Enumeration of bigrassmannian permutations in Bruhat order

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Main Thm

$$\beta(x) = \sum (x(i) - x(j)).$$

where $1 \leq i < j \leq n$ and $x(i) > x(j)$.

Plan

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1. Bruhat order
2. Bigrassmannian Permutations
3. Edelman order
4. Main Thm

Let $w, x \in S_n$.

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Define Bruhat order $w \leq x$ if ${}^{\exists}$ a path

$$w \to x_1 \to x_2 \to \cdots \to x_k \to x.$$

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$\bullet B(34512) = ? \implies$ Edelman order

($B(S_n)$ has nice properties in this order)

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$$x = \quad 42513 \quad = \begin{array}{llll} 4 & & & \\ 2 & 4 & & \\ 2 & 4 & 5 & \\ 1 & 2 & 4 & 5. \end{array}$$

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Ex.

$$x = \quad 42513 \quad = \quad \begin{matrix} 4 \\ 2\ 4 \\ 2\ 4\ 5 \\ 1\ 2\ 4\ 5. \end{matrix}$$

with $x_{11} = 4$, $x_{21} = 2 \ldots$

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Edelman order

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Def [Reading 02]

$(a, b, c) \in \mathbb{N}^3$ with $1 \leq b \leq a \leq n-1$ and $b+1 \leq c \leq n-a+b$,

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Def [Reading 02]

$(a, b, c) \in \mathbb{N}^3$ with $1 \leq b \leq a \leq n-1$ and $b+1 \leq c \leq n-a+b$, define

$$J_{abc} = \min\{x \in S_n \mid x_{ab} \geq c\}.$$

Ex. (n=5)

$$J_{324} \quad = \quad \begin{array}{llll} \_\_ & & & \\ \_\_ & \_\_ & & \\ \_\_ & \color{red}{4} & \_\_ & \\ \_\_ & \_\_ & \_\_ & \_\_ \end{array}$$

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Prop [Lascoux-Scützenberger, Reading]

$w \in B(S_n) \iff w = J_{abc} \ {}^{\exists}(a, b, c)$ as above.

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$c \leq x_{ab} \iff J_{abc} \leq x \iff J_{abc} \in B(x).$

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$$\beta(x) = \sum_{a=1}^{n-1} \sum_{b=1}^{a} (x_{ab} - b)$$

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$$= \sum \begin{pmatrix} 3 & & & \\ 1 & 2 & & \\ 1 & 2 & 2 & \\ 0 & 0 & 1 & 1 \end{pmatrix} = 13.$$

Lemma 2

For $i < j$, we have

$$\beta(x) - \beta(xt_{ij}) = (j - i)(x(i) - x(j)).$$

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&= \sum_{a=1}^{n} x(a)(n-a) - \sum_{a=1}^{n} w(a)(n-a) \\
&= x(i)(n-i) + x(j)(n-j) \\
&\quad -(w(i)(n-i) + w(j)(n-j))
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Inductive hypothesis:

$$\beta(w) = \sum_{(i,j)\in I(w)} w(i) - w(j).$$

By Lemma 2,

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Ex

$$\begin{aligned}\beta(42513) &= (4-2)+(4-1)+(4-3)\\ &\quad +(2-1)+(5-1)+(5-3)\\ &= 13.\end{aligned}$$

S_n	L
Coxeter group	distributive lattice
Bruhat	Edelman
ℓ	β
$\lvert I(x) \rvert$	$\sum x(i) - x(j)$
$\lvert\{t \mid tx < x\}\rvert$	$\sum_{tx<x} \mathrm{dp}(t)$ $(\mathrm{dp}(t) = \frac{\ell(t)-1}{2})$
$n(n-1)/2$	$(n+1)n(n-1)/6$
bigrassmannian	join-irreducible