

Parking and Trees

António Guedes de Oliveira Michel Las Vergnas

CMUP, Universidade do Porto CNRS

Séminaire Lotharingien de Combinatoire 65

Parking functions - definition 1

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- s.t., if $\{a_1, a_2, \dots, a_n\}_{\leq} := \{f(1), f(2), \dots, f(n)\}$,

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$$\begin{array}{cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ f : & 1 & 8 & 5 & 2 & 7 & 4 & 4 & 8 & 1 \end{array}$$

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$$P(f) : \begin{matrix} f: & \cdots & k & \cdots &) \\ P(f): & \underbrace{\cdots}_{k, k+1, \dots, n+1} & \underbrace{n+1}_{\cdots} & \cdots &) \end{matrix} \implies |P(f)^{-1}([k-1])| < k-1$$

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The sets

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Parking functions

Parking functions and trees

$$f : \quad \begin{matrix} 1 \\ 8 \\ 5 \\ 2 \\ 7 \\ 4 \\ 4 \\ 8 \\ 1 \end{matrix}$$

The sets

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Parking functions

Parking functions and trees

$$\begin{array}{ll} f : & \begin{matrix} 1 \\ 8 \\ 5 \\ 2 \\ 7 \\ 4 \\ 4 \\ 8 \\ 1 \end{matrix} \\ \sigma : & \begin{matrix} 1 \end{matrix} \end{array}$$

$$\sigma^{-1} : \quad \begin{matrix} \boxed{1} \\ \square \end{matrix} \quad \xrightarrow{\hspace{2cm}} \quad \xrightarrow{\hspace{2cm}}$$

The sets

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Parking functions

Parking functions and trees

$$\begin{array}{cccccccccc} f : & \textcolor{red}{1} & \textcolor{red}{2} & \textcolor{red}{3} & \textcolor{blue}{4} & \textcolor{blue}{5} & \textcolor{blue}{6} & \textcolor{red}{7} & \textcolor{blue}{8} & \textcolor{red}{9} \\ \sigma : & 1 & 8 & & & & 4 & & 8 & 1 \end{array}$$

$$\sigma^{-1} : \quad \boxed{1} \quad \square \quad \square \quad \square \quad \square \quad \square \quad \square \quad \boxed{2} \quad \square$$

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$$f : \begin{array}{cccccccccc} 1 & 8 & 5 & 2 & 7 & 4 & 7 & 8 & 1 \end{array}$$

$$\sigma : \begin{array}{cccccccccc} 1 & 8 & 5 & 2 & 7 & 4 & 6 & 9 & 3 \end{array}$$

$$\sigma^{-1} : \begin{array}{cccccccccc} 1 & 4 & 9 & 6 & 3 & 7 & 5 & 2 & 8 \end{array}$$

The sets

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$$\begin{array}{cccccccccc} f : & \textcolor{red}{1} & \textcolor{red}{8} & \textcolor{red}{5} & \textcolor{red}{2} & \textcolor{red}{7} & \textcolor{red}{4} & \textcolor{red}{7} & \textcolor{red}{8} & \textcolor{red}{1} \\ \sigma : & 1 & 8 & 5 & 2 & 7 & 4 & 6 & 9 & 3 \end{array}$$

$$\sigma^{-1} : \quad 1 \quad 4 \quad 9 \quad 6 \quad 3 \quad 7 \quad 5 \quad 2 \quad 8 \quad \textcolor{red}{10}$$

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$$\sigma^{-1} : \quad \overbrace{1 \quad 4} \quad 9 \quad 6 \quad 3 \quad 7 \quad 5 \quad 2 \quad 8 \quad \textcolor{red}{10}$$

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$$\sigma^{-1} : \quad \overbrace{1 \quad 4 \quad 9}^{\text{3 nodes}} \quad 6 \quad 3 \quad 7 \quad 5 \quad 2 \quad 8 \quad \textcolor{red}{10}$$

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graph LR; 1 --- 4 --- 9 --- 6 --- 3 --- 7 --- 5 --- 2 --- 8 --- 10
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The sets

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```
graph LR; 1[1] --- 4[4]; 4 --- 9[9]; 9 --- 6[6]; 6 --- 3[3]; 3 --- 7[7]; 7 --- 5[5]; 5 --- 2[2]; 2 --- 8[8]; 8 --- 10[10]
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A diagram illustrating the relationship between a parking function f and its inverse image σ^{-1} under a permutation σ . The numbers 1 through 10 are arranged in a row. Brackets above the first three numbers group them together. A red arc connects the 9 in f to the 6 in σ^{-1} , indicating they are paired under the inverse map.

Parking functions and trees

$$\begin{array}{ll} f : & \begin{array}{cccccccccc} \textcolor{red}{1} & \textcolor{red}{2} & \textcolor{red}{3} & \textcolor{red}{4} & \textcolor{red}{5} & \textcolor{red}{6} & \textcolor{red}{7} & \textcolor{red}{8} & \textcolor{red}{9} \\ 1 & 8 & 5 & 2 & 7 & 4 & 4 & 8 & 1 \end{array} \\ \sigma : & \begin{array}{cccccccccc} 1 & 8 & 5 & 2 & 7 & 4 & 6 & 9 & 3 \end{array} \end{array}$$

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```
graph LR; S1[1, 4] --- S2[9]; S2 --- S3[6, 3, 7]; S3 --- S4[5, 2, 8];
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The sets

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graph LR; 1[1] --- 4[4]; 4 --- 9[9]; 9 --- 6[6]; 6 --- 3[3]; 3 --- 7[7]; 7 --- 5[5]; 5 --- 2[2]; 2 --- 8[8]; 8 --- 10[10]
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```
graph LR; 1[1] --- 4[4]; 4 --- 9[9]; 9 --- 6[6]; 6 --- 3[3]; 3 --- 7[7]; 7 --- 5[5]; 5 --- 2[2]; 2 --- 8[8]; 8 --- 10[10]
```

The sets

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Parking functions

Parking functions and trees

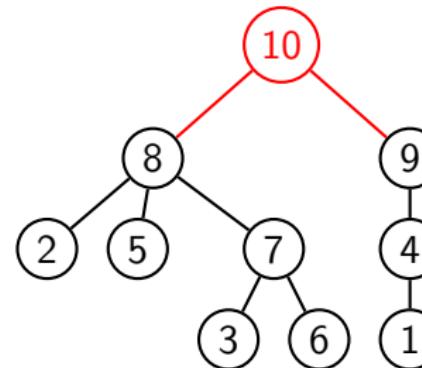
$$\begin{array}{ll} f : & \begin{array}{cccccccccc} 1 & 8 & 5 & 2 & 7 & 4 & 7 & 8 & 1 \\ 1 & 8 & 5 & 2 & 7 & 4 & 6 & 9 & 3 \end{array} \end{array}$$

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Parking functions and trees

$$\begin{array}{ll} f : & \begin{array}{cccccccccc} \textcolor{red}{1} & \textcolor{red}{2} & \textcolor{red}{3} & \textcolor{red}{4} & \textcolor{red}{5} & \textcolor{red}{6} & \textcolor{red}{7} & \textcolor{red}{8} & \textcolor{red}{9} \\ 1 & 8 & 5 & 2 & 7 & 4 & 4 & 8 & 1 \end{array} \\ \sigma : & \begin{array}{cccccccccc} 1 & 8 & 5 & 2 & 7 & 4 & 6 & 9 & 3 \end{array} \end{array}$$

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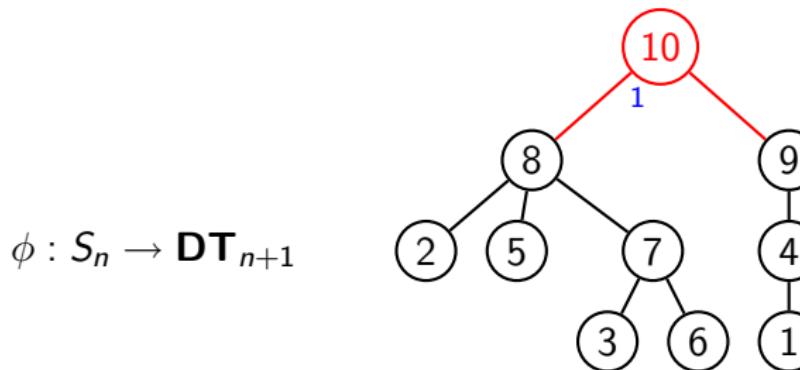


$$\phi : S_n \rightarrow \mathbf{DT}_{n+1}$$

Parking functions and trees

$$\begin{array}{ll} f : & \begin{array}{cccccccccc} \textcolor{red}{1} & \textcolor{red}{2} & \textcolor{red}{3} & \textcolor{red}{4} & \textcolor{red}{5} & \textcolor{red}{6} & \textcolor{red}{7} & \textcolor{red}{8} & \textcolor{red}{9} \\ 1 & 8 & 5 & 2 & 7 & 4 & 4 & 8 & 1 \end{array} \\ \sigma : & \begin{array}{cccccccccc} 1 & 8 & 5 & 2 & 7 & 4 & 6 & 9 & 3 \end{array} \end{array}$$

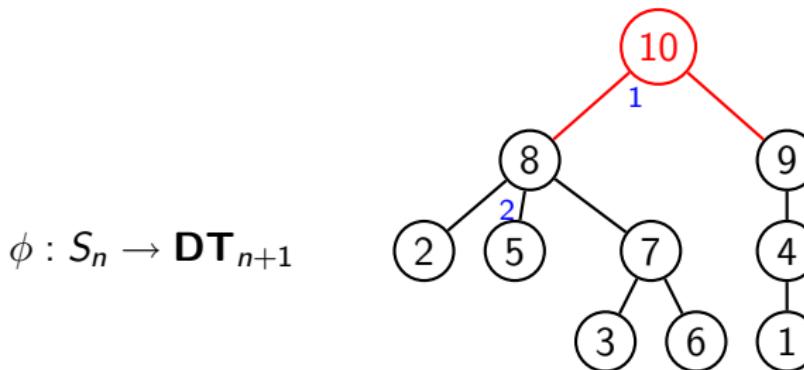
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Parking functions and trees

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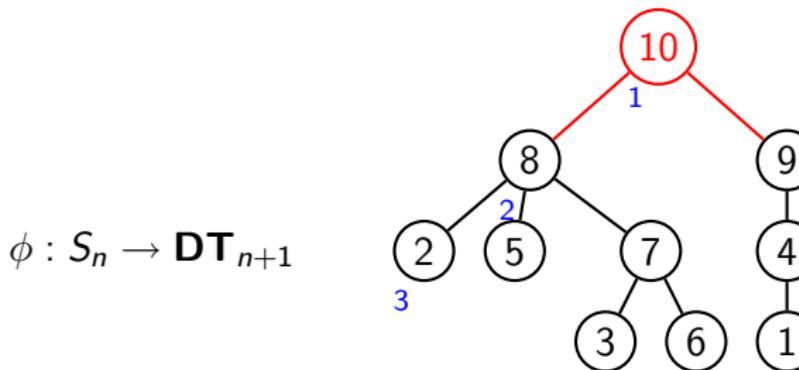
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Parking functions and trees

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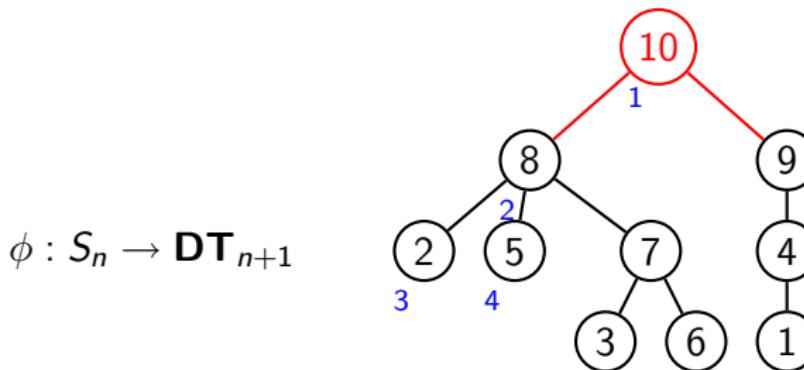
The sets
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Parking functions

Parking functions and trees

$$\begin{array}{ll} f : & \begin{array}{cccccccccc} \textcolor{red}{1} & \textcolor{red}{2} & \textcolor{red}{3} & \textcolor{red}{4} & \textcolor{red}{5} & \textcolor{red}{6} & \textcolor{red}{7} & \textcolor{red}{8} & \textcolor{red}{9} \\ 1 & 8 & 5 & 2 & 7 & 4 & 4 & 8 & 1 \end{array} \\ \sigma : & \begin{array}{cccccccccc} 1 & 8 & 5 & 2 & 7 & 4 & 6 & 9 & 3 \end{array} \end{array}$$

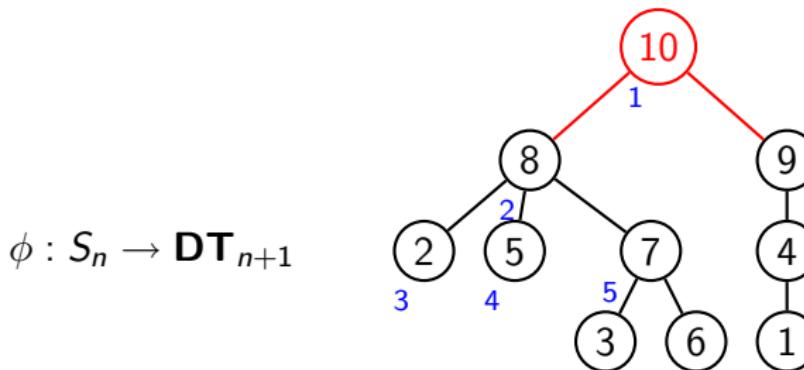
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Parking functions and trees

$$\begin{array}{ll} f : & \begin{array}{cccccccccc} \textcolor{red}{1} & \textcolor{red}{2} & \textcolor{red}{3} & \textcolor{red}{4} & \textcolor{red}{5} & \textcolor{red}{6} & \textcolor{red}{7} & \textcolor{red}{8} & \textcolor{red}{9} \\ 1 & 8 & 5 & 2 & 7 & 4 & 4 & 8 & 1 \end{array} \\ \sigma : & \begin{array}{cccccccccc} 1 & 8 & 5 & 2 & 7 & 4 & 6 & 9 & 3 \end{array} \end{array}$$

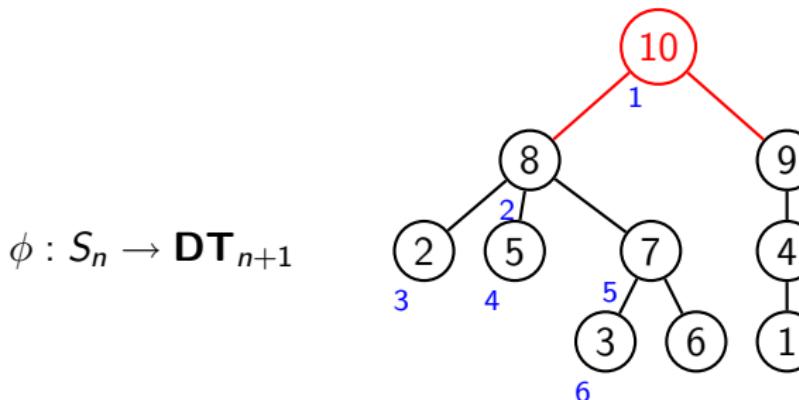
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Parking functions and trees

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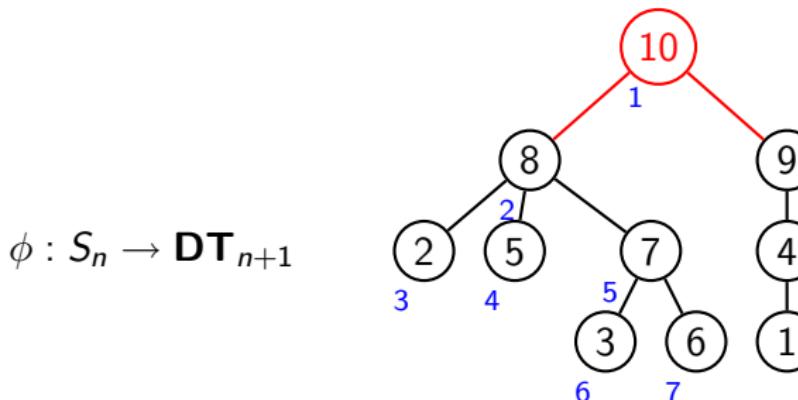
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Parking functions and trees

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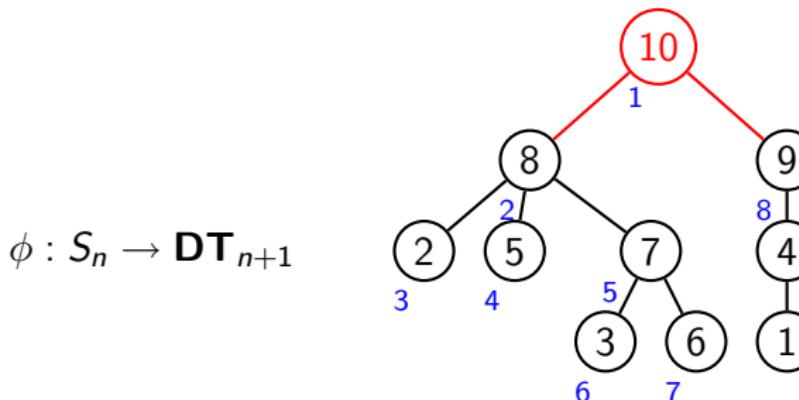
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Parking functions and trees

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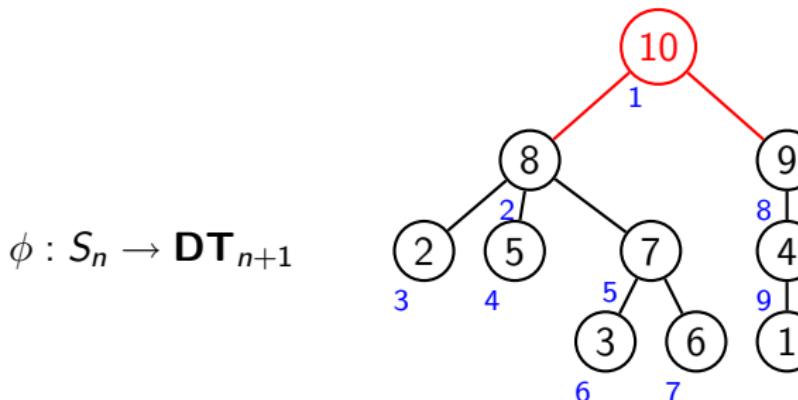
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Parking functions and trees

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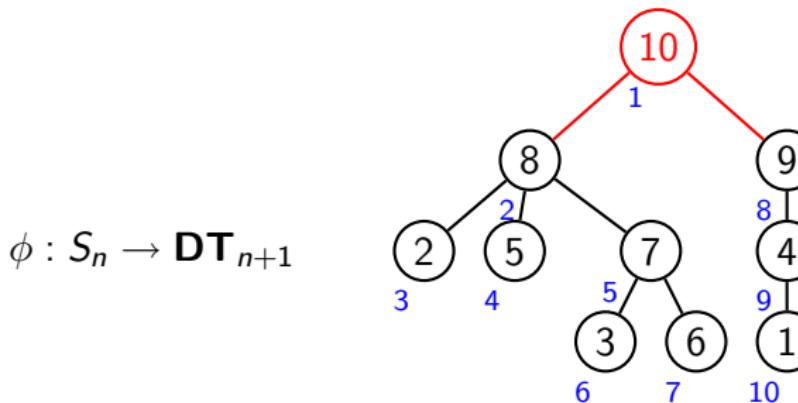
The sets
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Parking functions

Parking functions and trees

$$\begin{array}{ll} f : & \begin{array}{cccccccccc} 1 & 8 & 5 & 2 & 7 & 4 & 7 & 8 & 1 \\ \textcolor{red}{1} & \textcolor{red}{2} & \textcolor{red}{3} & \textcolor{red}{4} & \textcolor{red}{5} & \textcolor{red}{6} & \textcolor{red}{7} & \textcolor{red}{8} & \textcolor{red}{9} \end{array} \\ \sigma : & \begin{array}{cccccccccc} 1 & 8 & 5 & 2 & 7 & 4 & 6 & 9 & 3 \\ \textcolor{blue}{1} & \textcolor{blue}{8} & \textcolor{blue}{5} & \textcolor{blue}{2} & \textcolor{blue}{7} & \textcolor{blue}{4} & \textcolor{blue}{6} & \textcolor{blue}{9} & \textcolor{blue}{3} \end{array} \end{array}$$

$$\sigma^{-1} : \quad \begin{array}{cccccccccc} 1 & 4 & 9 & 6 & 3 & 7 & 5 & 2 & 8 & 10 \\ \textcolor{black}{1} & \textcolor{black}{4} & \textcolor{black}{9} & \textcolor{black}{6} & \textcolor{black}{3} & \textcolor{black}{7} & \textcolor{black}{5} & \textcolor{black}{2} & \textcolor{black}{8} & \textcolor{red}{10} \end{array}$$



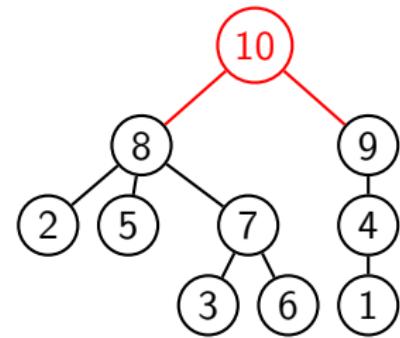
The sets

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Labeled trees

Labeled trees and parking functions

1 8 5 2 7 4 6 9 3 \mapsto



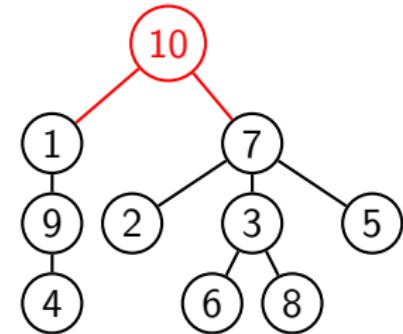
The sets

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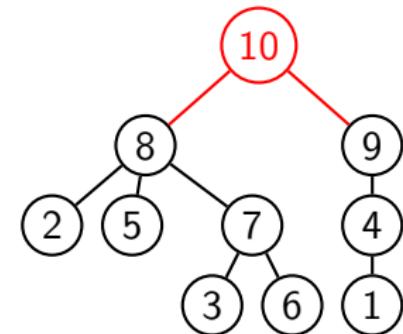
Labeled trees

Labeled trees and parking functions

$f:$ 1 8 5 2 7 4 4 8 1 \mapsto



$\sigma = P(f):$ 1 8 5 2 7 4 6 9 3 \mapsto



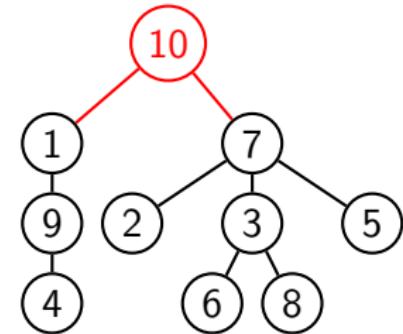
The sets

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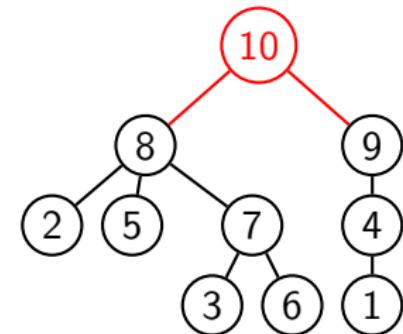
Labeled trees

Labeled trees and parking functions

$f:$ 1 8 5 2 7 4 4 8 1 \mapsto



$\sigma = P(f):$ 1 8 5 2 7 4 6 9 3 \mapsto



$\text{pr}(f):$ 0 0 0 0 0 0 2 1 2 $\rightarrow \Sigma^5$

The sets

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Labeled trees

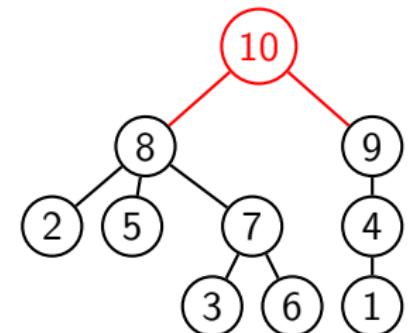
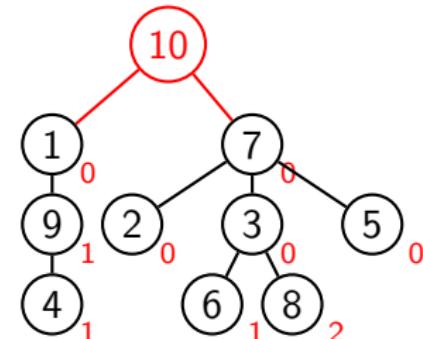
Labeled trees and parking functions

$f:$ 1 8 5 2 7 4 4 8 1 \mapsto

Inversions: 91 41 83 87 63

$\sigma = P(f):$ 1 8 5 2 7 4 6 9 3 \mapsto

$\text{pr}(f):$ 0 0 0 0 0 0 2 1 2 $\rightarrow \Sigma^5$



Theorem (AGO, Michel Las Vergnas)

There exists a bijection

$$\psi : \mathbf{PF}_n \rightarrow \mathbf{T}_{n+1} \quad \text{s.t.}$$

- $\#\text{Inv}(\psi(f)) = \#\text{Probes}(f);$

a

b

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^aKreweras'80

b

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- $\psi|_{S_n} = \phi : S_n \rightarrow \mathbf{DT}_{n+1}$;

^aKreweras'80

b

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^bHeesung Shin, SLC 61, Curia

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- $\mathbf{dp}_{\psi(f)} = \mathbf{dp}_f.$

^aKreweras'80

^bHeesung Shin, SLC 61, Curia

The sets

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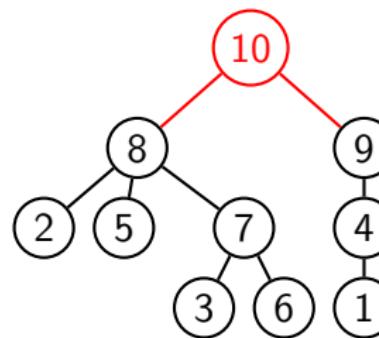
Labeled trees

Probes and Inversions

$$\sigma : \begin{array}{cccccccccc} 1 & 8 & 5 & 2 & 7 & 4 & 6 & 9 & 3 \end{array}$$

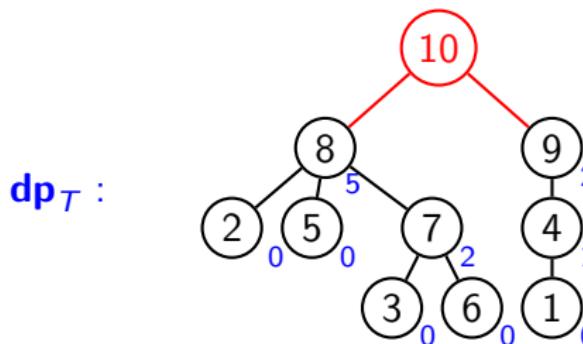
Probes and Inversions

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Probes and Inversions

$\sigma : \quad 1 \quad 8 \quad 5 \quad 2 \quad 7 \quad 4 \quad 6 \quad 9 \quad 3$



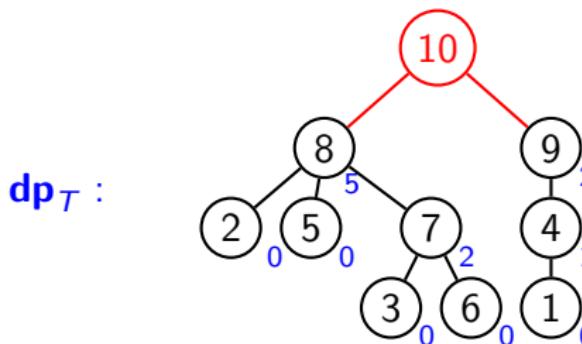
The sets

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Labeled trees

Probes and Inversions

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \sigma : & 1 & 8 & 5 & 2 & 7 & 4 & 6 & 9 & 3 \end{array}$$

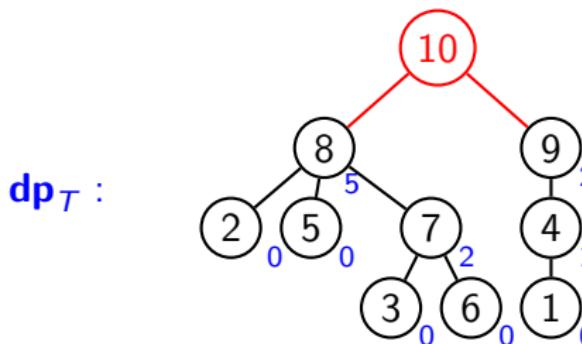


Probes and Inversions

1 2 3 4 5 6 7 8 9

$\sigma :$ 1 8 5 2 7 4 6 9 3

$\mathbf{dp}_\sigma :$ 0 0 0 1 0 0 2 5 2



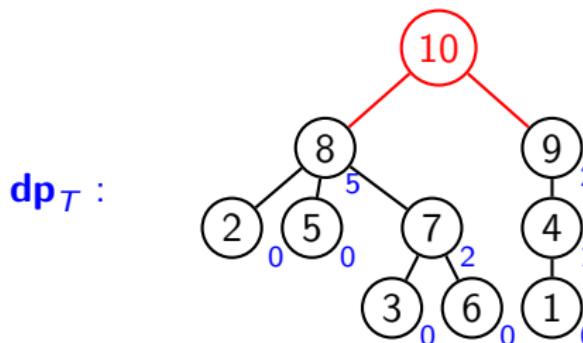
Probes and Inversions

$$\mathbf{dp}_\sigma(i) := \sigma(i) - \min \{k \in \sigma([i]): \{k, k+1, \dots, \sigma(i)\} \subset \sigma([i])\}$$

1 2 3 4 5 6 7 8 9

$\sigma :$ 1 8 5 2 7 4 6 9 3

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Probes and Inversions

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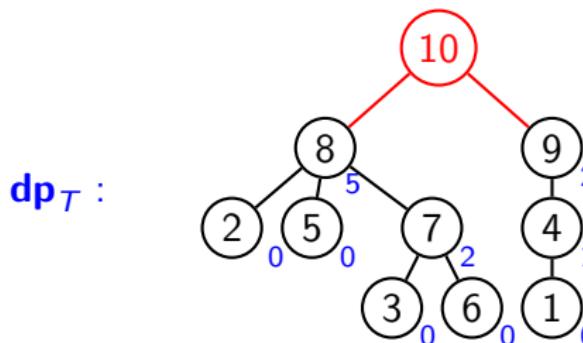
1 2 3 4 5 6 7 8 9

$\sigma :$ 1 8 5 2 7 4 6 9 3

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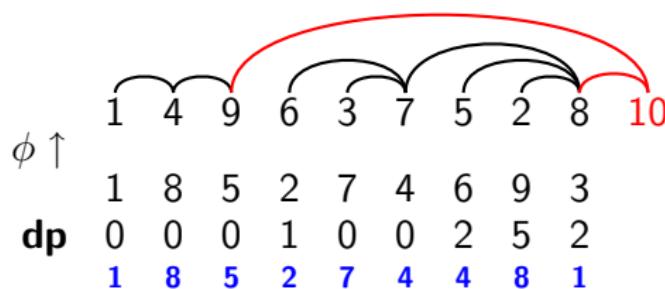
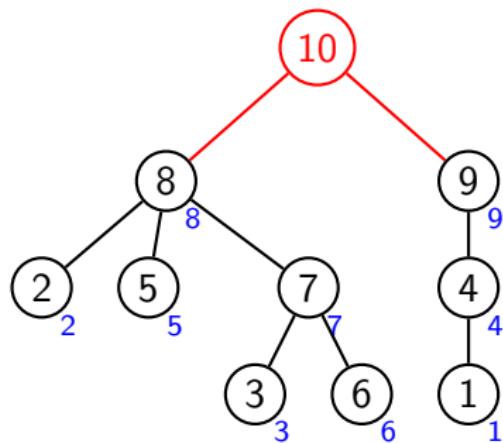
1 8 5 1 7 4 4 4 1



The sets

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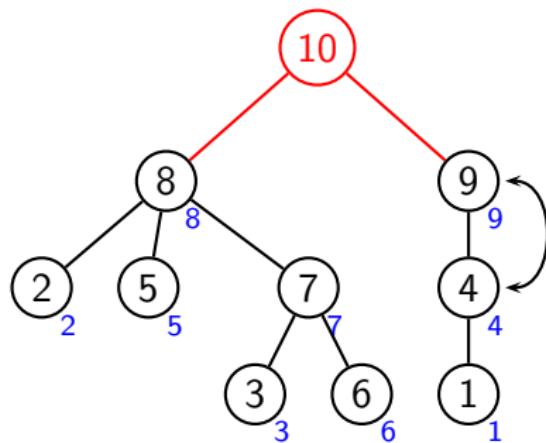
Trees vs. Parking Functions



The sets

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Trees vs. Parking Functions

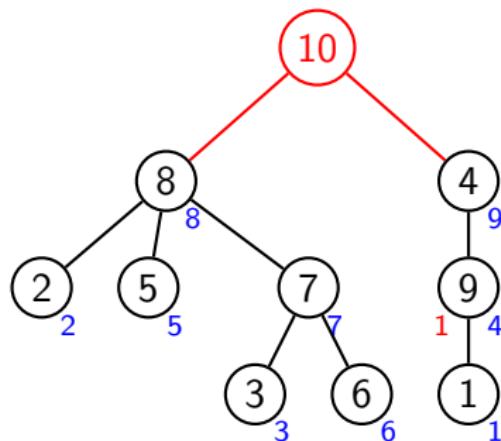


$\phi \uparrow$	1	4	9	6	3	7	5	2	8	10
	1	4	9	6	3	7	5	2	8	
	1	8	5	2	7	4	6	9	3	
dp	0	0	0	1	0	0	2	5	2	
	1	8	5	2	7	4	4	8	1	

The sets

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Trees vs. Parking Functions

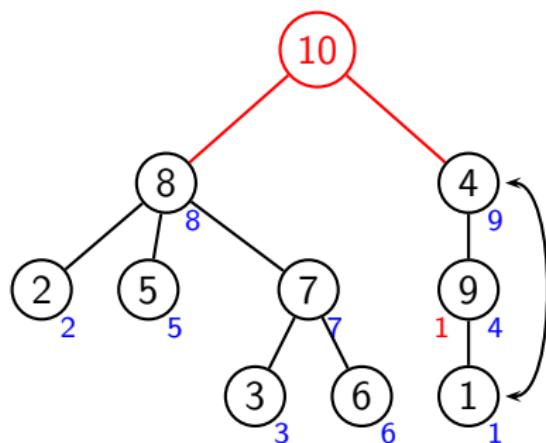


	1	9	4	6	3	7	5	2	8	10
$\phi \uparrow$	1	4	9	6	3	7	5	2	8	10
	1	8	5	2	7	4	6	9	2	
dp	0	0	0	1	0	0	2	5	1	
	1	8	5	2	7	4	4	8	1	

The sets

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Trees vs. Parking Functions

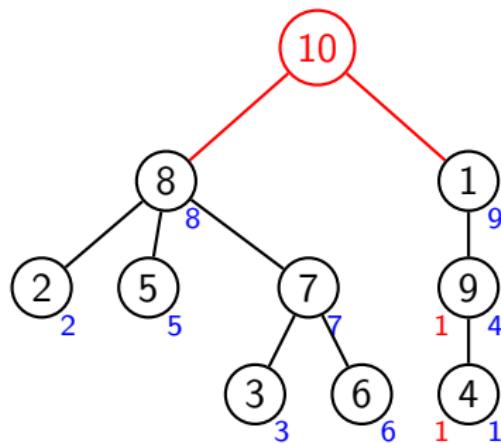


	1	9	4	6	3	7	5	2	8	10
$\phi \uparrow$	1	4	9	6	3	7	5	2	8	
	1	8	5	2	7	4	6	9	2	
dp	0	0	0	1	0	0	2	5	1	
	1	8	5	2	7	4	4	8	1	

The sets

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Trees vs. Parking Functions

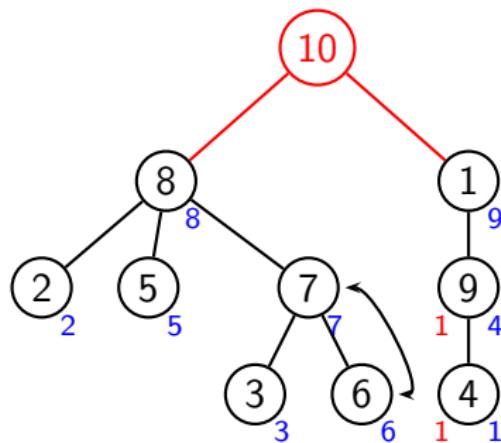


$\phi \uparrow$	4	9	1	6	3	7	5	2	8	10	
	1	4	9	6	3	7	5	2	8	1	
dp	0	0	0	1	0	0	2	5	0		
	1	8	5	2	7	4	4	8	1		

The sets

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Trees vs. Parking Functions

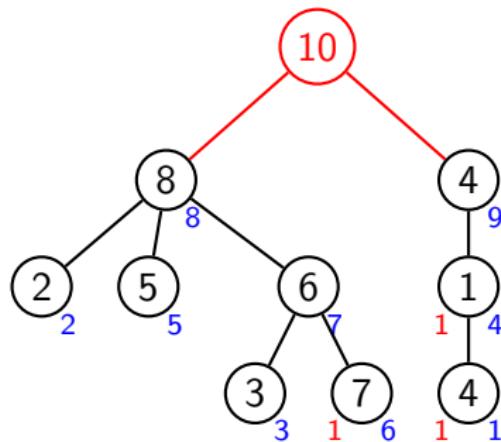


	4	9	1	6	3	7	5	2	8	10
$\phi \uparrow$	1	4	9	6	3	7	5	2	8	
	1	8	5	2	7	4	6	9	1	
dp	0	0	0	1	0	0	2	5	0	
	1	8	5	2	7	4	4	8	1	

The sets

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Trees vs. Parking Functions

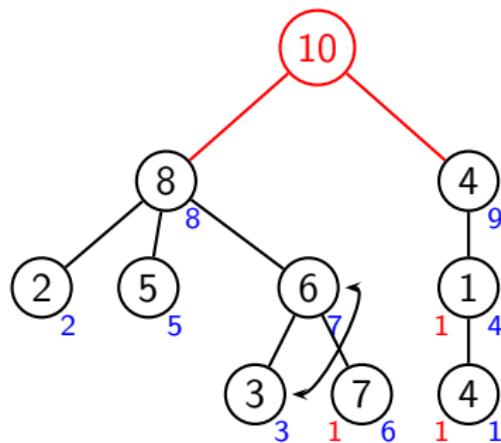


$\phi \uparrow$	4	9	1	7	3	6	5	2	8	10
	1	4	9	6	3	7	5	2	8	
dp	0	0	0	1	0	0	1	5	0	
	1	8	5	2	7	4	4	8	1	

The sets

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Trees vs. Parking Functions

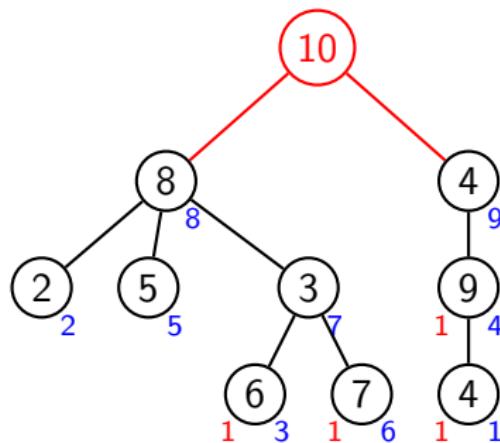


	4	9	1	7	3	6	5	2	8	10
$\phi \uparrow$	1	4	9	6	3	7	5	2	8	
	1	8	5	2	7	4	5	9	1	
dp	0	0	0	1	0	0	1	5	0	
	1	8	5	2	7	4	4	8	1	

The sets

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Trees vs. Parking Functions

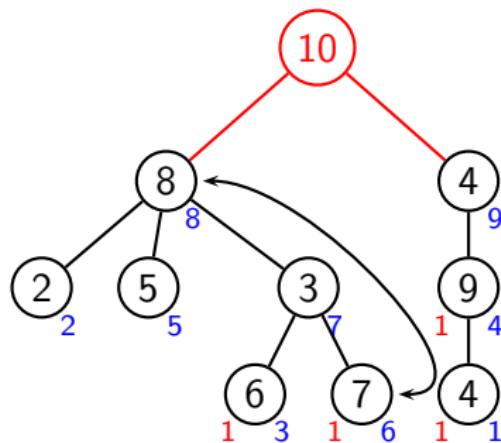


$\phi \uparrow$	4	9	1	7	6	3	5	2	8	10
	1	4	9	6	3	7	5	2	8	1
dp	0	0	0	1	0	0	0	5	0	
	1	8	5	2	7	4	4	8	1	

The sets

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Trees vs. Parking Functions

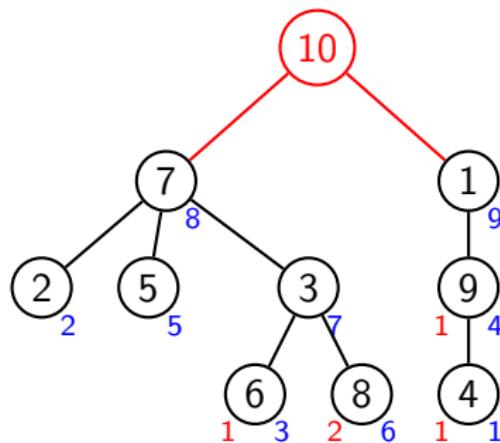


$\phi \uparrow$	4	9	1	7	6	3	5	2	8	10
	1	4	9	6	3	7	5	2	8	
	1	8	5	2	7	4	4	9	1	
dp	0	0	0	1	0	0	0	5	0	
	1	8	5	2	7	4	4	8	1	

The sets

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Trees vs. Parking Functions

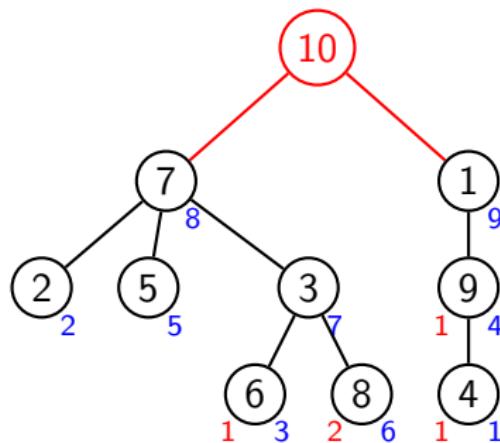


$\phi \uparrow$	4	9	1	8	6	3	5	2	7	10
	1	4	9	6	3	7	5	2	8	1
dp	0	0	0	1	0	0	0	4	0	
	1	8	5	2	7	4	4	8	1	

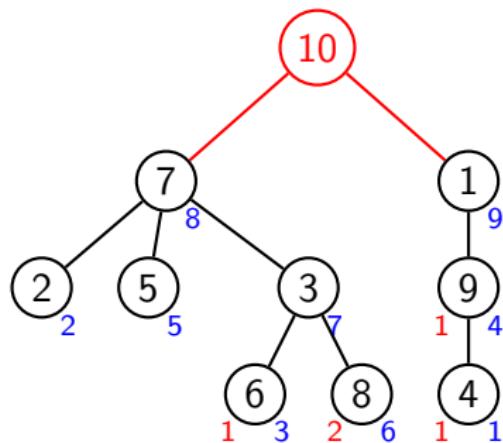
The sets

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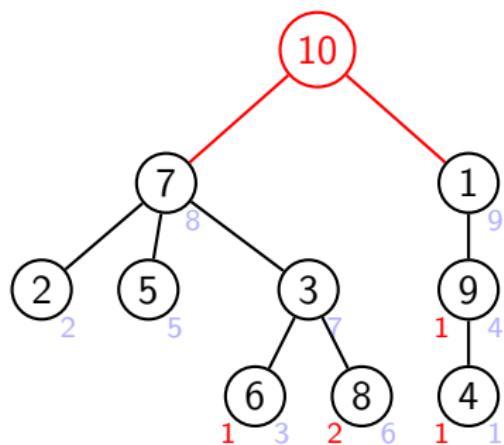
Trees vs. Parking Functions



$\phi \uparrow$	4	9	1	8	6	3	5	2	7	10
	1	4	9	6	3	7	5	2	8	
dp	0	0	0	1	0	0	0	4	0	
	1	8	5	2	7	4	4	8	1	

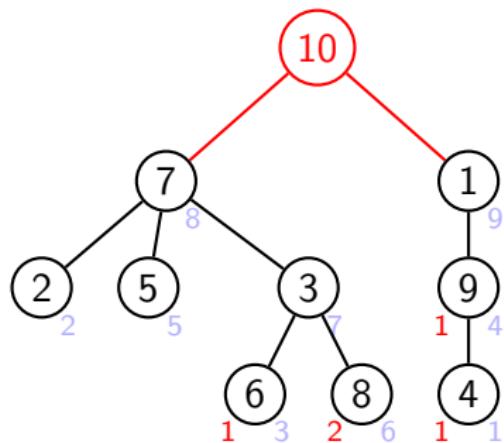


4 9 1 8 6 3 5 2 7
1 4 9 6 3 7 5 2 8



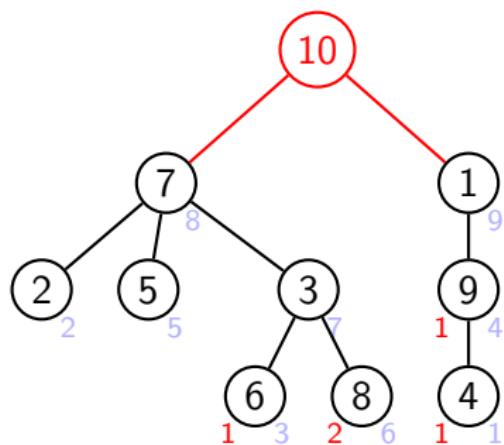
$$\pi^{-1} \downarrow \quad \begin{matrix} 4 & 9 & 1 & 8 & 6 & 3 & 5 & 2 & 7 \\ 1 & 4 & 9 & 6 & 3 & 7 & 5 & 2 & 8 \end{matrix}$$

Trees vs. Parking Functions



91 41 83 87 63

$$\pi^{-1} \downarrow \quad \begin{matrix} 4 & 9 & 1 & 8 & 6 & 3 & 5 & 2 & 7 \\ 1 & 4 & 9 & 6 & 3 & 7 & 5 & 2 & 8 \end{matrix}$$



91

41

83

87

63

$$\pi^{-1} \downarrow \begin{array}{cccccccccc} 4 & 9 & 1 & 8 & 6 & 3 & 5 & 2 & 7 \\ 1 & 4 & 9 & 6 & 3 & 7 & 5 & 2 & 8 \end{array}$$

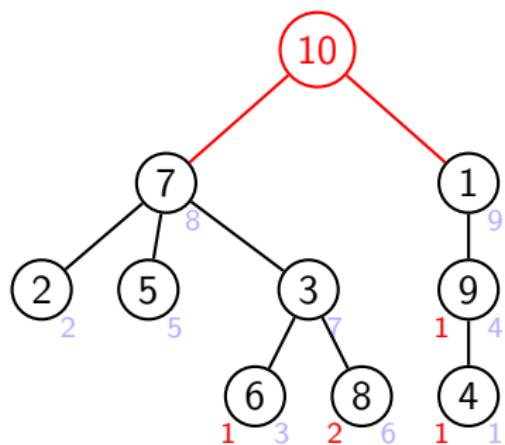
$$\beta =$$

$$1$$

The sets

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Trees vs. Parking Functions



91 41 83 87 63

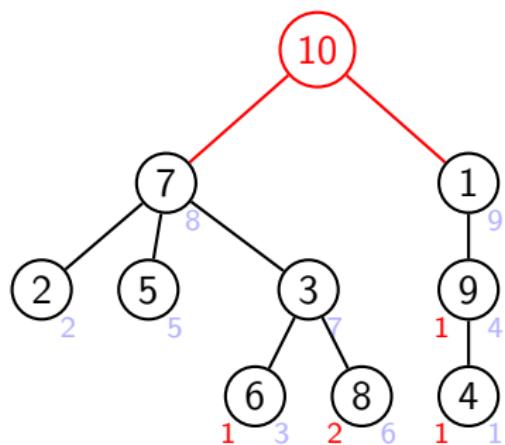
$$\pi^{-1} \downarrow \quad \begin{matrix} 4 & 9 & 1 & 8 & 6 & 3 & 5 & 2 & 7 \\ 1 & 4 & 9 & 6 & 3 & 7 & 5 & 2 & 8 \end{matrix}$$

$\beta =$ 9 1

The sets

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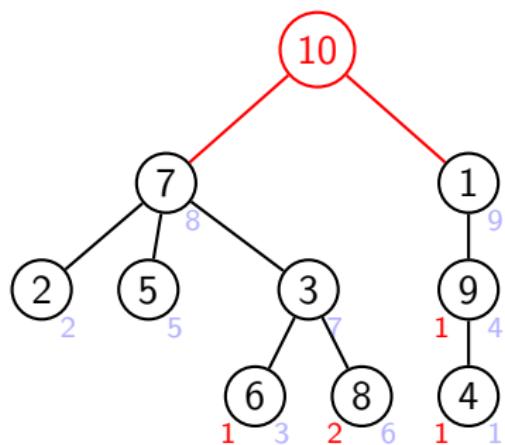
Trees vs. Parking Functions



91 41 83 87 63

$$\pi^{-1} \downarrow \quad \begin{matrix} 4 & 9 & 1 & 8 & 6 & 3 & 5 & 2 & 7 \\ 1 & 4 & 9 & 6 & 3 & 7 & 5 & 2 & 8 \end{matrix}$$

$\beta =$ 4 9 1



91

41

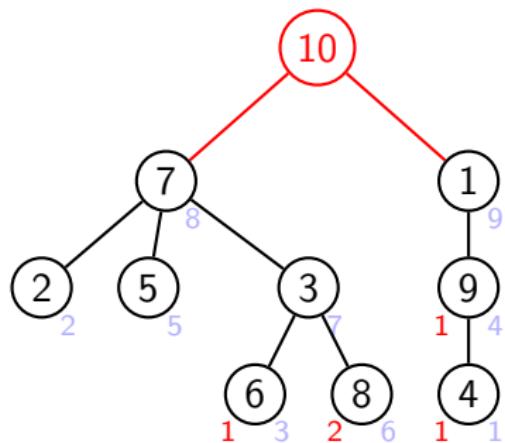
83

87

63

$$\pi^{-1} \downarrow \begin{array}{cccccccccc} 4 & 9 & 1 & 8 & 6 & 3 & 5 & 2 & 7 \\ 1 & 4 & 9 & 6 & 3 & 7 & 5 & 2 & 8 \end{array}$$

$$\beta = \begin{array}{cccccc} 7 & 4 & 9 & 1 \end{array}$$



91

41

83

87

63

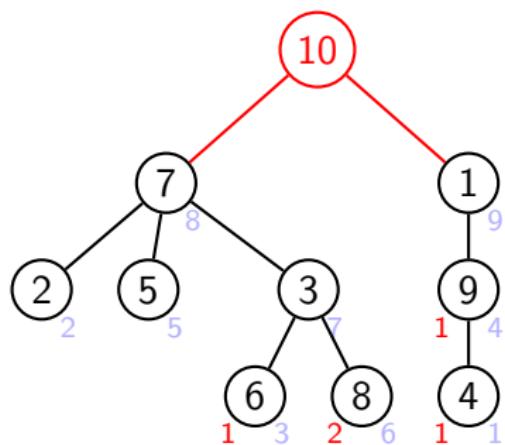
$$\pi^{-1} \downarrow \begin{array}{cccccccccc} 4 & 9 & 1 & 8 & 6 & 3 & 5 & 2 & 7 \\ 1 & 4 & 9 & 6 & 3 & 7 & 5 & 2 & 8 \end{array}$$

$$\beta = \quad 2 \ 7 \ 4 \ 9 \ 1$$

The sets

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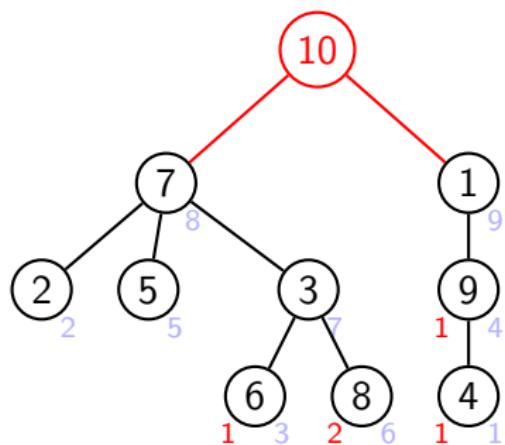
Trees vs. Parking Functions



91 41 83 87 63

$$\pi^{-1} \downarrow \quad \begin{matrix} 4 & 9 & 1 & 8 & 6 & 3 & 5 & 2 & 7 \\ 1 & 4 & 9 & 6 & 3 & 7 & 5 & 2 & 8 \end{matrix}$$

$\beta = \quad 3 \ 2 \ 7 \ 4 \ 9 \ 1$



91

41

83

87

63

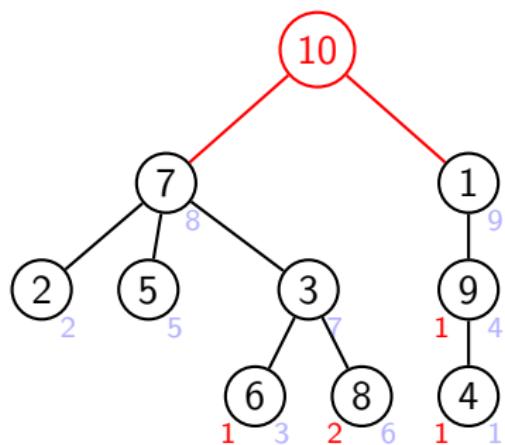
$$\pi^{-1} \downarrow \quad \begin{matrix} 4 & 9 & 1 & 8 & 6 & 3 & 5 & 2 & 7 \\ 1 & 4 & 9 & 6 & 3 & 7 & 5 & 2 & 8 \end{matrix}$$

$$\beta = \quad 6 \ 3 \ 2 \ 7 \ 4 \ 9 \ 1$$

The sets

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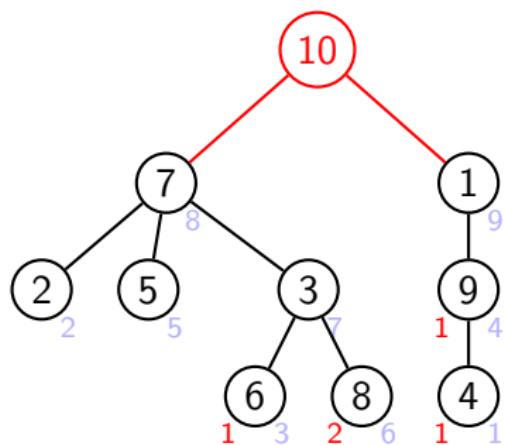
Trees vs. Parking Functions



91 41 83 87 63

$$\pi^{-1} \downarrow \quad \begin{matrix} 4 & 9 & 1 & 8 & 6 & 3 & 5 & 2 & 7 \\ 1 & 4 & 9 & 6 & 3 & 7 & 5 & 2 & 8 \end{matrix}$$

$\beta = \quad 8 \ 6 \ 3 \ 2 \ 7 \ 4 \ 9 \ 1$



91

41

83

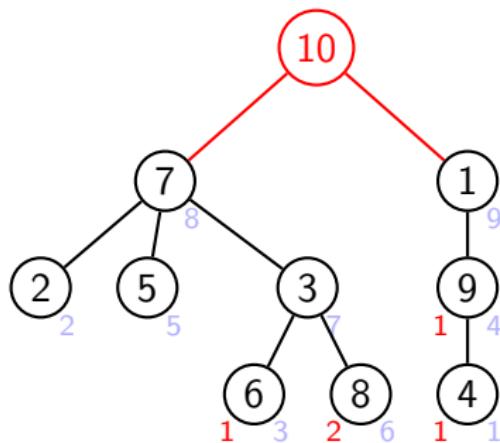
87

63

$$\pi^{-1} \downarrow \quad \begin{matrix} 4 & 9 & 1 & 8 & 6 & 3 & 5 & 2 & 7 \\ 1 & 4 & 9 & 6 & 3 & 7 & 5 & 2 & 8 \end{matrix}$$

$$\beta = 5 \ 8 \ 6 \ 3 \ 2 \ 7 \ 4 \ 9 \ 1$$

Trees vs. Parking Functions



$$91 \quad \beta(8)\beta(9)$$

$$41 \quad \beta(7)\beta(9)$$

$$83 \quad \beta(2)\beta(4)$$

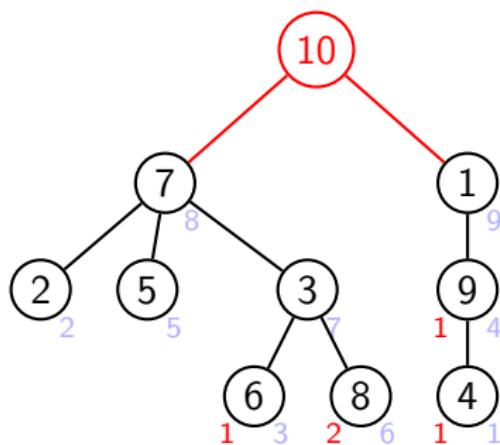
$$87 \quad \beta(2)\beta(6)$$

$$63 \quad \beta(3)\beta(4)$$

$$\pi^{-1} \downarrow \quad \begin{matrix} 4 & 9 & 1 & 8 & 6 & 3 & 5 & 2 & 7 \\ 1 & 4 & 9 & 6 & 3 & 7 & 5 & 2 & 8 \end{matrix}$$

$$\beta = 5 \ 8 \ 6 \ 3 \ 2 \ 7 \ 4 \ 9 \ 1$$

Trees vs. Parking Functions



$$91 \quad \beta(8)\beta(9)$$

$$41 \quad \beta(7)\beta(9)$$

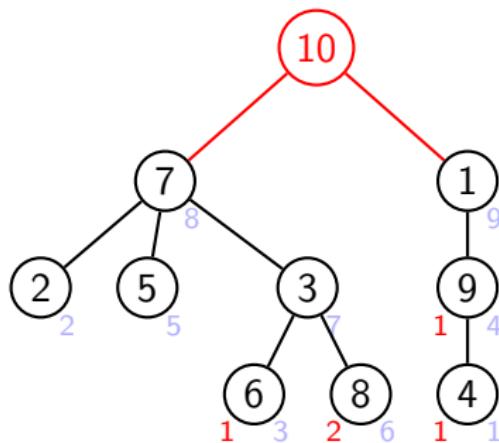
$$83 \quad \beta(2)\beta(4)$$

$$87 \quad \beta(2)\beta(6)$$

$$63 \quad \beta(3)\beta(4)$$

$$\pi^{-1} \downarrow \quad \begin{matrix} 4 & 9 & 1 & 8 & 6 & 3 & 5 & 2 & 7 \\ 1 & 4 & 9 & 6 & 3 & 7 & 5 & 2 & 8 \end{matrix}$$

$$\beta = 5 \ 8 \ 6 \ 3 \ 2 \ 7 \ 4 \ 9 \ 1$$

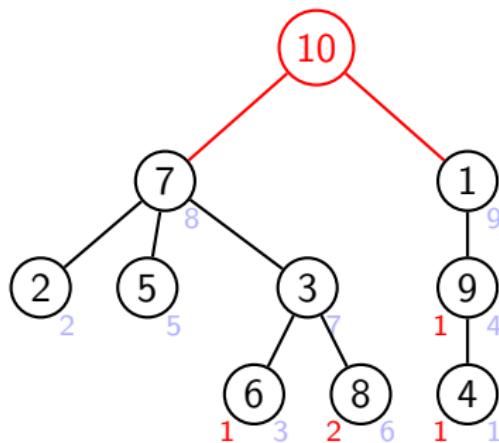


$$\begin{array}{c}
 91 \quad 41 \quad 83 \quad 87 \quad 63 \\
 \beta(8)\beta(9) \quad \beta(7)\beta(9) \quad \beta(2)\beta(4) \quad \beta(2)\beta(6) \quad \beta(3)\beta(4) \\
 (41)
 \end{array}$$

$$\pi^{-1} \downarrow \quad \begin{matrix} 4 & 9 & 1 & 8 & 6 & 3 & 5 & 2 & 7 \\ 1 & 4 & 9 & 6 & 3 & 7 & 5 & 2 & 8 \end{matrix}$$

$$\beta = 5 \ 8 \ 6 \ 3 \ 2 \ 7 \ 4 \ 9 \ 1$$

Trees vs. Parking Functions



$$\begin{matrix} 91 \\ \beta(8)\beta(9) \\ (41) \end{matrix}$$

$$\begin{matrix} 41 \\ \beta(7)\beta(9) \\ (91) \end{matrix}$$

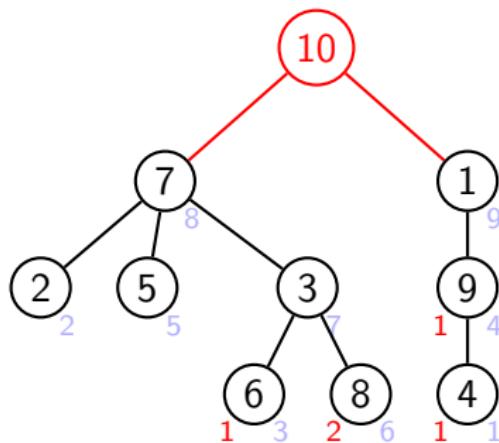
$$83$$

$$87$$

$$63$$

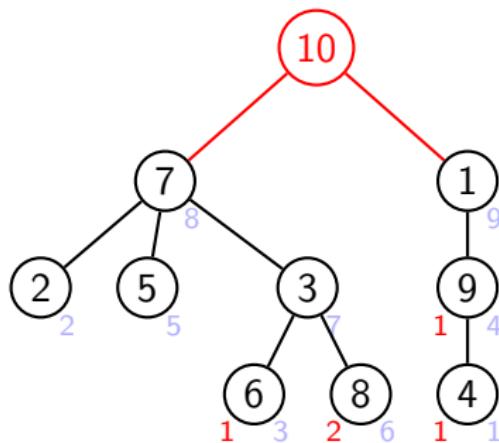
$$\pi^{-1} \downarrow \quad \begin{matrix} 4 & 9 & 1 & 8 & 6 & 3 & 5 & 2 & 7 \\ 1 & 4 & 9 & 6 & 3 & 7 & 5 & 2 & 8 \end{matrix}$$

$$\beta = 5 \ 8 \ 6 \ 3 \ 2 \ 7 \ 4 \ 9 \ 1$$



$$\begin{array}{c}
 91 \qquad 41 \qquad 83 \qquad 87 \qquad 63 \\
 \beta(8)\beta(9) \qquad \beta(7)\beta(9) \qquad \beta(2)\beta(4) \qquad \beta(2)\beta(\textcolor{red}{6}) \qquad \beta(3)\beta(4) \\
 (41) \qquad (91) \qquad (\textcolor{red}{87}) \\
 \pi^{-1} \downarrow \qquad \begin{matrix} 4 & 9 & 1 & 8 & 6 & 3 & 5 & 2 & 7 \\ \textcolor{blue}{1} & \textcolor{blue}{4} & \textcolor{blue}{9} & \textcolor{blue}{6} & \textcolor{blue}{3} & \textcolor{blue}{7} & \textcolor{blue}{5} & \textcolor{blue}{2} & \textcolor{blue}{8} \end{matrix}
 \end{array}$$

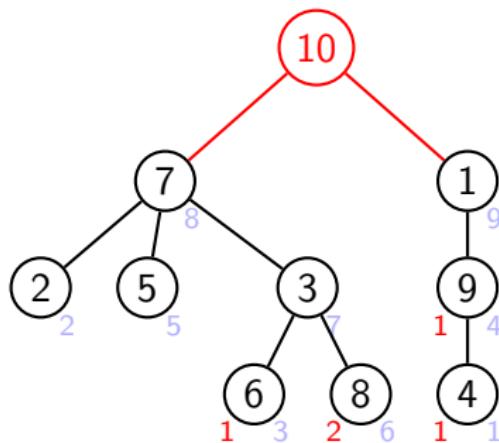
$$\beta = 5 \ 8 \ 6 \ 3 \ 2 \ 7 \ 4 \ 9 \ 1$$



$$\begin{array}{c}
 91 \qquad 41 \qquad 83 \qquad 87 \qquad 63 \\
 \beta(8)\beta(9) \qquad \beta(7)\beta(9) \qquad \beta(2)\beta(4) \qquad \beta(2)\beta(6) \qquad \beta(3)\beta(4) \\
 (41) \qquad (91) \qquad (87)
 \end{array}$$

$$\pi^{-1} \downarrow \quad \begin{matrix} 4 & 9 & 1 & 8 & 6 & 3 & 5 & 2 & 7 \\ 1 & 4 & 9 & 6 & 3 & 7 & 5 & 2 & 8 \end{matrix}$$

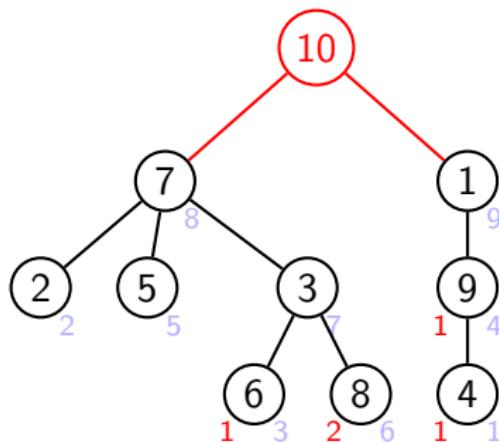
$$\beta = 5 \ 8 \ 6 \ 3 \ 2 \ 7 \ 4 \ 9 \ 1$$



$$\begin{array}{ccccc}
 91 & 41 & 83 & 87 & 63 \\
 \beta(8)\beta(9) & \beta(7)\beta(9) & \beta(2)\beta(4) & \beta(2)\beta(6) & \beta(3)\beta(4) \\
 (41) & (91) & (87) & (63) &
 \end{array}$$

$$\pi^{-1} \downarrow \quad \begin{matrix} 4 & 9 & 1 & 8 & 6 & 3 & 5 & 2 & 7 \\ 1 & 4 & 9 & 6 & 3 & 7 & 5 & 2 & 8 \end{matrix}$$

$$\beta = 5 \ 8 \ 6 \ 3 \ 2 \ 7 \ 4 \ 9 \ 1$$



$$\begin{matrix} 91 \\ \beta(8)\beta(9) \\ (41) \end{matrix}$$

$$\begin{matrix} 41 \\ \beta(7)\beta(9) \\ (91) \end{matrix}$$

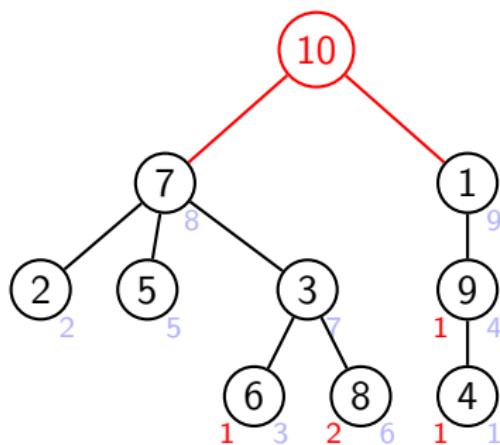
$$\begin{matrix} 83 \\ \beta(2)\beta(4) \\ (87) \end{matrix}$$

$$\begin{matrix} 87 \\ \beta(2)\beta(6) \\ (63) \end{matrix}$$

$$\begin{matrix} 63 \\ \beta(3)\beta(4) \\ (83) \end{matrix}$$

$$\pi^{-1} \downarrow \quad \begin{matrix} 4 & 9 & 1 & 8 & 6 & 3 & 5 & 2 & 7 \\ 1 & 4 & 9 & 6 & 3 & 7 & 5 & 2 & 8 \end{matrix}$$

$$\beta = 5 \ 8 \ 6 \ 3 \ 2 \ 7 \ 4 \ 9 \ 1$$



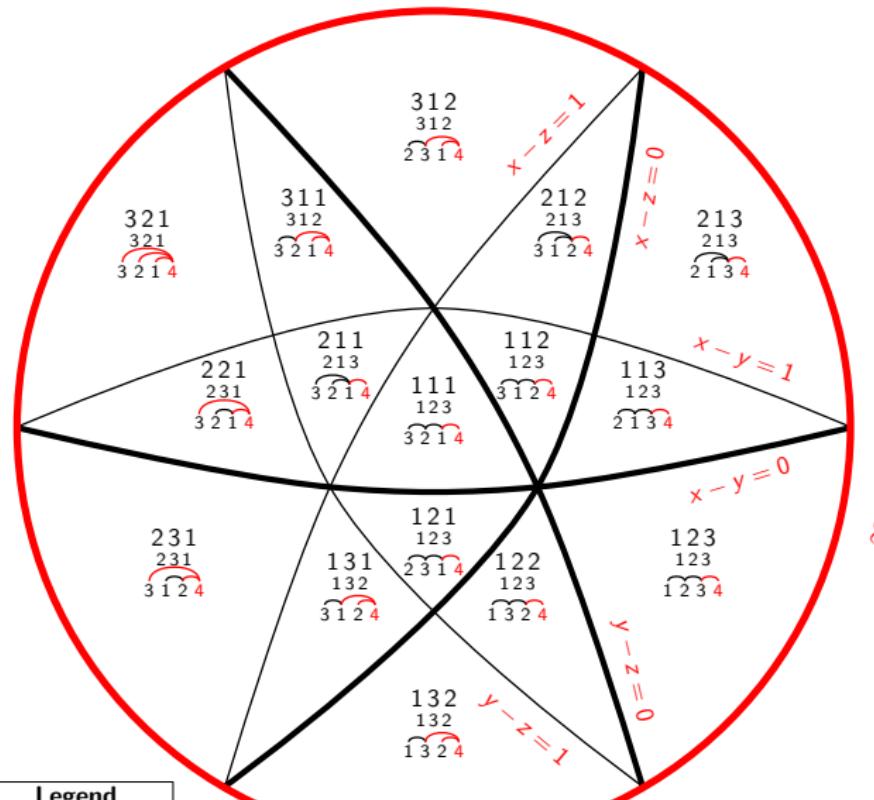
$$\begin{array}{ccccc}
 91 & 41 & 83 & 87 & 63 \\
 \beta(8)\beta(9) & \beta(7)\beta(9) & \beta(2)\beta(4) & \beta(2)\beta(6) & \beta(3)\beta(4) \\
 (41) & (91) & (87) & (63) & (83) \\
 \pi^{-1} \downarrow & 4 & 9 & 1 & 8 & 6 & 3 & 5 & 2 & 7 \\
 & \textcolor{blue}{1} & \textcolor{blue}{4} & \textcolor{blue}{9} & \textcolor{blue}{6} & \textcolor{blue}{3} & \textcolor{blue}{7} & \textcolor{blue}{5} & \textcolor{blue}{2} & \textcolor{blue}{8}
 \end{array}$$

$$\beta = 5 \ 8 \ 6 \ 3 \ 2 \ 7 \ 4 \ 9 \ 1$$

The sets

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Trees vs. Parking Functions



Parking functions - the number

1 2 3 4 5 6 7 8 9

$$f : \quad 1 \quad 8 \quad 5 \quad 2 \quad 7 \quad 4 \quad 4 \quad 8 \quad 1$$

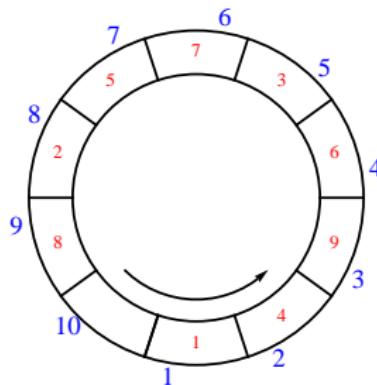
$$\sigma = P(f) : \quad 1 \quad 8 \quad 5 \quad 2 \quad 7 \quad 4 \quad 6 \quad 9 \quad 3$$

$$\sigma^{-1} : \boxed{1} \quad \boxed{4} \quad \boxed{9} \quad \boxed{6} \quad \boxed{3} \quad \boxed{7} \quad \boxed{5} \quad \boxed{2} \quad \boxed{8}$$



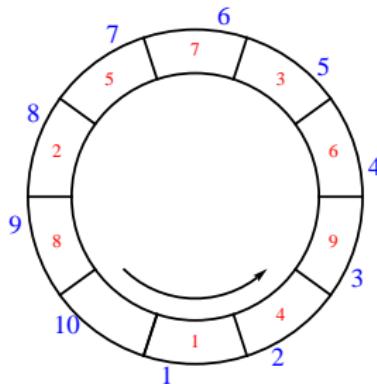
Parking functions - the number

$$\begin{array}{cccccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
 f : & 1 & 8 & 5 & 2 & 7 & 4 & 4 & 8 & 1 \\
 \sigma = P(f) : & 1 & 8 & 5 & 2 & 7 & 4 & 6 & 9 & 3 \\
 \sigma^{-1} : & \boxed{1} & \boxed{4} & \boxed{9} & \boxed{6} & \boxed{3} & \boxed{7} & \boxed{5} & \boxed{2} & \boxed{8} \\
 & \longrightarrow & & \longrightarrow & & & & & &
 \end{array}$$



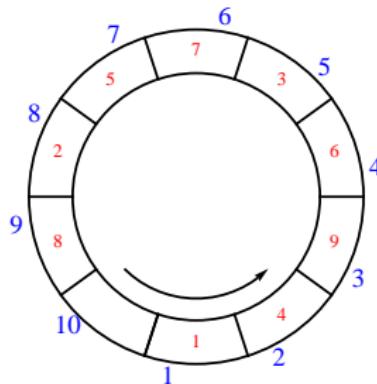
Parking functions - the number

	1	2	3	4	5	6	7	8	9
$f :$	1	8	5	2	7	4	4	8	1
$\sigma = P(f) :$	1	8	5	2	7	4	6	9	3



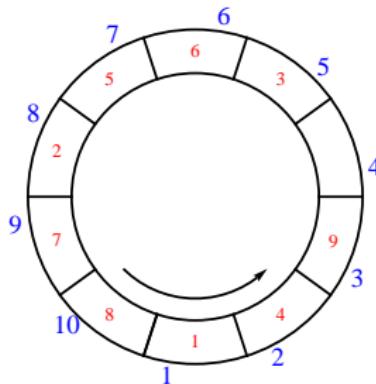
Parking functions - the number

	1	2	3	4	5	6	7	8	9
					5	5			
$f :$	1	8	5	2	7	4	4	8	1
$\sigma = P(f) :$	1	8	5	2	7	4	6	9	3
						6	9	10	



Parking functions - the number

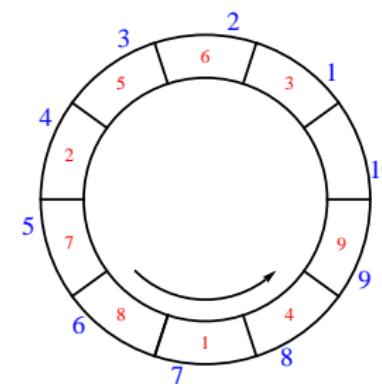
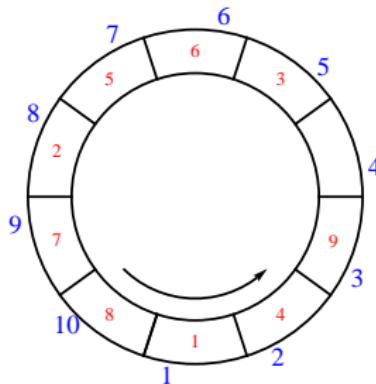
	1	2	3	4	5	6	7	8	9
$f :$	1	8	5	2	7	5	5	8	1
$\sigma = P(f) :$	1	8	5	2	7	6	9	10	3



Parking functions - the number

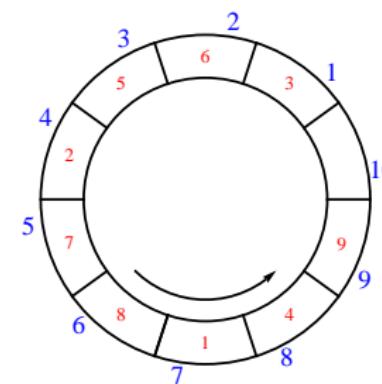
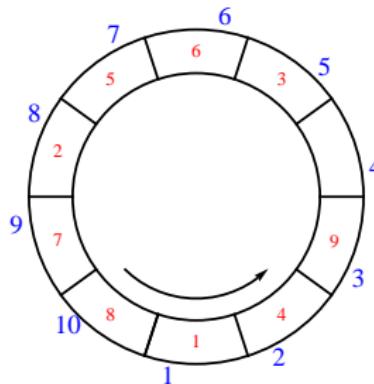
1	2	3	4	5	6	7	8	9
---	---	---	---	---	---	---	---	---

$f :$	1	8	5	2	7	5	5	8	1
$\sigma = P(f) :$	1	8	5	2	7	6	9	10	3



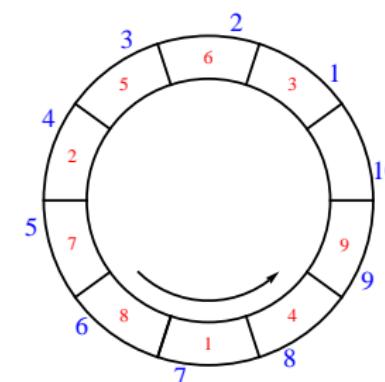
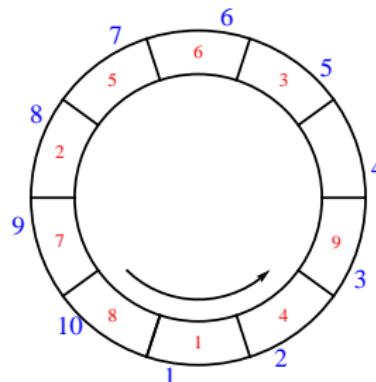
Parking functions - the number

$g \in \text{PF}_9$	1 7	2 4	3 1	4 8	5 3	6 2	7 2	8 5	9 8
$f :$	1	8	5	2	7	5	5	8	1
$\sigma = P(f) :$	1	8	5	2	7	6	9	10	3



Parking functions - the number

$g \in \mathbf{PF}_9$	1 7	2 4	3 1	4 8	5 3	6 2	7 2	8 5	9 8
$f :$	1	8	5	2	7	5	5	8	1
$\sigma = P(f) :$	1	8	5	2	7	6	9	10	3



$$|\mathbf{PF}_n| = (n+1)^n / (n+1) = (n+1)^{n-1} = |\mathbf{T}_{n+1}|$$