On the full interval linear expansion of a skew Schur function

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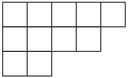
Outline

- Partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$,
  $n = \sum \lambda_i$ weight, $\ell(\lambda) = \ell$ length

  Example: $\lambda \quad = (5, 4, 2) \qquad =$ 

- If $\mu \subseteq \lambda$ then the skew diagram λ/μ is obtained by removing from the diagram of λ the boxes of μ

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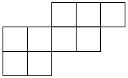
  Example: $\lambda/\mu = (5,4,2)/(2) =$ 

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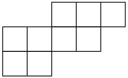
  Example: $\lambda/\mu = (5,4,2)/(2) =$ 

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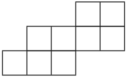
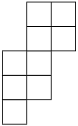
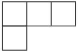
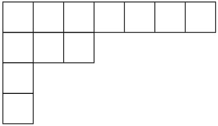
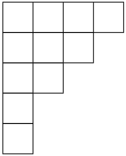
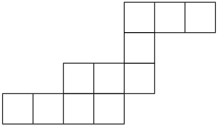
$(\lambda/\mu)^\pi =$  , $(\lambda/\mu)' =$  , $\lambda^* =$ 

$$s_{in} = (2,2,1,3), \quad m^n - \text{shortness of } \mu \text{ is } 1$$

$$s_{out} = (2,1,2,1,1,1), \quad m^n - \text{shortness of } \lambda^* \text{ is } 1$$

$\lambda = (4, 2, 0, 0), \mu = (3, 1, 1, 0).$

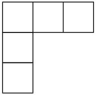
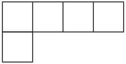
- $\lambda + \mu = (4 + 3, 2 + 1, 1) =$ 
- $\lambda \cup \mu = (4, 3, 2, 1, 1) =$ 
- $\lambda^{\pi} \bullet \mu =$ 

Dominance order on partitions λ, μ having the same weight: $\lambda \preceq \mu$ if

$$\mu_1 + \mu_2 + \cdots + \mu_i \leq \lambda_1 + \lambda_2 + \cdots + \lambda_i$$

for $i = 1, 2, \ldots, \min\{\ell(\mu), \ell(\lambda)\}$.

- $\mu \preceq \nu$ iff μ is obtained by "lowering" at least one box of ν.
- $\mu \lhd \nu$, iff ν is obtained by lifting exactly one box of μ to the next available position such that the transfer must be from some μ_{i+1} to μ_i , or from μ'_{i-1} to μ'_i.

$\mu =$  $\lhd \nu =$  but $\mu \not\lhd \eta =$  .

- Semistandard Young tableau $T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 3 \\ \hline 3 & 3 & 4 \\ \cline{1-3} 4 \\ \cline{1-1} \end{array}$
  $\lambda \quad = (4, 3, 1)$
  content$=(1, 1, 4, 2)$ reading word $w(T) = 33214334$

- The Schur function s_λ in the variables $x = (x_1, x_2, \ldots)$ is then defined by

$$s_\lambda = \sum_{SSYT\ T} x_1^{\#1's \text{ in } T} x_2^{\#2's \text{ in } T} \ldots$$

Example: $s_\lambda = x_1^1 x_2^1 x_3^4 x_4^2 + \cdots$

- Semistandard Young tableau $T = \begin{array}{cccc} & & 3 & 3 \\ 3 & 3 & 4 & \\ 4 & & & \end{array}$
  $\lambda/\mu = (4,3,1)/(2)$
  content$=(0,0,4,2)$ reading word $w(T) = 334334$

- The skew Schur function $s_{\lambda/\mu}$ in the variables $x = (x_1, x_2, \ldots)$ is then defined by

$$s_{\lambda/\mu} = \sum_{SSYT\ T} x_1^{\#1's \text{ in } T} x_2^{\#2's \text{ in } T} \cdots$$

Example: $s_{\lambda/\mu} = \quad x_3^4 x_4^2 + \cdots$

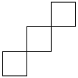
- Littlewood-Richardson Rule

$$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu\nu} s_{\nu}$$

where $c^{\lambda}_{\mu\nu}$ is the number of SSYT of shape λ/μ and content ν whose reading word is a lattice permutation (LR tableaux).

1122132 is a lattice permutation, but 122213 is not.

- $\mathrm{supp}(\lambda/\mu) = \{\nu' : c^{\lambda}_{\mu\nu} > 0\}$ support of $s_{\lambda/\mu}$ (or λ/μ)

If $\lambda/\mu = (3,2,1)/(2,1) =$  then $s_{\lambda/\mu} = s_{(1,1,1)} + 2s_{(2,1)} + s_{(3)}$ and $\mathrm{supp}(\lambda/\mu) = \{(1,1,1),(2,1),(3)\}$.

Littlewood- Richardson coefficients satisfy a number of symmetry properties, including:

- $c^{\lambda}_{\mu\nu} = c^{\lambda}_{\nu\mu}$ and $c^{\lambda'}_{\mu'\nu'} = c^{\lambda}_{\mu\nu}$;

It is useful to note that

- $s_{\lambda/\mu} = s_{(\lambda/\mu)^{\pi}}$ and $s_{\lambda/\mu} = s_{\widehat{\lambda}/\widehat{\mu}}$

where $\widehat{\lambda}/\widehat{\mu}$ is the skew diagram obtained from λ/μ by deleting any empty rows and any empty columns.

$w^r = 2312211$ the reverse lattice permutation $\quad P(w) = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$

$$w^r = \text{shuffle}(321, 21, 21) = 2312211$$

$$w = 1122132 = \text{shuffle}(123, 12, 12)$$

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- The set of reverse lattice permutations with the same content m forms a single plactic class whose unique tableau is the Yamanouchi tableau of shape m.
- Moreover, this set is equal to all shuffles of the columns of the only tableau in that class.

$T =$

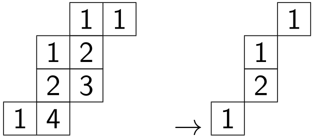
		1	1
	1	2	
	2	3	
1	4		

LR tableau with content $(4, 2, 1, 1) = (4, 2, 1, 1)'$

$S_k = (y_1, y_2, \ldots, y_k)$ is a k-string of T if $y_1 < \cdots < y_k$ and the rightmost box in row y_i is labeled with i

$S_4 = (1, 2, 3, 4)$

- A tableau with content $m = (m_1, m_2, \ldots, m_s)'$ is a LR tableau if and only if it has a complete sequence of strings $S_{m_1} \leq S_{m_2} \leq \cdots \leq S_{m_s}$.

$T =$ 

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$S_4 = (1, 2, 3, 4) \leq S_2 = (1, 3)$

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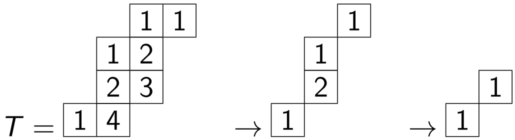

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$S_4 = (1,2,3,4) \leq S_2 = (1,3) \leq S_1 = (2)$

$w(T) = 11213241$

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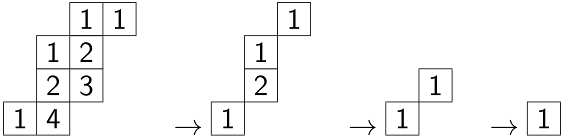

LR tableau with content $(4,2,1,1) = (4,2,1,1)'$

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$S_4 = (1,2,3,4) \leq S_2 = (1,3) \leq S_1 = (2) \ \leq S_1 = (4)$ complete sequence of strings

$w(T) = 11213241$

- A tableau with content $m = (m_1, m_2, \ldots, m_s)'$ is a LR tableau if and only if it has a complete sequence of strings $S_{m_1} \leq S_{m_2} \leq \cdots \leq S_{m_s}$.

$$\lambda/\mu = \begin{array}{|c|c|c|c|}\hline \cdot & \cdot & \square & \square \\ \cdot & \square & \square & \\ \cdot & \square & \square & \\ \square & \square & & \\ \hline\end{array}, \quad T(\mathbf{w}) = \begin{array}{cccc} & & 1 & 1 \\ & 1 & 2 & \\ & 2 & 3 & \\ 1 & 3 & & \end{array}, \quad T(\mathbf{n}) = \begin{array}{cccc} & & 1 & 1 \\ & 2 & 2 & \\ & 3 & 3 & \\ 4 & 4 & & \end{array}$$

$$\mathbf{w} = (3,3,1,1) \preceq \mathbf{n} = (4,4)$$

- $\mathrm{supp}(\lambda/\mu) \subseteq [\mathbf{w}, \mathbf{n}]$ with $\mathbf{w}, \mathbf{n} \in \mathrm{supp}(\lambda/\mu)$ and $c^{\lambda}_{\mu\mathbf{w}'} = c^{\lambda}_{\mu\mathbf{n}'} = 1$.
- $[\mathbf{w}, \mathbf{n}]$ is called the Schur interval of $s_{\lambda/\mu}$

Question: Characterize the skew Schur functions whose support is equal to $[\mathbf{w}, \mathbf{n}]$.

$$\lambda/\mu = \begin{array}{|c|c|c|c|}\hline & & \square & \square \\ \hline & \square & \square & \\ \hline & \square & \square & \\ \hline \square & \square & & \\ \hline\end{array}\ , \quad T(\mathbf{w}) = \begin{array}{cccc} & & 1 & 1 \\ & 1 & 2 & \\ & 2 & 3 & \\ 1 & 3 & & \end{array}\ , \quad T(\mathbf{n}) = \begin{array}{cccc} & & 1 & 1 \\ & 2 & 2 & \\ & 3 & 3 & \\ 4 & 4 & & \end{array}$$

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- $\mathrm{supp}(\lambda/\mu) \subseteq [\mathbf{w}, \mathbf{n}]$ with $\mathbf{w}, \mathbf{n} \in \mathrm{supp}(\lambda/\mu)$ and $c^{\lambda}_{\mu \mathbf{w}'} = c^{\lambda}_{\mu \mathbf{n}'} = 1$.
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Question: Characterize the skew Schur functions whose support is equal to $[\mathbf{w}, \mathbf{n}]$.

Question: Characterize the multiplicity-free skew Schur functions whose support is equal to $[\mathbf{w}, \mathbf{n}]$.

Theorem (Gutschwager '06, Thomas and Yong '05, King et al '09)

The basic skew Schur function $s_{\lambda/\mu}$ is multiplicity-free if and only if one or more of the following is true:

- *R*0 μ or λ^* is the zero partition 0;
- *R*1 μ or λ^* is a rectangle of m^n-shortness 1;
- *R*2 μ is a rectangle of m^n-shortness 2 and λ^* is a fat hook;
- *R*3 μ is a rectangle and λ^* is a fat hook of m^n-shortness 1;
- *R*4 μ and λ^* are rectangles.

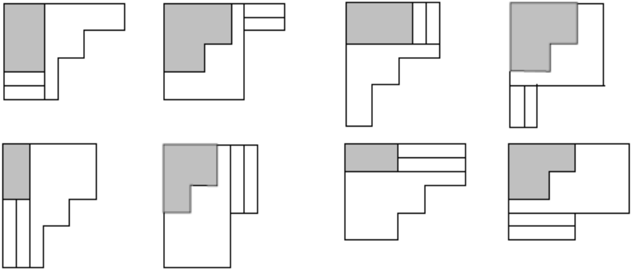
Corollary (Stembridge '00)

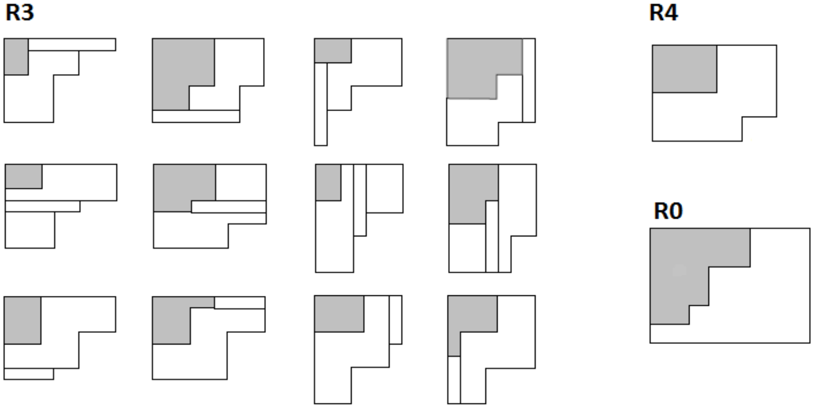
The product $s_\mu s_\nu$ is multiplicity-free if and only if

- (*i*) μ or ν is a one-line rectangle, or
- (*ii*) μ is a two line rectangle and ν is a fat hook, or
- (*iii*) μ is a rectangle an ν is a near rectangle, or
- (*iv*) μ and ν are rectangles.

R1

R2

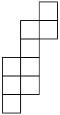



R3
R4
R0

Fix a skew diagram λ/μ and let T be a LR tableau with shape λ/μ and content $(m_1, \ldots, m_t)'$

- Rule 1. $$T = \begin{array}{ccc} & 1 & 1 \\ & 2 & 2 \\ 1 & 3 & 3 \\ 2 & 4 & \\ 5 & & \end{array} \rightarrow T_1 = \begin{array}{ccc} & 1 & 1 \\ & 2 & 2 \\ 1 & 3 & 3 \\ 4 & 4 & \\ 5 & & \end{array}$$
$$m = (5,3,2)' \lhd m_1 = (5,4,1)'$$

- Rule 2. $$T = \begin{array}{ccc} & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & \\ 3 & 4 & \\ 5 & & \end{array} \rightarrow T_1 = \begin{array}{ccc} & 1 & 1 \\ 1 & 2 & 2 \\ 3 & 3 & \\ 4 & 4 & \\ 5 & & \end{array}$$
$$m = (5,3,2)' \lhd m_1 = (5,4,1)'$$

$\lambda/\mu =$  , $\mathbf{w} = (4,3,2) \preceq \mathbf{n} = (6,3)$

- $T(\mathbf{w}) = \begin{array}{ccc} & & 1 \\ & \color{red}1 & 2 \\ & \color{red}2 & \\ \color{gray}1 & 3 & \\ \color{gray}2 & 4 & \\ \color{gray}3 & & \end{array} \xrightarrow{R2} \begin{array}{ccc} & & 1 \\ & \color{red}1 & 2 \\ & 3 & \\ \color{gray}1 & 4 & \\ \color{gray}2 & 5 & \\ \color{gray}3 & & \end{array} = \overline{T}$

  $\mathbf{w}$=(4,3,2) $\qquad \preceq$(5,3,1)

- $T(\mathbf{w}) = \begin{array}{ccc} & & 1 \\ & \color{red}1 & 2 \\ & \color{red}2 & \\ \color{blue}1 & 3 & \\ \color{blue}2 & 4 & \\ \color{red}3 & & \end{array} \xrightarrow{R1} \begin{array}{ccc} & & 1 \\ & \color{red}1 & 2 \\ & \color{red}2 & \\ \color{blue}1 & 3 & \\ \color{blue}2 & 4 & \\ 5 & & \end{array} \xrightarrow{R1} \begin{array}{ccc} & & 1 \\ & \color{red}1 & 2 \\ & \color{red}2 & \\ \color{blue}1 & 3 & \\ \color{red}3 & 4 & \\ 5 & & \end{array} \xrightarrow{R2} \begin{array}{ccc} & & 1 \\ & \color{red}1 & 2 \\ & \color{red}2 & \\ \color{red}3 & 3 & \\ \color{red}4 & 4 & \\ 5 & & \end{array}$

  $\mathbf{w}$=(4,3,2) $\qquad \lhd$(5,2,2) $\qquad \lhd$(5,3,1) $\qquad \lhd$(5,4)

$T(\mathbf{w}) = \begin{array}{ccc} & & 1 \\ & \color{red}1 & 2 \\ & \color{red}2 & \\ \color{blue}1 & 3 & \\ \color{blue}2 & 4 & \\ \color{red}3 & & \end{array} \xrightarrow{R2} \begin{array}{ccc} & & 1 \\ & \color{red}1 & 2 \\ & \color{red}2 & \\ \color{blue}1 & 3 & \\ \color{red}3 & 4 & \\ \color{red}4 & & \end{array}$

$\mathbf{w}$=(4,3,2) $\qquad \lhd$(4,4,1)

- $\overline{T} = \begin{array}{c}\begin{array}{ccc} & & 1\\ & 1 & 2\\ & 3 & \\ 1 & 4 & \\ 2 & 5 & \\ 3 & & \end{array}\\ (5,3,1)\end{array} \xrightarrow{R1} \begin{array}{c}\begin{array}{ccc} & & 1\\ & 1 & 2\\ & 3 & \\ 1 & 4 & \\ 2 & 5 & \\ 6 & & \end{array}\\ \lhd(6,2,1)\end{array} \xrightarrow{R1} \begin{array}{c}\begin{array}{ccc} & & 1\\ & 1 & 2\\ & 3 & \\ 2 & 4 & \\ 3 & 5 & \\ 6 & & \end{array}\\ \lhd(6,3)=\mathbf{n}\end{array}$
- $\overline{T} = \begin{array}{c}\begin{array}{ccc} & & 1\\ & 1 & 2\\ & 3 & \\ 1 & 4 & \\ 2 & 5 & \\ 3 & & \end{array}\\ (5,3,1)\end{array} \xrightarrow{R1} \begin{array}{c}\begin{array}{ccc} & & 1\\ & 1 & 2\\ & 3 & \\ 2 & 4 & \\ 3 & 5 & \\ 4 & & \end{array}\\ \lhd(5,4)\end{array}$

$$\begin{aligned}[\mathbf{w}, \mathbf{n}] &= \{(4,3,2), (4,4,1), (5,2,2), (5,3,1), (6,2,1), (5,4), (6,3)\}\\ &= \mathrm{supp}(\lambda/\mu)\end{aligned}$$

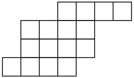
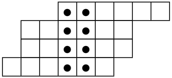
Lemma

- $\mathbf{w} = \mathbf{n}$ if and only if $\lambda/\mu = \nu$ or $\lambda/\mu = \nu^\pi$.
- (van Willigenburg '04) $s_{\lambda/\mu} = s_\nu$ if and only if $\lambda/\mu = \nu$ or $\lambda/\mu = \nu^\pi$.

Lemma

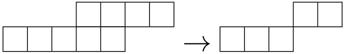
Let v, w be partitions with n entries.
(a) If $\lambda/\mu = v^\pi \bullet w$ and $\beta/\alpha = [v + (x^n)]^\pi \bullet w$, then $c \in \mathrm{supp}(\beta/\alpha)$ if and only if $c = b \cup (n^x)$ with $b \in \mathrm{supp}(\lambda/\mu)$.
(b) If $\lambda/\mu = (v^\pi \bullet w)'$ and $\beta/\alpha = ([v + (x^n)]^\pi \bullet w)'$, then $c \in \mathrm{supp}(\beta/\alpha)$ if and only if $c = b + (x^n)$ with $b \in \mathrm{supp}(\lambda/\mu)$.

$v = (3, 2, 2, 0), w = (4, 2, 2, 1)$ with $n = 4$

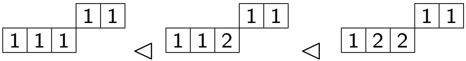
$v^\pi \bullet w =$  , $[v + (2^4)]^\pi \bullet w =$ 

Lemma

If λ/μ is a two row skew diagram then its support is the entire Schur interval.

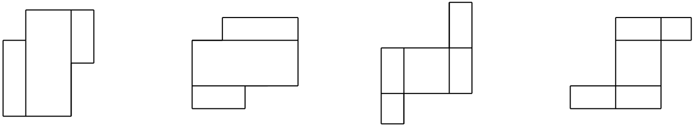


$= \lambda/\mu$

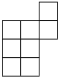


$\mathbf{w} = (1^5) \lhd b = (2, 1^3) \lhd \mathbf{n} = (2, 2, 1)$

$[\mathbf{w}, \mathbf{n}] = \{\mathbf{w}, b, \mathbf{n}\} = \mathrm{supp}(\lambda/\mu).$

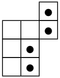


Example

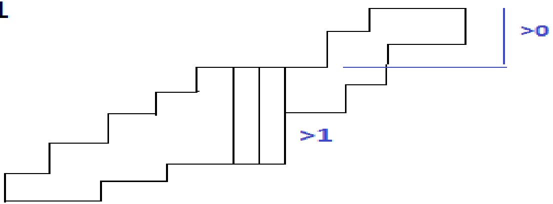
$\lambda/\mu =$  $, \mathbf{w} = (3,3,2)$ and $\mathbf{n} = (4,3,1)$

$$T(\mathbf{w}) = \underset{\mathbf{w}=(3,3,2)}{\begin{array}{ccc} & & 1 \\ 1 & 1 & 2 \\ 2 & 2 & \\ 3 & 3 & \end{array}} \xrightarrow{R1} \underset{\preceq(4,3,1)=\mathbf{n}}{\begin{array}{ccc} & & 1 \\ 1 & 1 & 2 \\ 2 & 3 & \\ 3 & 4 & \end{array}} = T(\mathbf{n}), \qquad \mathrm{supp}(\lambda/\mu) = \{\mathbf{w}, \mathbf{n}\}$$

$$\xi := (4,2,2) \in [\mathbf{w}, \mathbf{n}] \quad \text{but } \xi \notin \mathrm{supp}(\lambda/\mu)$$

$\lambda^1/\mu =$  $, \mathbf{w}^1 = (3,1) = \mathbf{n}^1$ and $(2,2) \prec \mathbf{w}^1$.

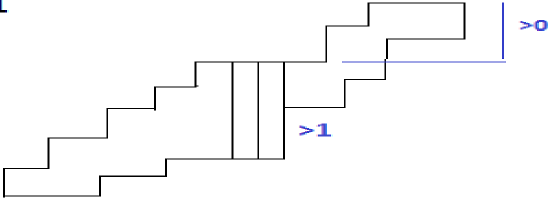
F1



Theorem

- If, up to a π-rotation and/or conjugation, λ/μ is an $F1$ configuration then $\mathrm{supp}(\lambda/\mu) \subsetneq [\mathbf{w}, \mathbf{n}]$.
- If $\ell(\lambda^1/\mu) \leq \ell(\lambda/\mu) - 2$ and λ^1/μ has at least one column of length ≥ 2 then $\mathrm{supp}(\lambda/\mu) \subsetneq [\mathbf{w}, \mathbf{n}]$.

F1



Theorem

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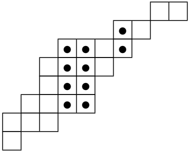
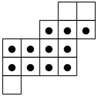
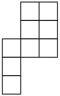
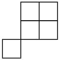
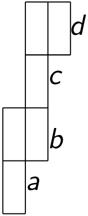
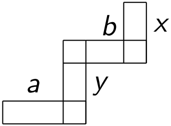
Examples:

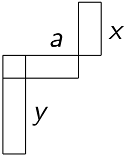
$F1$  , $F1^{\pi'}$  , $F1^{\pi}$  , $F1'$ 



$G2$



$G3$

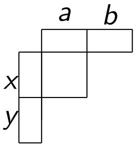


$G4$

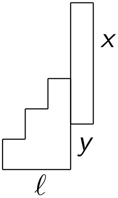
Proposition

1. The support of $G2$ is the entire Schur interval iff $a \leq c+1$ and $d \leq b+1$.
2. The support of $G3$ is the entire Schur interval iff $a = x = 1$, or $a = 1$ and $x \leq y+1$, or $a \leq b+1$ and $x = 1$
3. The support of $G4$ is the entire Schur interval iff $a = 1$ and $x \leq y+1$ or $a \geq 2$ and $x = 1$.

G6



G8



Proposition

1. The support of $G6$ is the entire Schur interval iff $b = y = 1$.
2. Let $\mathbf{w} = (n_1, \ldots, n_\ell) \cup (x)$ be the minimal filling of $G8$. The support of $G8$ is the entire Schur interval iff $\ell = 3, x = n_2, n_3 = 1, x + y = n_1 + 1$ and (i) $n_2 \leq y$ and $k \geq 1$; or (ii) $n_2 > y$, where k is the number of rows that the last two columns share.

Theorem

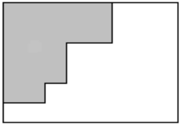
The basic skew Schur function $s_{\lambda/\mu}$ is multiplicity-free and its support is the entire Schur interval $[\mathbf{w}, \mathbf{n}]$ if and only if, up to a block of maximal width or maximal length, and up to a π-rotation and/or conjugation, one or more of the following is true:

(*i*) μ or λ^* is the zero partition 0.

(*ii*) λ/μ is a two column or a two row diagram.

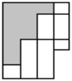
(*iii*) λ/μ is an $A2, A3, A4, A6$ or $A8$ configuration.

R0

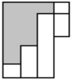


R4

R1



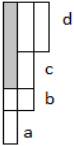
or



with $\ell(c2)=\ell(c4)$ and $k>0$

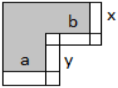
R3

A2:



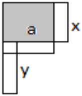
$a \leq c+1$ and $b \leq d+1$

A3:



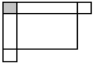
$a=x=1$, or
$a=1$ and $x \leq y+1$, or
$x=1$ and $a \leq b+1$

A4:



$a=1$ and $x \leq y+1$, or
$a>1$ and $x=1$

A6:



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