, 6, 4, 11

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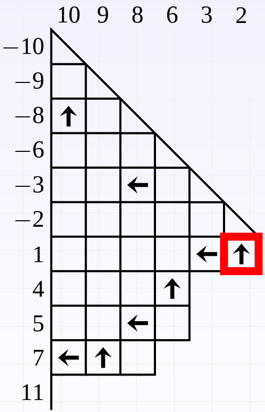
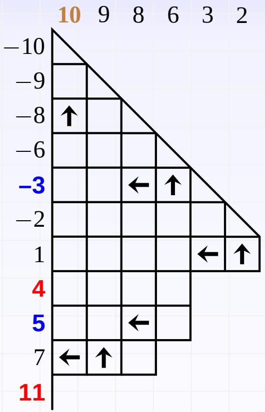


$9, 7, 10, \mathbf{2}, 1, -3, 5 - 8, 6, 4, 11$

Some properties of the type B bijection

Proposition



Then,

$$\pi = \overbrace{9, 7, 10}^{\sigma}, \overbrace{6, 2, 1, -3, 5}^{\tau}, \overbrace{-8}^{m}, \overbrace{4, 11}^{\rho}$$

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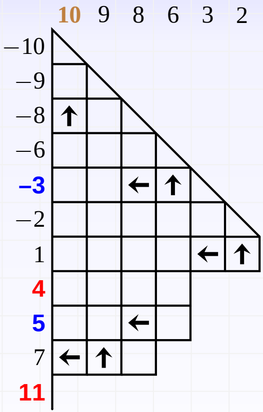

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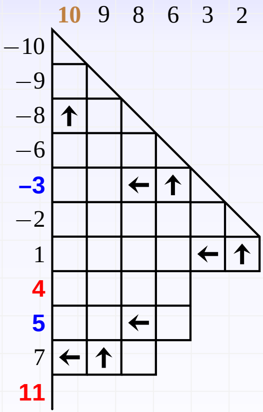

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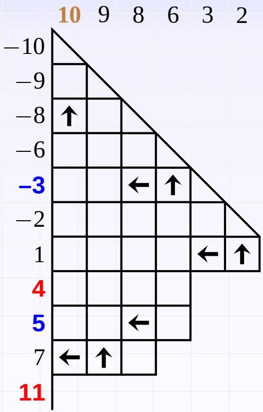

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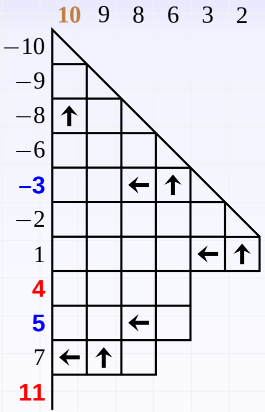

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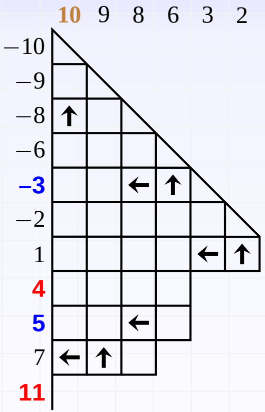

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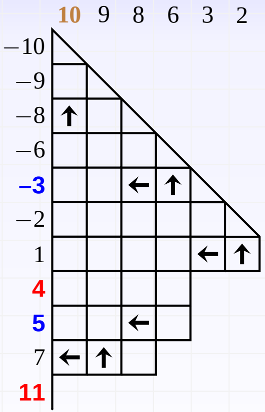

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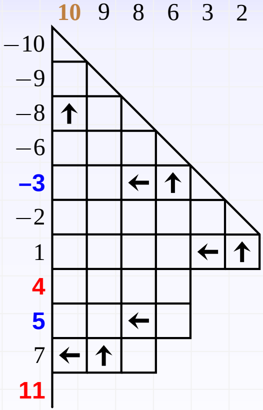

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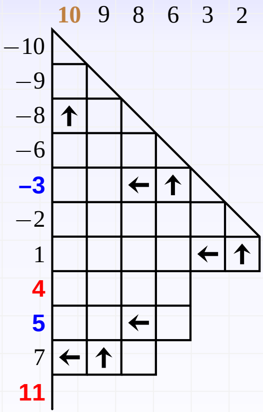

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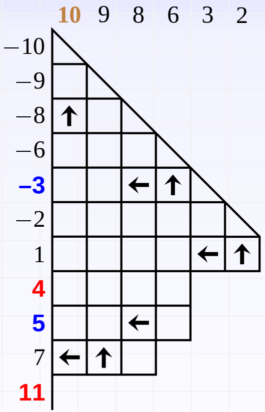

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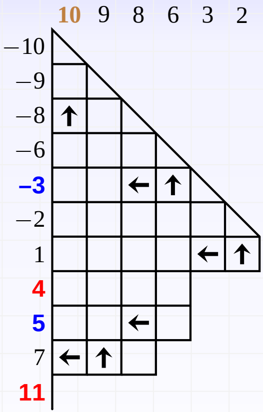
4. *Rows with* $\leftarrow$ *in Column* $|m|$ $\leftrightarrow$ *RL-minima of* τ



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Generalization

Theorem

$$\sum_{T \in \mathcal{PT}_B(n)} x^{\mathrm{urr}(T)-1} y^{\mathrm{top}_{0,1}(T)} z^{\mathrm{diag}(T)} = (1+z)^n (x+y)_{n-1}$$

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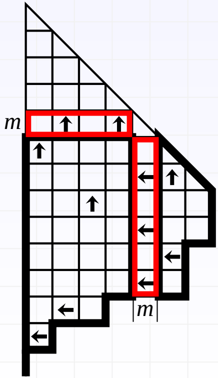
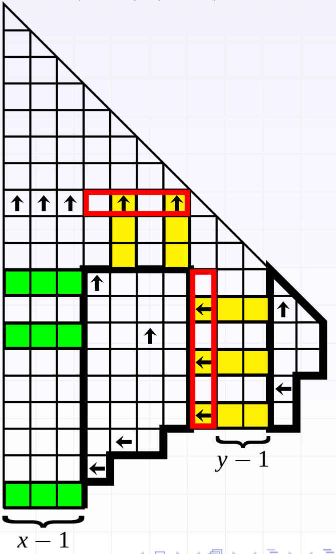
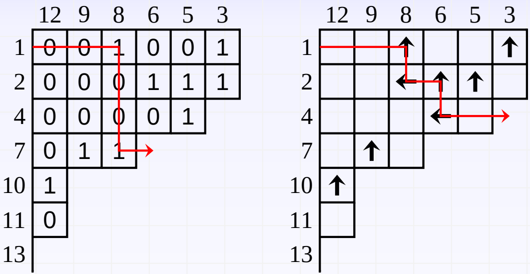




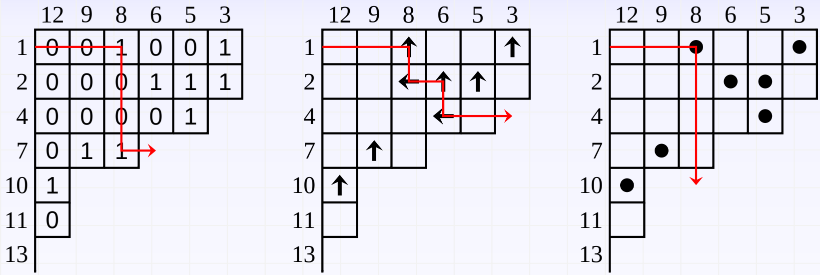
Zigzag maps

	12	9	8	6	5	3
1	0	0	1	0	0	1
2	0	0	0	1	1	1
4	0	0	0	0	1	
7	0	1	1			
10	1					
11	0					
13						

Zigzag maps


12 9 8 6 5 3
1 0 0 1 0 0 1
2 0 0 0 1 1 1
4 0 0 0 0 1
7 0 1 1
10 1
11 0
13
12 9 8 6 5 3
1
2
4
7
10
11
13

Zigzag maps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11
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13

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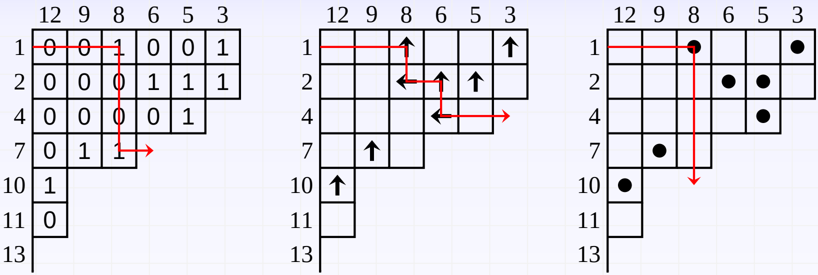

Theorem

The zigzag map on the alternative representation is the same as $\varphi \circ \Phi$.

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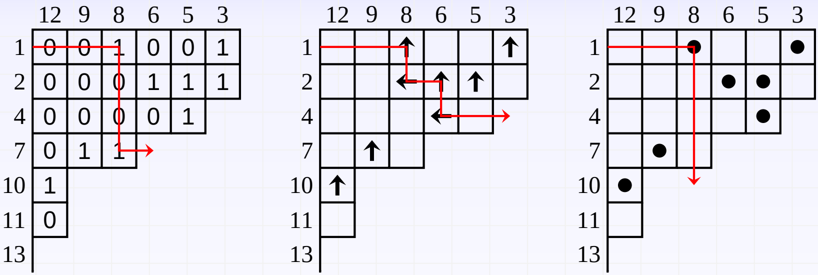

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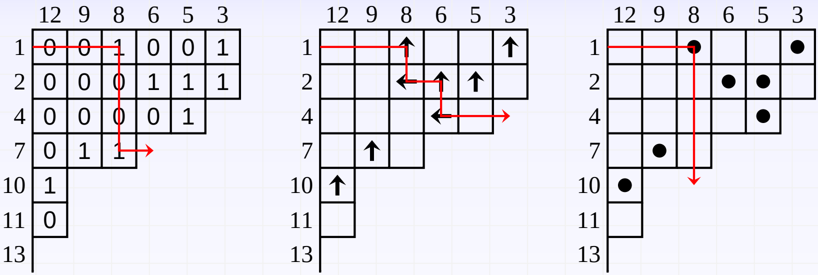
Example

Zigzag maps



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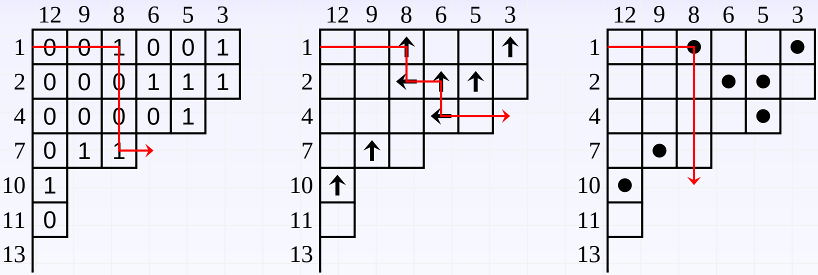
Example

- $\Phi(T) = 4, 6, 5, 2, 8, 3,$ **1**$, 9,$ **7**$, 11, 12,$ **10**

Zigzag maps



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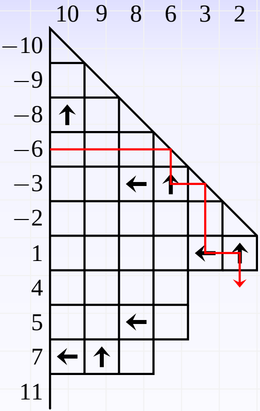
Example

- $\Phi(T) = 4, 6, 5, 2, 8, 3, \mathbf{1}, 9, \mathbf{7}, 11, 12, \mathbf{10}$
- $\varphi \circ \Phi(T) = (4, 6, 5, 2, 8, 3, 1)(9, 7)(11, 12, 10)$

The zigzag map on the type B alternative representation



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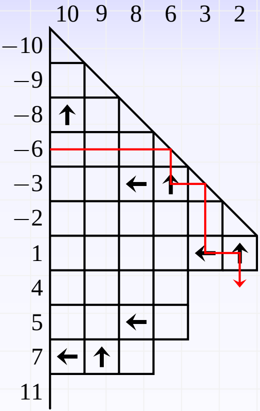

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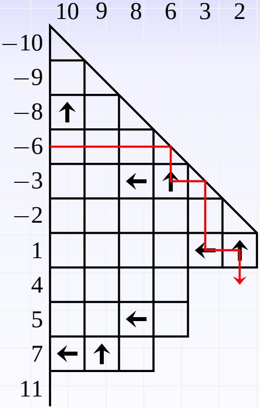

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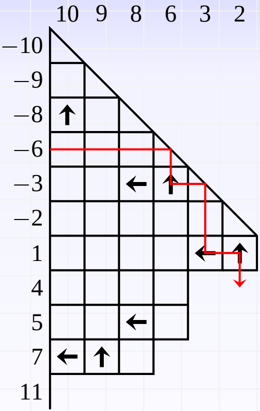

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- $\varphi \circ \Phi_B(T) = (9, 7, 10, 6, 2, 1, -3, 5, -8)(4)(11)$

Further study

- Find a combinatorial proof of the following:

$$\sum_{T \in \mathcal{PT}(n)} \operatorname{sgn}(T) = \frac{(1+i)^n + (1-i)^n}{2} = \begin{cases} (-1)^k \cdot 2^{2k}, & \text{if } n = 4k \text{ or } n = 4k+1, \\ 0, & \text{if } n = 4k+2, \\ (-1)^{k+1} \cdot 2^{2k+1}, & \text{if } n = 4k+3. \end{cases}$$

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$$\left| \sum_{T \in \mathcal{PT}(n)} \operatorname{sgn}(T) \right| = \left| \sum_{T \in \mathcal{SYT}(n)} \operatorname{sgn}(T) \right| = 2^{\lfloor \frac{n}{2} \rfloor}.$$

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Thank you for your attention!