A cyclic sieving phenomenon in Catalan Combinatorics

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Non-crossing partitions

- **Non-crossing partitions** can be defined for any Coxeter group W as

$$NC(W,c) := \{\omega \in W : \ell_T(\omega) + \ell_T(c\omega^{-1}) = \ell_T(c)\},$$

where c is a Coxeter element and where $\ell_T(\omega)$ denotes the absolute length on W,

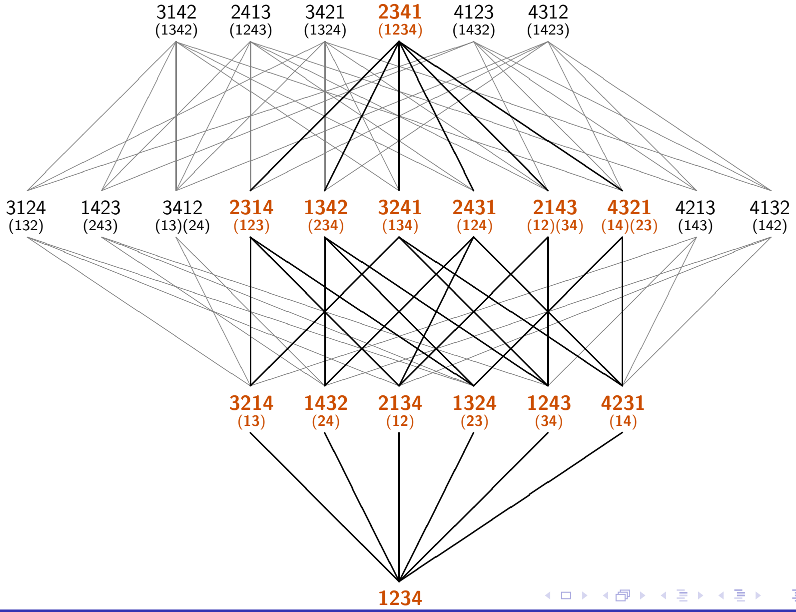
- for $W = \mathfrak{S}_n$ being the **symmetric group**,

$$NC(\mathfrak{S}_n) = \{\omega = c_1 \dots c_k \in \mathfrak{S}_n : c_i \text{ increasing}; \ c_i, c_j \text{ non-crossing}\},$$

where the Coxeter element $c = (1, \dots, n)$ and where $\omega = c_1 \dots c_k$ is the cycle notation of ω.

We focus on $\mathfrak{S}_n$ (and keep the general case in mind).

Example: $NC(\mathfrak{S}_4)$

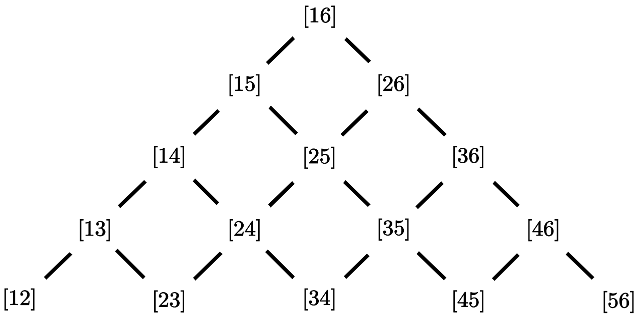


Non-nesting partitions

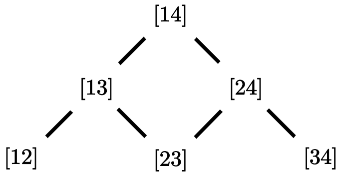
- **Non-nesting partitions** can be defined for any crystallographic Coxeter group W as

$$NC(W) := \{A \subset \Phi^+ : A \text{ antichain}\},$$

where Φ^+ denotes the root poset of W,
- for $W = \mathfrak{S}_n$ being the symmetric group, $NN(\mathfrak{S}_n)$ is the set of all antichains in



Example: $NN(\mathfrak{S}_4)$



$$\emptyset, [12], [23], [34], [13], [24], [14],$$

$$[12] - [23], [12] - [34], [23] - [34], [12] - [24],$$

$$[13] - [34], [13] - [24], [12] - [23] - [34]$$

Non-crossing and non-nesting partitions

Theorem (Athanasiadis 2004)

For any crystallographic Coxeter group W of rank ℓ, the number of non-crossing and of non-nesting partitions coincide with the W-Catalan number,

$$\left|NC(W)\right| = \left|NN(W)\right| = \mathrm{Cat}(W) := \prod_{i=1}^{\ell} \frac{d_i + h}{d_i},$$

where $d_1 \leq \ldots \leq d_\ell$ denote the degrees of the fundamental invariants of W.

An important open problem

Different groups found explicit bijections between non-crossing and non-nesting partitions for various types, but the general connection is still open:

Open Problem

Find a type-independent bijection between $NC(W)$ and $NN(W)$.

Further refinements

Define a cyclic group action on $NC(W)$ by **Kreweras complementation**,

$$\mathbf{K}(\omega) := c\ \omega^{-1}.$$

Observe that $\mathbf{K}^2(\omega) = c\ \omega\ c^{-1}$ is conjugation by c.

Theorem (CSP on non-crossing partitions)

$$\begin{aligned} \left|\{\omega \in NC(W) : \mathbf{K}^k(\omega) = \omega\}\right| &= \mathrm{Cat}(W;\zeta_d^k) \\ &:= \prod_{i=1}^{\ell} \frac{[d_i + h]_q}{[d_i]_q}\Bigg|_{q=\zeta_d^k}, \end{aligned}$$

where the product $\mathrm{Cat}(W;q)$ *is a q-analogue of the W-Catalan number* $\mathrm{Cat}(W)$ *and where* $d = 2h$ *is the order of the cyclic group.*

Further refinements

- The theorem is an instance of the **cyclic sieving phenomenon** (which you all know from the SLC 62 one year ago in Heilsbronn, Germany).

- It was proved accidentally for the symmetric group by C. Heitsch,
- and it was recently proved in much more generality by C. Krattenthaler in an unpublished manuscript and by him together with T. Müller in two articles on 134 pages,
- it generalizes a theorem by D. Bessis and V. Reiner on the CSP by conjugation.

Example (continued): $NC(\mathfrak{S}_4)$

$$
\begin{aligned}
(12) &\mapsto (1234)(12) = (134) \\
&\mapsto (1234)(143) = (23) \\
&\mapsto (1234)(23) = (124) \\
&\mapsto (1234)(142) = (34) \\
&\mapsto (1234)(34) = (123) \\
&\mapsto (1234)(132) = (14) \\
&\mapsto (1234)(14) = (234) \\
&\mapsto (1234)(243) = (12)
\end{aligned}
$$

$$
\begin{aligned}
(13) &\mapsto (1234)(13) = (14)(23) \\
&\mapsto (1234)(14)(23) = (24) \\
&\mapsto (1234)(24) = (12)(34) \\
&\mapsto (1234)(12)(34) = (13)
\end{aligned}
$$

$$
\begin{aligned}
() &\mapsto (1234)() = (1234) \\
&\mapsto (1234)(4321) = ()
\end{aligned}
$$

$\mathrm{Cat}(\mathfrak{S}_n; q) = 1 + q^2 + q^3 + 2q^4 + q^5 + 2q^6 + q^7 + 2q^8 + q^9 + q^{10} + q^{12}$
evaluated at 8-th roots of unity gives

$$
\begin{cases}
14 = |NC| & \text{if } \zeta = 1 \\
6 = |NC^{\mathbf{K}^4}| & \text{if } \zeta = -1 \\
2 = |NC^{\mathbf{K}^2}| = |NC^{\mathbf{K}^6}| & \text{if } \zeta = \pm\mathbf{i} \\
0 = |NC^{\mathbf{K}}| = |NC^{\mathbf{K}^3}| = |NC^{\mathbf{K}^5}| = |NC^{\mathbf{K}^7}| & \text{otherwise}
\end{cases}
$$

Further refinements

Define a cyclic group action on $NC(W)$ by **Panyushev complementation**

$$\mathbf{P}(A) := \min\{t \in \Phi^+ : t \not\leq a \text{ for some } a \in A\} \in NN(W).$$

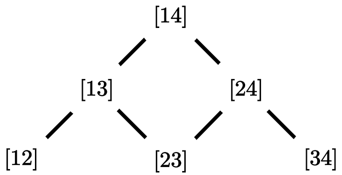
Conjecture (V. Reiner, SLC 62)

$$\left|\{\omega \in NN(W) : \mathbf{P}^k(\omega) = \omega\}\right| = \mathrm{Cat}(W; \zeta_d^k).$$

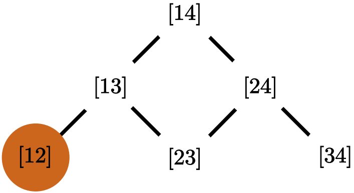
Or equivalently, there exists a bijection $\psi : NN(W) \xrightarrow{\sim} NC(W)$ such that

$$\psi \circ \mathbf{P} = \mathbf{K} \circ \psi.$$

Example (continued): $NN(\mathfrak{S}_4)$

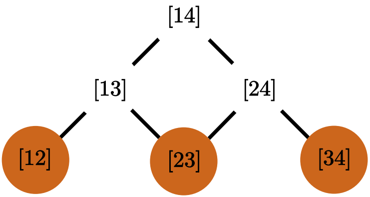


Example (continued): $NN(\mathfrak{S}_4)$



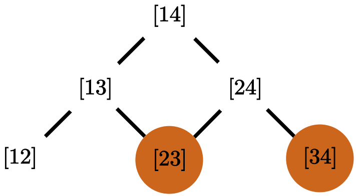
12

Example (continued): $NN(\mathfrak{S}_4)$



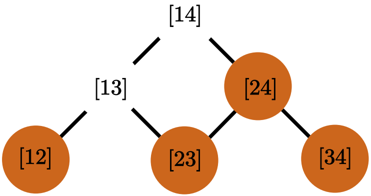
$12 \mapsto 23, 34$

Example (continued): $NN(\mathfrak{S}_4)$



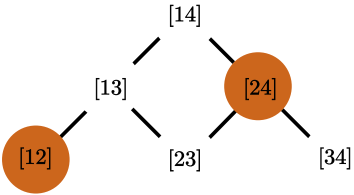
$12 \mapsto 23, 34$

Example (continued): $NN(\mathfrak{S}_4)$



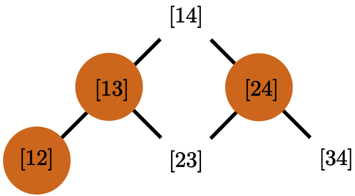
$$12 \mapsto 23, 34 \mapsto 12, 24$$

Example (continued): $NN(\mathfrak{S}_4)$



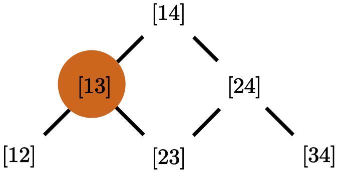
$$12 \mapsto 23, 34 \mapsto 12, 24$$

Example (continued): $NN(\mathfrak{S}_4)$



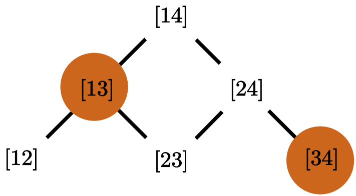
$12 \mapsto 23, 34 \mapsto 12, 24 \mapsto 13$

Example (continued): $NN(\mathfrak{S}_4)$



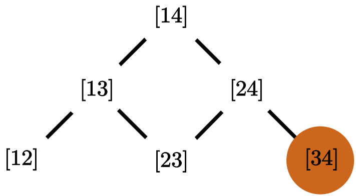
$$12 \mapsto 23, 34 \mapsto 12, 24 \mapsto 13$$

Example (continued): $NN(\mathfrak{S}_4)$



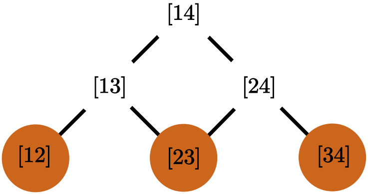
$$12 \mapsto 23, 34 \mapsto 12, 24 \mapsto 13 \mapsto 34$$

Example (continued): $NN(\mathfrak{S}_4)$



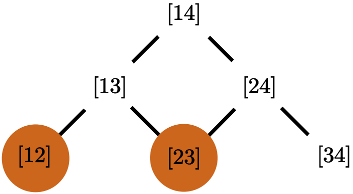
$$12 \mapsto 23, 34 \mapsto 12, 24 \mapsto 13 \mapsto 34$$

Example (continued): $NN(\mathfrak{S}_4)$



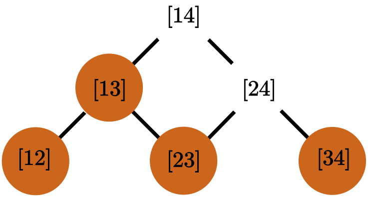
$12 \mapsto 23, 34 \mapsto 12, 24 \mapsto 13 \mapsto 34 \mapsto 12, 23$

Example (continued): $NN(\mathfrak{S}_4)$



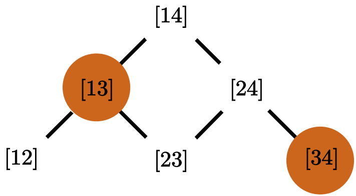
$$12 \mapsto 23, 34 \mapsto 12, 24 \mapsto 13 \mapsto 34 \mapsto 12, 23$$

Example (continued): $NN(\mathfrak{S}_4)$



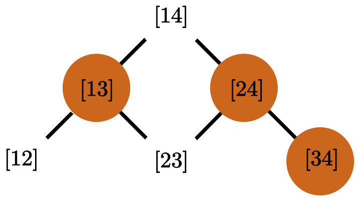
$$12 \mapsto 23, 34 \mapsto 12, 24 \mapsto 13 \mapsto 34 \mapsto 12, 23 \mapsto 13, 34$$

Example (continued): $NN(\mathfrak{S}_4)$



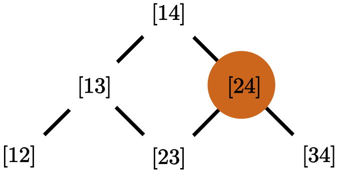
$$12 \mapsto 23, 34 \mapsto 12, 24 \mapsto 13 \mapsto 34 \mapsto 12, 23 \mapsto 13, 34$$

Example (continued): $NN(\mathfrak{S}_4)$



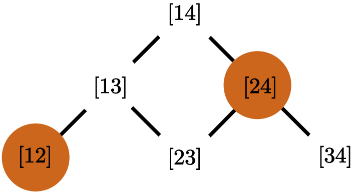
$$12 \mapsto 23, 34 \mapsto 12, 24 \mapsto 13 \mapsto 34 \mapsto 12, 23 \mapsto 13, 34 \mapsto 24$$

Example (continued): $NN(\mathfrak{S}_4)$



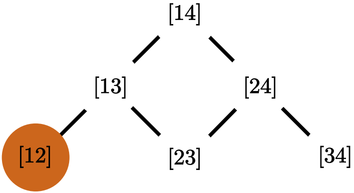
$$12 \mapsto 23, 34 \mapsto 12, 24 \mapsto 13 \mapsto 34 \mapsto 12, 23 \mapsto 13, 34 \mapsto 24$$

Example (continued): $NN(\mathfrak{S}_4)$



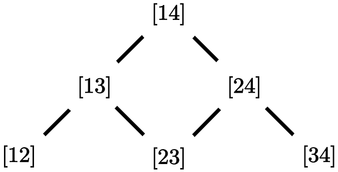
$$12 \mapsto 23, 34 \mapsto 12, 24 \mapsto 13 \mapsto 34 \mapsto 12, 23 \mapsto 13, 34 \mapsto 24 \mapsto 12$$

Example (continued): $NN(\mathfrak{S}_4)$



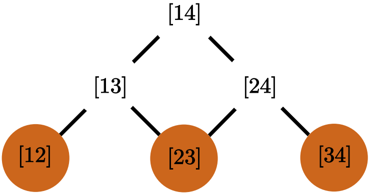
$$12 \mapsto 23, 34 \mapsto 12, 24 \mapsto 13 \mapsto 34 \mapsto 12, 23 \mapsto 13, 34 \mapsto 24 \mapsto 12$$

Example (continued): $NN(\mathfrak{S}_4)$



$$12 \mapsto 23, 34 \mapsto 12, 24 \mapsto 13 \mapsto 34 \mapsto 12, 23 \mapsto 13, 34 \mapsto 24 \mapsto 12$$

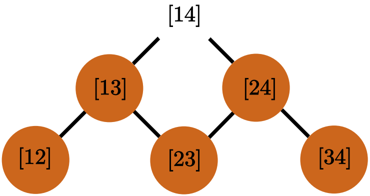
Example (continued): $NN(\mathfrak{S}_4)$



$12 \mapsto 23, 34 \mapsto 12, 24 \mapsto 13 \mapsto 34 \mapsto 12, 23 \mapsto 13, 34 \mapsto 24 \mapsto 12$

$12, 23, 34$

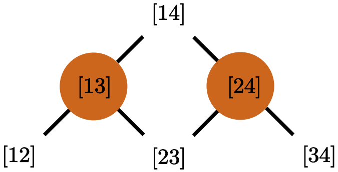
Example (continued): $NN(\mathfrak{S}_4)$



$12 \mapsto 23, 34 \mapsto 12, 24 \mapsto 13 \mapsto 34 \mapsto 12, 23 \mapsto 13, 34 \mapsto 24 \mapsto 12$

$12, 23, 34 \mapsto 13, 24$

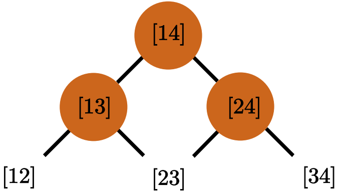
Example (continued): $NN(\mathfrak{S}_4)$



$12 \mapsto 23, 34 \mapsto 12, 24 \mapsto 13 \mapsto 34 \mapsto 12, 23 \mapsto 13, 34 \mapsto 24 \mapsto 12$

$12, 23, 34 \mapsto 13, 24$

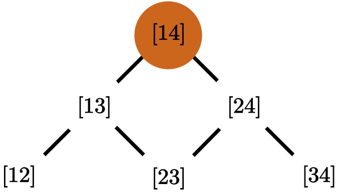
Example (continued): $NN(\mathfrak{S}_4)$



$$12 \mapsto 23, 34 \mapsto 12, 24 \mapsto 13 \mapsto 34 \mapsto 12, 23 \mapsto 13, 34 \mapsto 24 \mapsto 12$$

$$12, 23, 34 \mapsto 13, 24 \mapsto 14$$

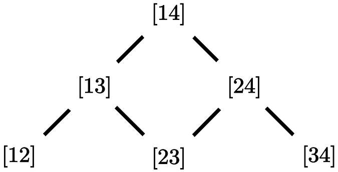
Example (continued): $NN(\mathfrak{S}_4)$



$$12 \mapsto 23, 34 \mapsto 12, 24 \mapsto 13 \mapsto 34 \mapsto 12, 23 \mapsto 13, 34 \mapsto 24 \mapsto 12$$

$$12, 23, 34 \mapsto 13, 24 \mapsto 14$$

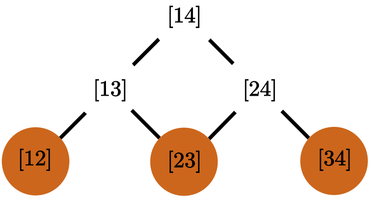
Example (continued): $NN(\mathfrak{S}_4)$



$$12 \mapsto 23, 34 \mapsto 12, 24 \mapsto 13 \mapsto 34 \mapsto 12, 23 \mapsto 13, 34 \mapsto 24 \mapsto 12$$

$$12, 23, 34 \mapsto 13, 24 \mapsto 14 \mapsto \emptyset$$

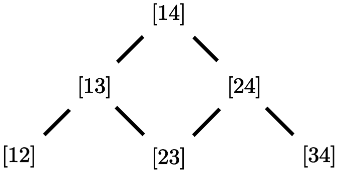
Example (continued): $NN(\mathfrak{S}_4)$



$12 \mapsto 23, 34 \mapsto 12, 24 \mapsto 13 \mapsto 34 \mapsto 12, 23 \mapsto 13, 34 \mapsto 24 \mapsto 12$

$12, 23, 34 \mapsto 13, 24 \mapsto 14 \mapsto \emptyset \mapsto 12, 23, 34$

Example (continued): $NN(\mathfrak{S}_4)$



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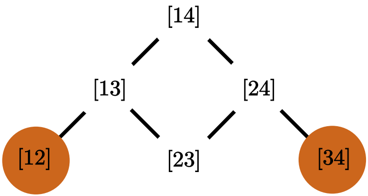
$$12, 23, 34 \mapsto 13, 24 \mapsto 14 \mapsto \emptyset \mapsto 12, 23, 34$$

Example (continued): $NN(\mathfrak{S}_4)$

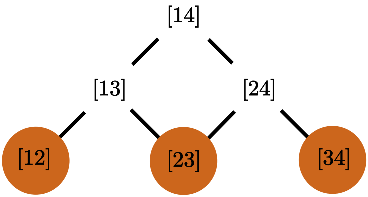


$12 \mapsto 23, 34 \mapsto 12, 24 \mapsto 13 \mapsto 34 \mapsto 12, 23 \mapsto 13, 34 \mapsto 24 \mapsto 12$

$12, 23, 34 \mapsto 13, 24 \mapsto 14 \mapsto \emptyset \mapsto 12, 23, 34$

$12, 34$

Example (continued): $NN(\mathfrak{S}_4)$

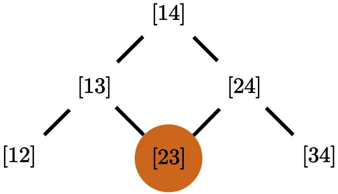


$12 \mapsto 23, 34 \mapsto 12, 24 \mapsto 13 \mapsto 34 \mapsto 12, 23 \mapsto 13, 34 \mapsto 24 \mapsto 12$

$12, 23, 34 \mapsto 13, 24 \mapsto 14 \mapsto \emptyset \mapsto 12, 23, 34$

$12, 34 \mapsto 23$

Example (continued): $NN(\mathfrak{S}_4)$

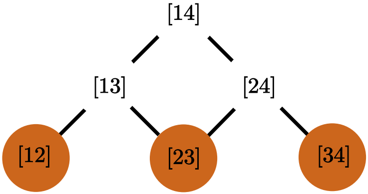


$12 \mapsto 23, 34 \mapsto 12, 24 \mapsto 13 \mapsto 34 \mapsto 12, 23 \mapsto 13, 34 \mapsto 24 \mapsto 12$

$12, 23, 34 \mapsto 13, 24 \mapsto 14 \mapsto \emptyset \mapsto 12, 23, 34$

$12, 34 \mapsto 23$

Example (continued): $NN(\mathfrak{S}_4)$

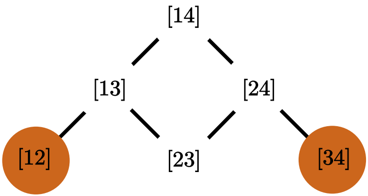


$$12 \mapsto 23, 34 \mapsto 12, 24 \mapsto 13 \mapsto 34 \mapsto 12, 23 \mapsto 13, 34 \mapsto 24 \mapsto 12$$

$$12, 23, 34 \mapsto 13, 24 \mapsto 14 \mapsto \emptyset \mapsto 12, 23, 34$$

$$12, 34 \mapsto 23 \mapsto 12, 34$$

Example (continued): $NN(\mathfrak{S}_4)$

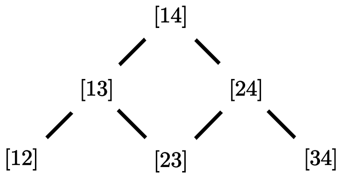


$$12 \mapsto 23, 34 \mapsto 12, 24 \mapsto 13 \mapsto 34 \mapsto 12, 23 \mapsto 13, 34 \mapsto 24 \mapsto 12$$

$$12, 23, 34 \mapsto 13, 24 \mapsto 14 \mapsto \emptyset \mapsto 12, 23, 34$$

$$12, 34 \mapsto 23 \mapsto 12, 34$$

Example (continued): $NN(\mathfrak{S}_4)$



$$12 \mapsto 23, 34 \mapsto 12, 24 \mapsto 13 \mapsto 34 \mapsto 12, 23 \mapsto 13, 34 \mapsto 24 \mapsto 12$$

$$12, 23, 34 \mapsto 13, 24 \mapsto 14 \mapsto \emptyset \mapsto 12, 23, 34$$

$$12, 34 \mapsto 23 \mapsto 12, 34$$

The type A: Non-crossing handshake configurations

- $\mathcal{T}_n$ set of non-crossing handshake configurations of $2n$,
- $\mathcal{C}_{2n}$-action on $\mathcal{T}_n$ by cyclic permutation of $\{1, \ldots, 2n\}$.

Theorem

- $\big(\mathcal{T}_n, \mathrm{Cat}(\mathfrak{S}_n; q), \mathcal{C}_{2n}\big)$ *exhibits the CSP.*
- *We can construct a bijection*

$$\psi_1 : \mathcal{T}_n \xrightarrow{\sim} NC(\mathfrak{S}_n),$$

such that $\psi_1 \circ c = \mathbf{K} \circ \psi_1$*, and a bijection*

$$\psi_2 : NN(\mathfrak{S}_n) \xrightarrow{\sim} \mathcal{T}_n,$$

such that $\psi_2 \circ \mathbf{P} = c \circ \psi_2$*.*

- *The construction can be easily generalized to type* B*.*

Towards a uniform bijection

- The bijection in type A can be generalized to type B, but so far **not** to type D,
- the exceptional types can be checked by computer.

What does this have to do with a uniform bijection?

Towards a uniform bijection

Theorem (Conjectured by D. Armstrong in all types)

Let L, R be a bipartition of the simple transpositions such that L and R pairwise commute, and let c be the associated bipartite Coxeter element. $\psi = \psi_1 \circ \psi_2 : NN(\mathfrak{S}_n) \tilde{\longrightarrow} NC(\mathfrak{S}_n, c = c_L c_R)$ is the unique bijection with the following inductive property:

$$\begin{aligned} \psi : \emptyset &\mapsto c_L, \\ \psi \circ \mathbf{P} &= \mathbf{K} \circ \psi, \\ \psi(I) &= \prod_{s \in L \setminus \operatorname{supp} I} s \qquad \psi\big|_{\operatorname{supp} I}(I). \end{aligned}$$

Remark

- The theorem gives an inductively defined uniform definition of a bijection between non-nesting and non-crossing partitions in types A and B.