Enumeration of snakes

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Alternating permutations

Euler numbers are: $\tan z + \sec z = \sum_{n=0}^{\infty} E_n \frac{z^n}{n!}$.
(1, 1, 1, 2, 5, 16, ...)

E_n is the number of alternating permutations in $\mathfrak{S}_n$ (such that $\sigma_1 > \sigma_2 < \sigma_3 > \dots \sigma_n$).

Richard P. Stanley, *A survey of alternating permutations*.

Definition

A signed permutation is $\pi = \pi_1 \dots \pi_n$*, such that* $\{\ |\pi_i|\ \} = \{1 \dots n\}$*. Example: 2,-1,-4,3.*
Also, group of permutations π *of* $\{-n, \dots, -1, 1, \dots, n\}$ *such that* $\pi(-i) = -\pi(i)$*.*

Is there an analog of alternating permutations in the context of signed permutations ?

Vladimir I. Arnold, *The calculus of snakes and the combinatorics of Bernoulli, Euler, and Springer numbers of Coxeter groups.*

Definition

A signed permutation π *is a snake of type B if*
$0 < \pi_1 > \pi_2 < \pi_3 > \ldots \pi_n$.
(convention: $\pi_0 = 0$*).*

The number of snakes $\pi_1 \ldots \pi_n$ is the "Euler number of the group B_n" [Arnol'd]: we can define a number $K(R)$ for each root system R (Springer number) such that

- $K(A_{n-1}) = \#$ alternating permutations in $\mathfrak{S}_n = E_n$
- $K(B_n) = \#$ snakes in $\mathfrak{S}_n^B = S_n$

Just as alternating permutations are related with tan and sec, with snakes we need to consider the successive derivatives of tan and sec.

Let $P_n(t)$, $Q_n(t)$, and $R_n(t)$ be polynomials such that:

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \tan x = P_n(\tan x),$$
$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \sec x = Q_n(\tan x) \sec x,$$
$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \sec^2 x = R_n(\tan x) \sec^2 x.$$

$\Big($note that $\tan' = \sec^2$, and it follows $P_{n+1} = (1 + t^2) R_n$ $\Big)$

Proposition (Hoffman)

$$P_{2n+1}(0) = E_{2n+1}, \qquad Q_{2n+1}(0) = 0,$$
$$P_{2n}(0) = 0, \qquad Q_{2n}(0) = E_{2n},$$
$$P_{n+1}(1) = 2^n E_n \qquad Q_n(1) = S_n.$$

$$P_{n+1} = (1+t^2)P'_n, \qquad P_0(t) = t,$$
$$Q_{n+1} = (1+t^2)Q'_n + tQ_n, \qquad Q_0(t) = 1,$$
$$R_{n+1} = (1+t^2)R'_n + 2tR_n, \qquad R_0(t) = 1.$$

There are combinatorial models of $P_n(t)$, $Q_n(t)$, and $R_n(t)$ in terms of

- snakes,
- cycle-alternating permutations,
- increasing trees and forrests,
- weighted Dyck prefixes,
- weighted Motzkin paths...

Definition

Let $\pi = \pi_1, \ldots, \pi_n$ be a signed permutation. Then $(\pi_0), \pi_1, \ldots, \pi_n, (\pi_{n+1})$ is a snake when $\pi_0 < \pi_1 > \pi_2 < \ldots \pi_{n+1}$. Different conventions on π_0 and π_{n+1} gives different types of snakes.

- $\mathcal{S}_n = \{$ snakes $(\pi_0), \pi_1, \ldots, \pi_n, (\pi_{n+1})$ with $\pi_0 = -(n+1),$ $\pi_{n+1} = (-1)^n(n+1)\}$
- $\mathcal{S}_n^0 = \{$ ... with $\pi_0 = 0$, $\pi_{n+1} = (-1)^n(n+1)\}$
- $\mathcal{S}_n^{00} = \{$ ... with $\pi_0 = \pi_{n+1} = 0\}$

Example

$(-4), -2, -3, 1, (-4) \in \mathcal{S}_n,$ $\quad$ $(0), 3, -1, 2, (-4) \in \mathcal{S}_n^0$

$(0), 4, -1, 3, -2, (0) \in \mathcal{S}_n^{00}$

Theorem

Let $\mathrm{sc}(\pi)$ *be the number of sign changes through* π*, i.e.* $\mathrm{sc}(\pi) = \#\{\ i \mid 0 \le i \le n,\ \pi_i \pi_{i+1} < 0\ \}$*. Then*

$$P_n(t) = \sum_{\pi \in \mathcal{S}_n} t^{\mathrm{sc}(\pi)}, \qquad Q_n(t) = \sum_{\pi \in \mathcal{S}_n^0} t^{\mathrm{sc}(\pi)}, \qquad R_n(t) = \sum_{\pi \in \mathcal{S}_{n+1}^{00}} t^{\mathrm{sc}(\pi)}.$$

Example

$Q_2(t) = 2t^2 + 1$,
the snakes are $(0), 2, -1, (3)$ and $(0), 2, 1, (3)$ and $(0), 1, -2, (3)$.

$R_1(t) = 2t$,
the snakes are $(0), 2, -1, (0)$ and $(0), 1, -2, (0)$.

Proof

We can check the recurrence relation, for example:

$$R_n = (1 + t^2)R'_{n-1} + 2tR_{n-1}.$$

Let $\pi \in \mathcal{S}^{00}_{n+1}$. We want to obtain $(1 + t^2)R'_{n-1} + 2tR_{n-1}$.

- Case where $\pi_1 = 1$. Let $\pi' = -\pi_2 \ldots -\pi_{n+1}$. We relabel ($2 \mapsto 1$, $3 \mapsto 2$, etc.), then $\pi' \in \mathcal{S}^{00}_n$, whence the term tR_{n-1}.
- Case where $\pi_{n+1} = \pm 1$. Let $\pi' = \pi_1 \ldots \pi_n$, we relabel, then $\pi' \in \mathcal{S}^{00}_n$, whence the term tR_{n-1}.
- Case where $\pi_j = \pm 1$ and $2 \leq j \leq n$. Then π_{j-1}, π_{j+1} have the same sign. We will obtain R'_{n-1} if π_j has also the same sign as π_{j-1}, π_{j+1}, and $t^2 R'_{n-1}$ otherwise.
  Let us suppose: π_{j-1}, π_j, π_{j+1} have the same sign. Let $\pi' = \pi_1 \ldots \pi_{j-1}, -\pi_{j+1} \ldots -\pi_{n+1}$.
  $\pi \mapsto (\pi', j)$ is bijective, whence the term R'_{n-1}.

Cycle-alternating permutations

Definition

Let $\mathcal{C}_n$ be the set of cycle-alternating signed permutations, i.e. such that $\forall i$, $\pi^{-1}(i) < i > \pi(i)$ or $\pi^{-1}(i) > i < \pi(i)$.

Example: (2,-4,1,-2,4,-1)(3,-5)(-3,5)

Recall that in signed permutations, we have two types of cycles:

- one-orbit cycles $(i_1, \ldots i_n, -i_1, \ldots, -i_n)$
- two-orbit cycles $(i_1, \ldots, i_n)(-i_1, \ldots, -i_n)$

Lemma

Let $\pi = \pi_1, \ldots, \pi_n$ be cycle-alternating, with only one cycle. Then π is a one-orbit cycle if and only if n is odd.

Theorem

Let $\mathrm{neg}(\pi) = \#\{\, i > 0 \mid \pi(i) < 0 \,\}$*, then*

$$P_n(t) = \sum_{\substack{\pi \in \mathcal{C}_{n+1} \\ \pi \text{ has only} \\ \text{one cycle}}} t^{\mathrm{neg}(\pi)}, \quad Q_n(t) = \sum_{\pi \in \mathcal{C}_n} t^{\mathrm{neg}(\pi)}, \quad R_n(t) = \sum_{\substack{\pi \in \mathcal{C}_{n+2} \\ \pi \text{ has only} \\ \text{one cycle} \\ \pi_1 > 1}} t^{\mathrm{neg}(\pi)}.$$

Proof

Bijections between snakes and cycle-alternating permutations.
Example in the case of P_n:

- (-4),3,-1,2,(-4) goes to (3,-1,2,-4)(-3,1,-2,4)
- (-5),1,-3,-2,-4,(5) goes to (1,-3,-2,-4,5,-1,3,2,4,-5)

Exponential generating functions

Theorem

$$\sum_{n=0}^{\infty} P_n(t)\frac{z^n}{n!} = \frac{\sin z + t\cos z}{\cos z - t\sin z}, \qquad \sum_{n=0}^{\infty} Q_n(t)\frac{z^n}{n!} = \frac{1}{\cos z - t\sin z},$$

$$\sum_{n=0}^{\infty} R_n(t)\frac{z^n}{n!} = \frac{1}{(\cos z - t\sin z)^2}.$$

Proof.

(case of Q_n, following [Hoffman]) Use Taylor expansion formula:

$$\sum_{n=0}^{\infty} Q_n(\tan u)\sec u\frac{z^n}{n!} = \sec(u+z) = \frac{1}{\cos u\cos z - \sin u\sin z} = \frac{\sec u}{\cos z - \tan u\sin z}.$$

□

The exponential generating functions can also be obtained combinatorially.

Using snakes, we have:

$$\sum_{n=0}^{\infty} Q_n(t)\frac{z^n}{n!} = \frac{\sec z}{1 - t\tan z}.$$

Using cycle-alternating permutations, we have directly:

$$\sum_{n=0}^{\infty} P_n(t)\frac{z^n}{n!} = \frac{\mathrm{d}}{\mathrm{d}z}\log\left(\sum_{n=0}^{\infty} Q_n(t)\frac{z^n}{n!}\right).$$

Using snakes, we have directly:

$$\sum_{n=0}^{\infty} R_n(t)\frac{z^n}{n!} = \left(\sum_{n=0}^{\infty} Q_n(t)\frac{z^n}{n!}\right)^2.$$

Differential equations

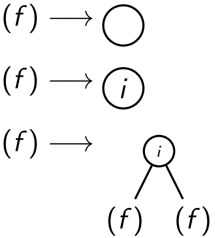
Let $f = \sum P_n \frac{z^n}{n!}$ and $g = \sum Q_n \frac{z^n}{n!}$. They satisfy:

$$\begin{cases} f' = 1 + f^2 & f(0) = t, \\ g' = fg & g(0) = 1. \end{cases}$$

From Leroux and Viennot's combinatorial theory of differential equations, it follows that P_n and Q_n count increasing trees.
Rewrite:

$$\begin{cases} f = t + z + \int f^2, \\ g = 1 + \int fg. \end{cases}$$

From $f = t + z + \int f^2$, $f(z)$ counts increasing trees produced by the rules:

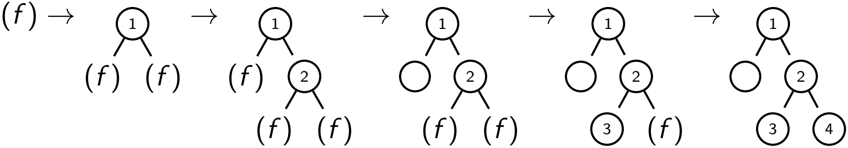


a leaf with no label (with "weight" t),

a leaf with integer label,

an internal node.

Example



Theorem

Let $\mathcal{T}_n$ be the set of complete binary trees, such that

- *n nodes are labelled with integers from 1 to n, but some leaves have no label,*
- *labels are increasing from the root to the leaves.*

Then, $\mathrm{em}(T)$ being the number of empty leaves in T, we have

$$P_n(t) = \sum_{T \in \mathcal{T}_n} t^{\mathrm{em}(T)}, \qquad Q_n(t) = \sum_{\substack{T \in \mathcal{T}_n \\ \text{the rightmost} \\ \text{leaf is empty}}} t^{\mathrm{em}(T)-1}$$

Conclusion

Whenever you know an interesting result about alternating permutations, try to generalize it to snakes.

Thanks for your attention !