Proof of the q-TSPP Conjecture

Christoph Koutschan
(collaboration with Manuel Kauers and Doron Zeilberger)

Research Institute for Symbolic Computation (RISC)
Johannes Kepler Universität Linz, Austria

13.09.2010
Séminaire Lotharingien de Combinatoire 65

Partition

$$17 =$$
$$5 + 4 + 3 + 2 + 1 + 1 + 1$$

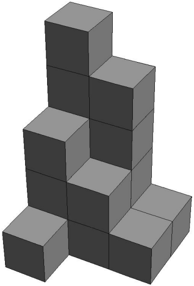
PP

Plane Partition

5	4	1
3	2	1
1		

- two-dimensional array $\pi = (\pi_{i,j})_{1 \leq i,j}$
- $\pi_{i,j} \in \mathbb{N}$ with finite sum $|\pi| = \sum \pi_{i,j}$
- $\pi_{i,j} \geq \pi_{i+1,j}$ and $\pi_{i,j} \geq \pi_{i,j+1}$

PP

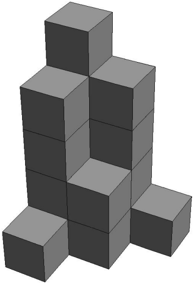


Plane Partition (3D Ferrers diagram)

- two-dimensional array $\pi = (\pi_{i,j})_{1 \leq i,j}$
- $\pi_{i,j} \in \mathbb{N}$ with finite sum $|\pi| = \sum \pi_{i,j}$
- $\pi_{i,j} \geq \pi_{i+1,j}$ and $\pi_{i,j} \geq \pi_{i,j+1}$

SPP

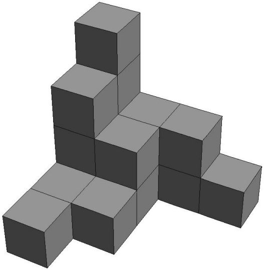
Symmetric Plane Partition



- two-dimensional array $\pi = (\pi_{i,j})_{1 \leq i,j}$
- $\pi_{i,j} \in \mathbb{N}$ with finite sum $|\pi| = \sum \pi_{i,j}$
- $\pi_{i,j} \geq \pi_{i+1,j}$ and $\pi_{i,j} \geq \pi_{i,j+1}$
- $\pi_{i,j} = \pi_{j,i}$

CSPP

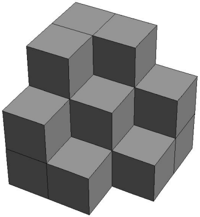
Cyclically Symmetric Plane Partition



- two-dimensional array $\pi = (\pi_{i,j})_{1 \leq i,j}$
- $\pi_{i,j} \in \mathbb{N}$ with finite sum $|\pi| = \sum \pi_{i,j}$
- $\pi_{i,j} \geq \pi_{i+1,j}$ and $\pi_{i,j} \geq \pi_{i,j+1}$
- cyclically symmetric

TSPP

Totally Symmetric Plane Partition



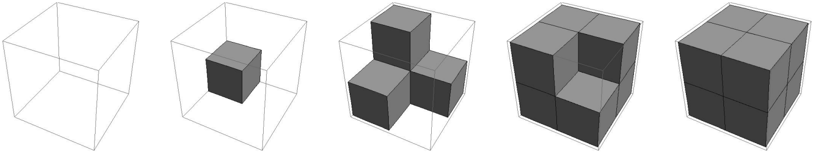
- two-dimensional array $\pi = (\pi_{i,j})_{1 \leq i,j}$
- $\pi_{i,j} \in \mathbb{N}$ with finite sum $|\pi| = \sum \pi_{i,j}$
- $\pi_{i,j} \geq \pi_{i+1,j}$ and $\pi_{i,j} \geq \pi_{i,j+1}$
- $\pi_{i,j} = \pi_{j,i}$
- cyclically symmetric

3D Ferrers diagram is invariant under the action of S_3.

TSPP Count

Theorem: There are $\displaystyle\prod_{1\leq i\leq j\leq k\leq n} \frac{i+j+k-1}{i+j+k-2}$ TSPPs in $[0,n]^3$.

Example: $(n=2)$



$$\prod_{1\leq i\leq j\leq k\leq 2} \frac{i+j+k-1}{i+j+k-2} = \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4} = 5.$$

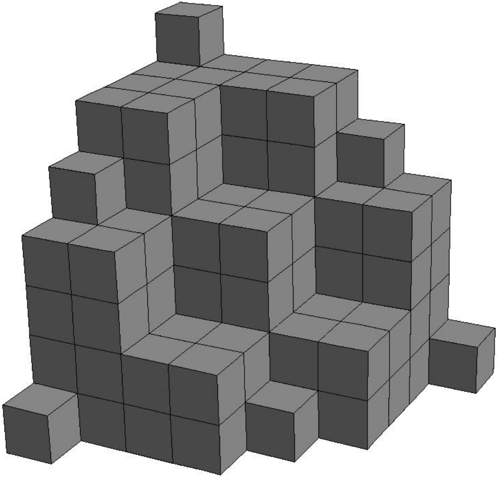
TSPP Count

Theorem: There are $$\prod_{1\le i\le j\le k\le n} \frac{i+j+k-1}{i+j+k-2}$$ TSPPs in $[0,n]^3$.

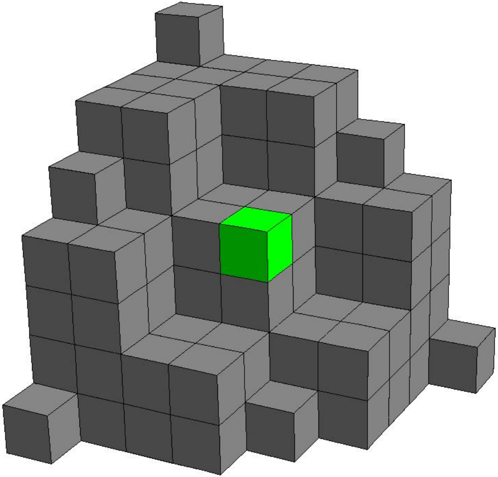
Proof: See

- John Stembridge, *The enumeration of totally symmetric plane partitions*, Advances in Mathematics **111** (1995), 227–243.
- George E. Andrews, Peter Paule, and Carsten Schneider, *Plane Partitions VI. Stembridge's TSPP theorem*, Advances in Applied Mathematics **34** (2005), 709–739.
- Christoph Koutschan, *Eliminating Human Insight: An Algorithmic Proof of Stembridge's TSPP Theorem*, Contemporary Mathematics **517** (2010), 219–230.

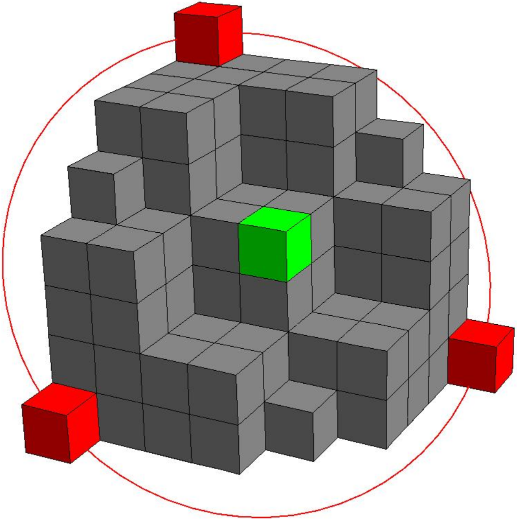
Example: TSPP and orbits



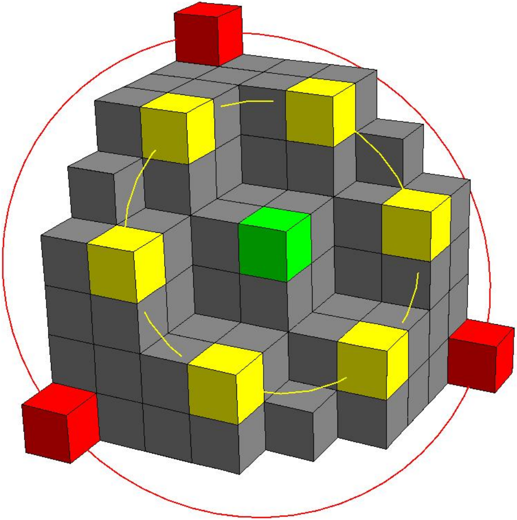
Example: TSPP and orbits



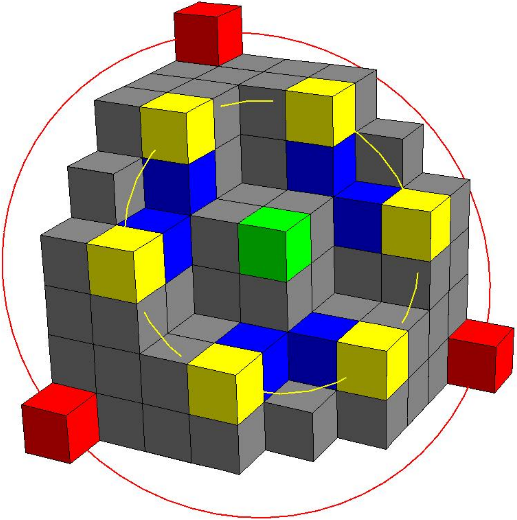
Example: TSPP and orbits



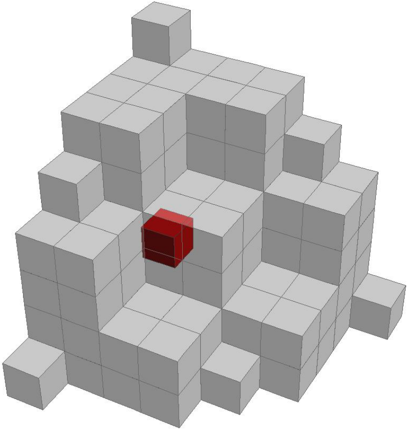
Example: TSPP and orbits



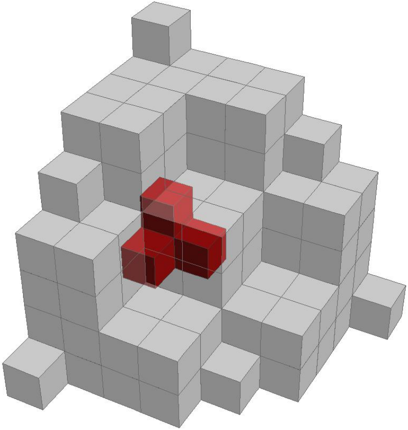
Example: TSPP and orbits



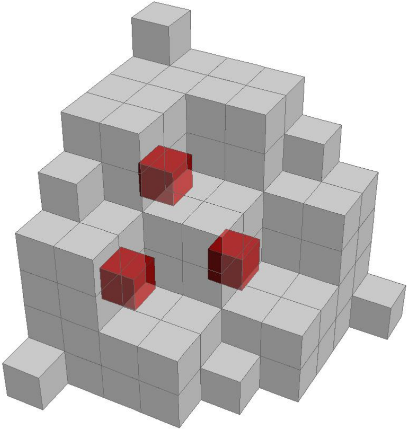
Orbits



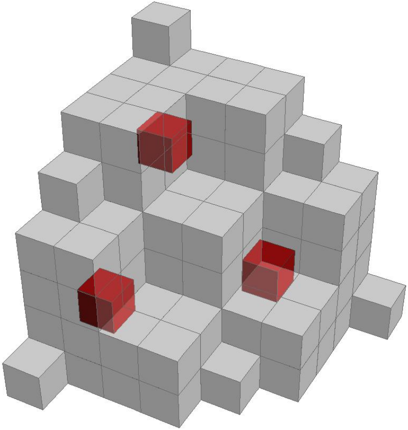
Orbits



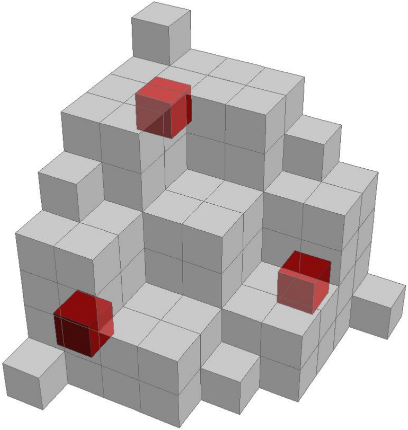
Orbits



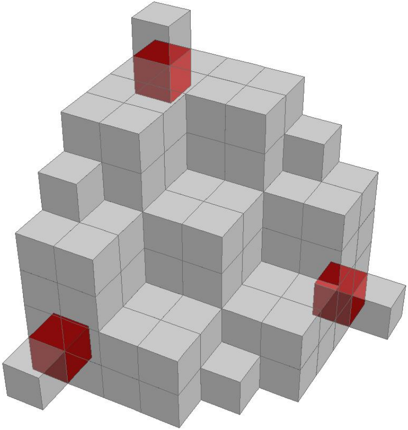
Orbits



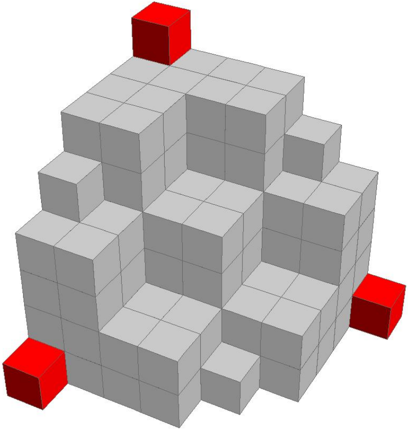
Orbits



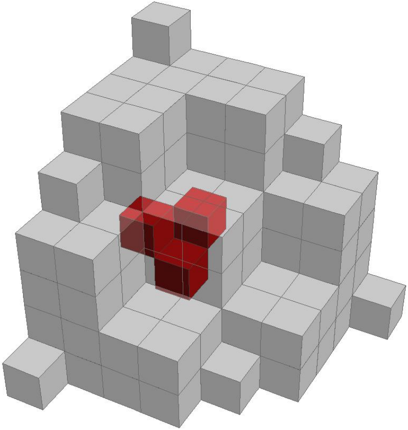
Orbits



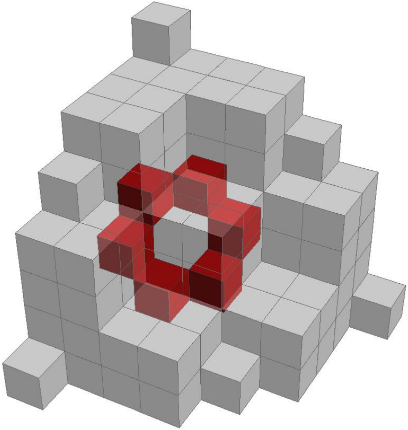
Orbits



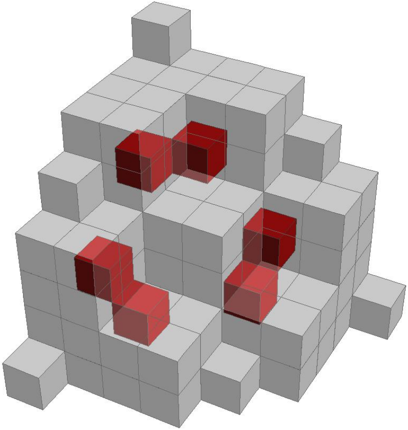
Orbits



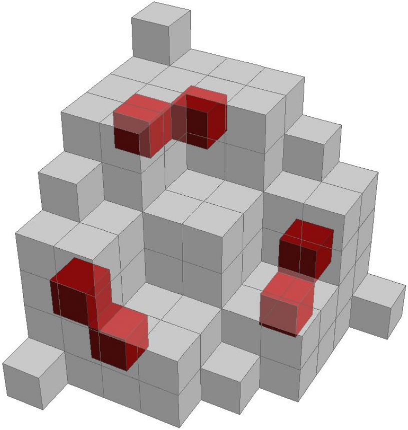
Orbits



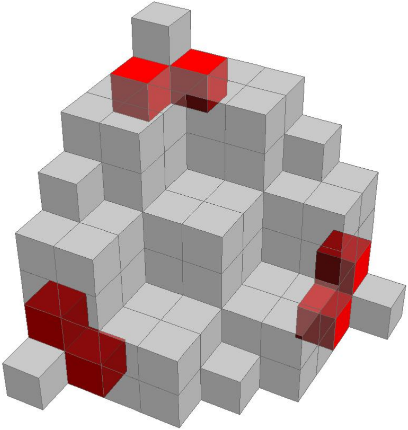
Orbits



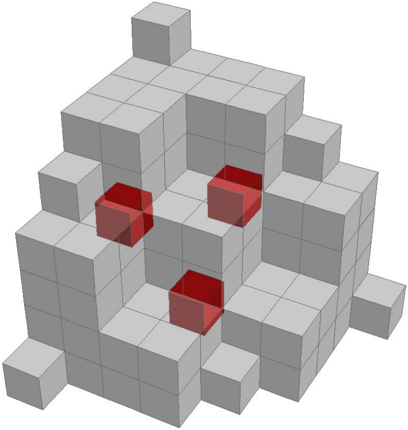
Orbits



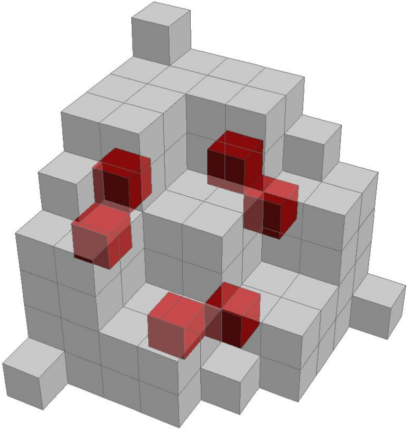
Orbits



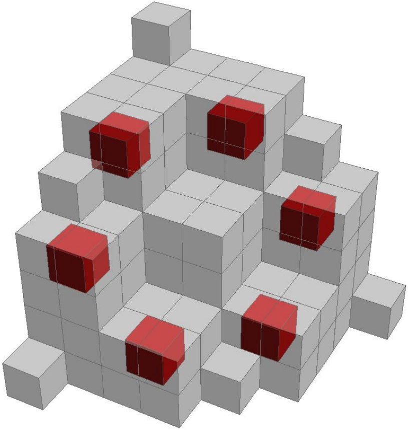
Orbits



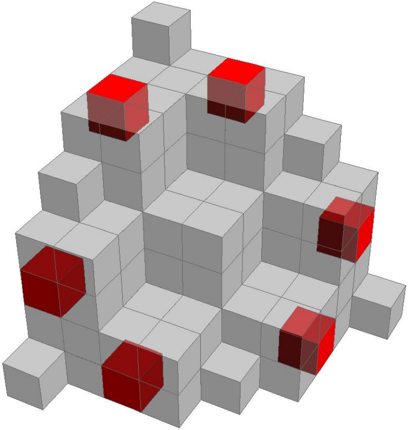
Orbits



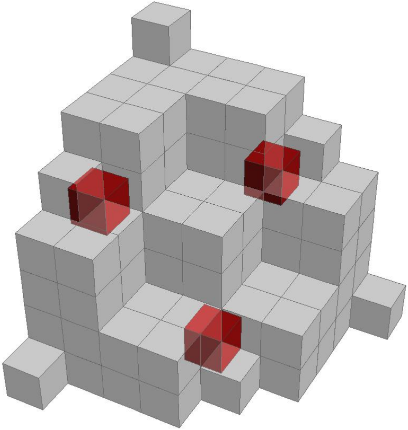
Orbits



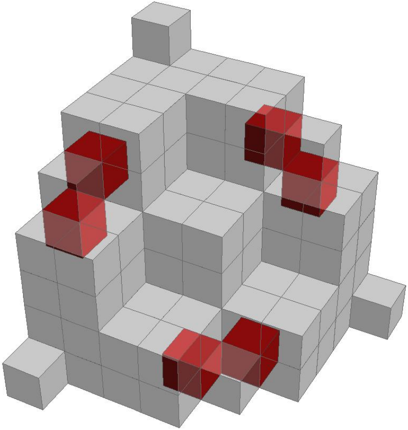
Orbits



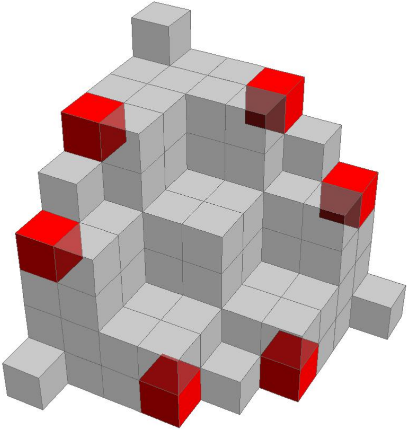
Orbits



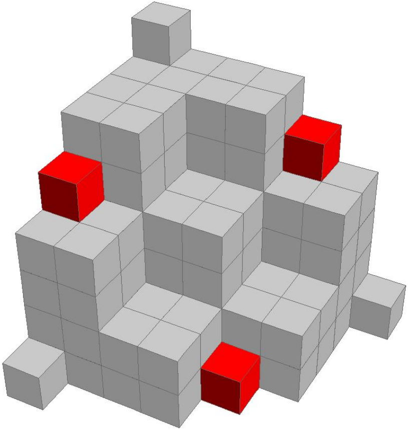
Orbits



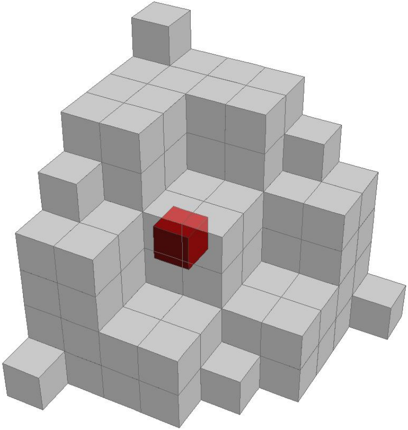
Orbits



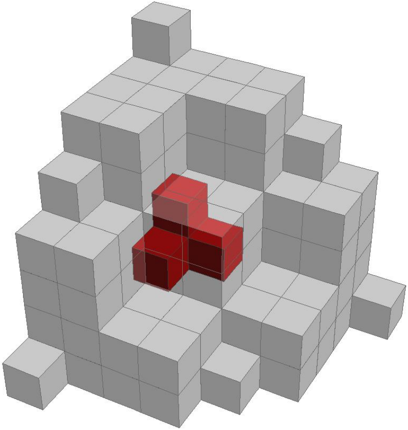
Orbits



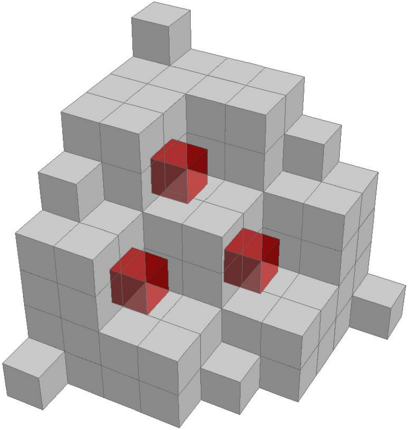
Orbits



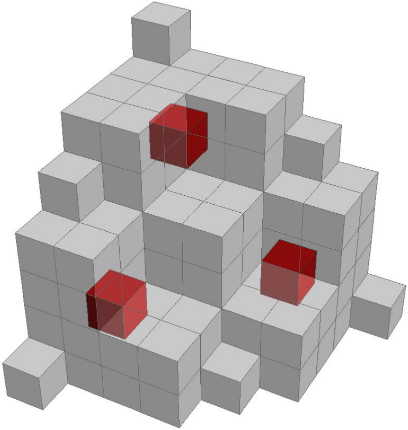
Orbits



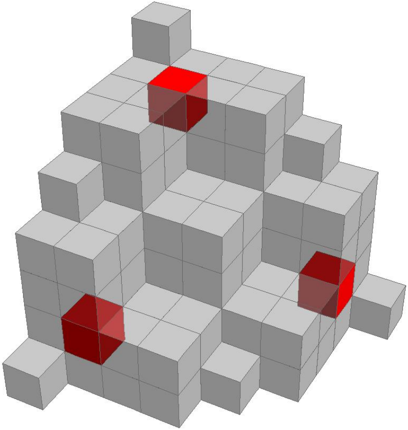
Orbits



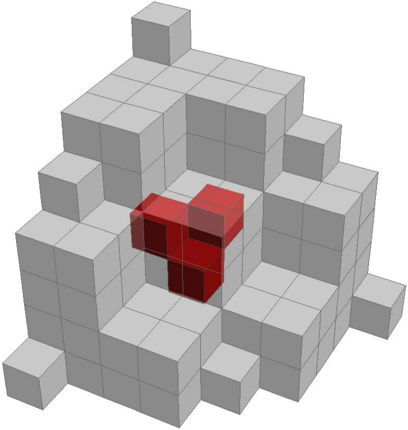
Orbits



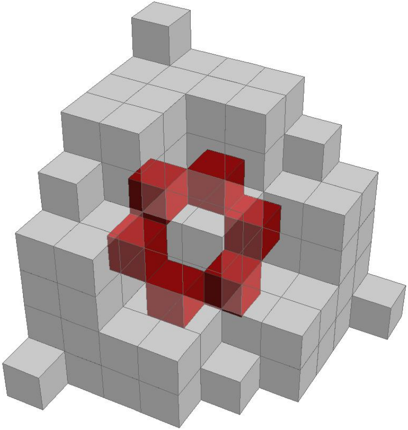
Orbits



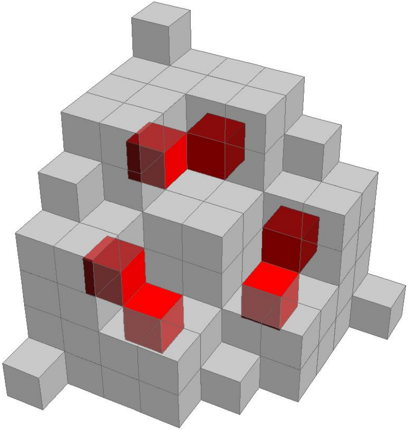
Orbits



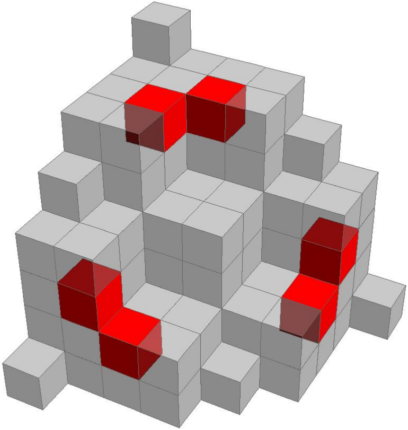
Orbits



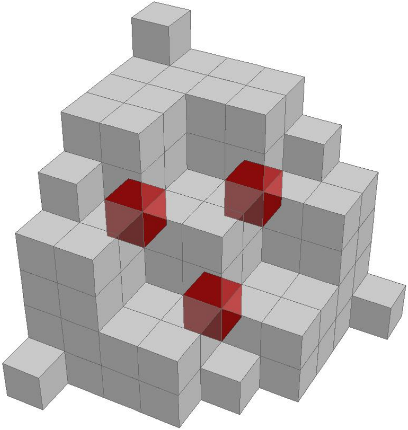
Orbits



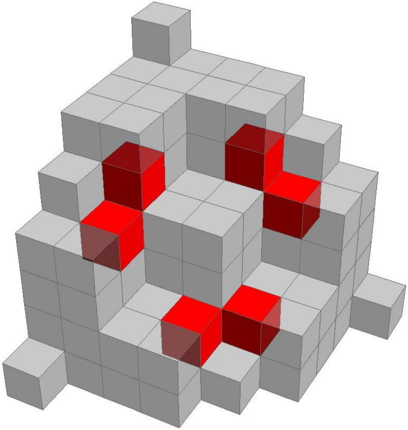
Orbits



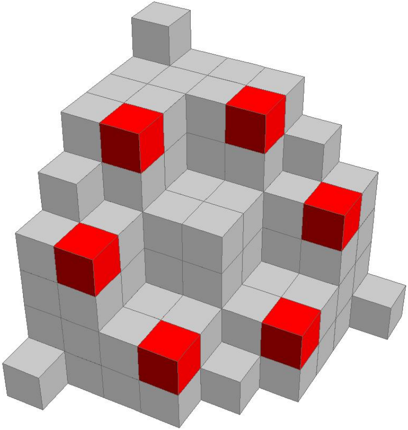
Orbits



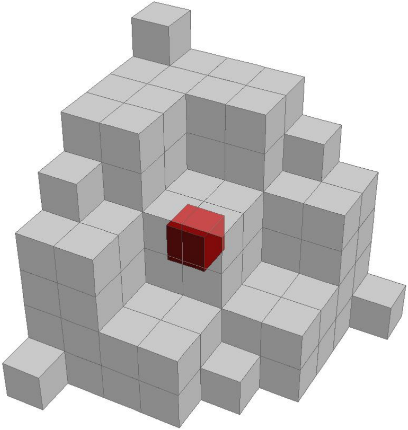
Orbits



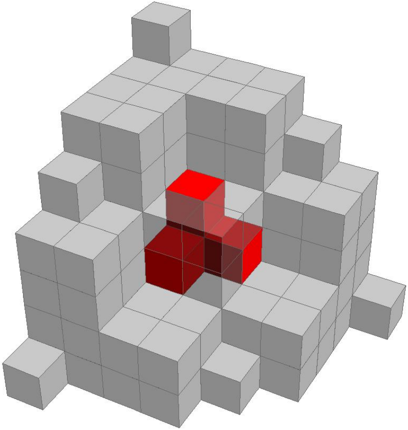
Orbits



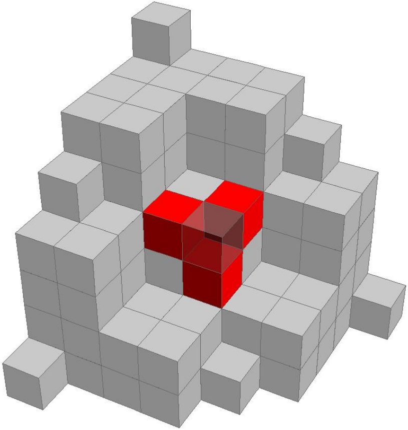
Orbits



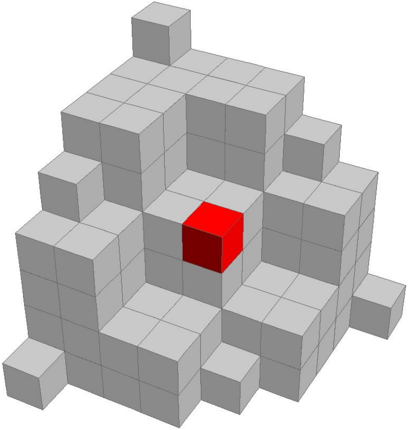
Orbits



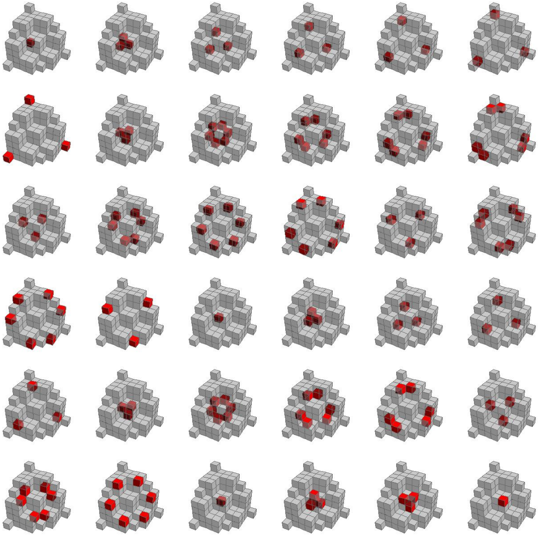
Orbits



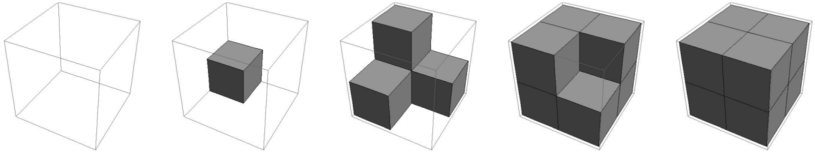
Orbits



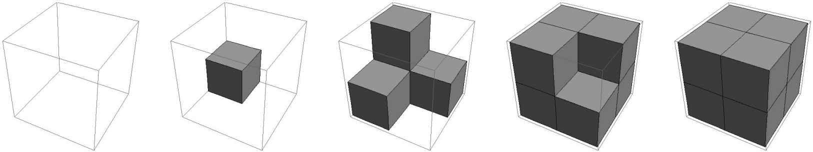
36 Orbits



Let $T(n)$ denote set of TSPPs with largest part at most n.

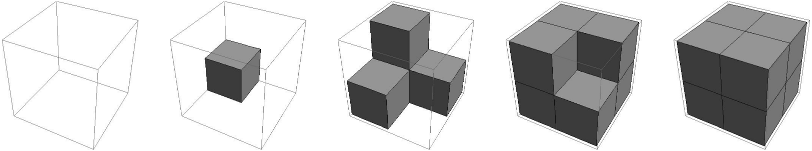


Let $T(n)$ denote set of TSPPs with largest part at most n.



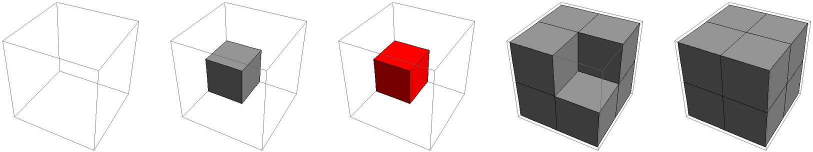
q^0

Let $T(n)$ denote set of TSPPs with largest part at most n.



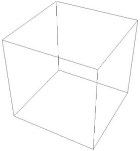
q^0 q^1

Let $T(n)$ denote set of TSPPs with largest part at most n.

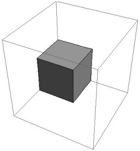


q^0 q^1

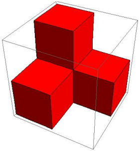
Let $T(n)$ denote set of TSPPs with largest part at most n.



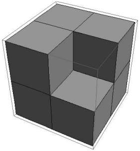
q^0

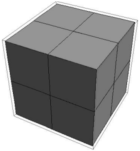


q^1

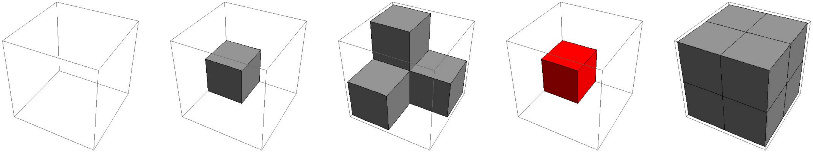


q^2



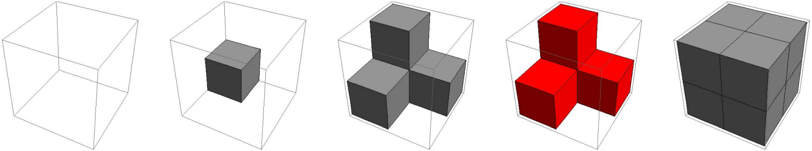


Let $T(n)$ denote set of TSPPs with largest part at most n.



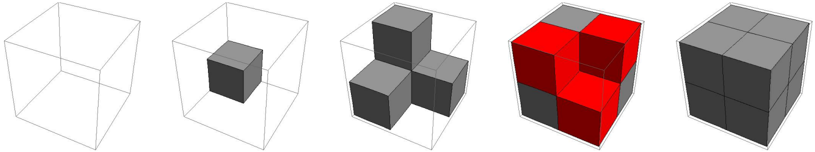
q^0 q^1 q^2

Let $T(n)$ denote set of TSPPs with largest part at most n.



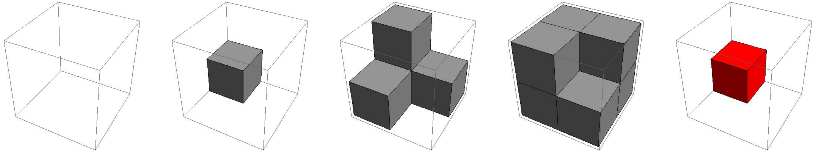
q^0 q^1 q^2

Let $T(n)$ denote set of TSPPs with largest part at most n.



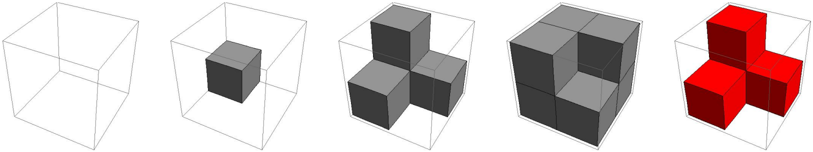
q^0 q^1 q^2 q^3

Let $T(n)$ denote set of TSPPs with largest part at most n.



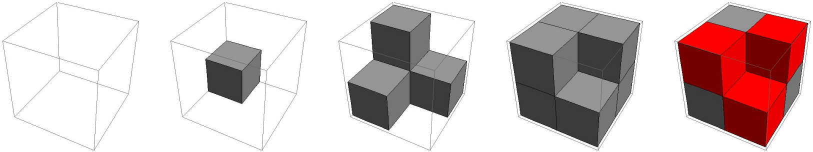
q^0 $\quad$ q^1 $\quad$ q^2 $\quad$ q^3

Let $T(n)$ denote set of TSPPs with largest part at most n.



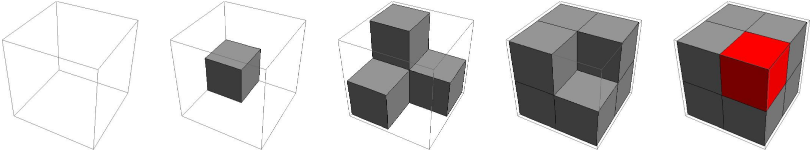
q^0 q^1 q^2 q^3

Let $T(n)$ denote set of TSPPs with largest part at most n.



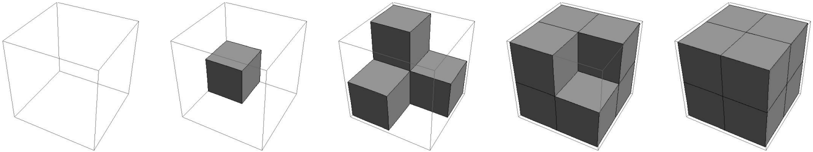
q^0 q^1 q^2 q^3

Let $T(n)$ denote set of TSPPs with largest part at most n.



q^0 q^1 q^2 q^3 q^4

Let $T(n)$ denote set of TSPPs with largest part at most n.



$$q^0 \quad + \quad q^1 \quad + \quad q^2 \quad + \quad q^3 \quad + \quad q^4 \quad =$$

$$= 1 + q + q^2 + q^3 + q^4 = \frac{1 - q^5}{1 - q}$$

q-TSPP conjecture: $$\sum_{\pi \in T(n)} q^{|\pi / S_3|} = \prod_{1 \leq i \leq j \leq k \leq n} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}}$$

Stembridge's theorem: $$|T(n)| = \prod_{1 \leq i \leq j \leq k \leq n} \frac{i + j + k - 1}{i + j + k - 2}$$

The q-TSPP conjecture

Conjectured independently by
George Andrews and David Robbins (ca. 1983)

Last surviving conjecture of the collection by Richard Stanley:
A baker's dozen of conjectures concerning plane partitions (1986)
(alternating sign matrix conjecture, TSPP conjecture, etc.)

Conjecture 7. (see [11, Case 4]). The number of totally symmetric plane partitions with largest part $\leq$ n is equal to

$$T_n = \prod_{1 \leq i \leq j \leq k \leq n} \frac{i+j+k-1}{i+j+k-2} .$$

The q-TSPP conjecture

Conjectured independently by
George Andrews and David Robbins (ca. 1983)

Last surviving conjecture of the collection by Richard Stanley:
A baker's dozen of conjectures concerning plane partitions (1986)
(alternating sign matrix conjecture, TSPP conjecture, etc.)

Note. All quantities arising in connection with Conjecture 7 have natural q-analogues. The q-analogue of T_n is

$$T_n(q) = \prod_{1 \le i \le j \le k \le n} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}$$

The q-analogue of the number of totally symmetric plane partitions with largest part $\le n$ is the polynomial $N_G'(B;q)$ defined in [11], where $B = B(n,n,n)$ and $G = S_3$.

All these problems had been solved, except one: q-TSPP.

Determinantal formulation

Also in Stanley's paper, we find:

<u>Note</u>. It is not hard to show that the number of totally symmetric plane partitions with largest part $\leq$ n is also equal to

• • •

c) the sum of the minors of all orders (including the void minor equal to 1) of the matrix whose (i,j)-entry is $\binom{i}{j}$ for $0 \leq i, j \leq n-1$.

Okada's determinant

Soichi Okada: *On the generating functions for certain classes of plane partitions*, Journal of Combinatorial Theory, Series A (1989).

Rewrite "the sum of all minors" as a single determinant!

The q-TSPP conjecture is true if

$$\det\left(a_{i,j}\right)_{1\le i,j\le n} = \prod_{1\le i\le j\le k\le n} \left(\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}\right)^2 =: b_n.$$

where

$$a_{i,j} := q^{i+j-1}\left(\begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_q + q\begin{bmatrix} i+j-1 \\ i \end{bmatrix}_q\right) + (1+q^i)\delta_{i,j} - \delta_{i,j+1}.$$

Zeilberger's holonomic ansatz

Doron Zeilberger: *The HOLONOMIC ANSATZ II. Automatic DISCOVERY(!) and PROOF(!!) of Holonomic Determinant Evaluations*, Annals of Combinatorics (2007).

Problem: Given $a_{i,j}$ and $b_n \neq 0$. Show $\det\,(a_{i,j})_{1\le i,j\le n} = b_n$.

Method: "Pull out of the hat" a function $c_{n,j}$ and prove

$$\begin{aligned} c_{n,n} &= 1 && (n \geq 1), \\ \sum_{j=1}^{n} c_{n,j} a_{i,j} &= 0 && (1 \leq i < n), \\ \sum_{j=1}^{n} c_{n,j} a_{n,j} &= \frac{b_n}{b_{n-1}} && (n \geq 1). \end{aligned}$$

Then $\det\,(a_{i,j})_{1\le i,j\le n} = b_n$ holds.

Zeilberger's holonomic ansatz

Laplace expansion w.r.t. the n-th row:

$$b_n \quad = \det\left(a_{i,j}\right)_{1\leq i,j\leq n} \qquad = \sum_{j=1}^{n} a_{n,j}\,(-1)^{n+j} M_{n,j}$$

Zeilberger's holonomic ansatz

Laplace expansion w.r.t. the n-th row:

$$\frac{b_n}{b_{n-1}} = \frac{\det\left(a_{i,j}\right)_{1\le i,j\le n}}{b_{n-1}} \qquad = \sum_{j=1}^{n} a_{n,j}\, \underbrace{\frac{(-1)^{n+j} M_{n,j}}{b_{n-1}}}_{=:\, c_{n,j}}$$

Zeilberger's holonomic ansatz

Laplace expansion w.r.t. the n-th row:

$$\frac{b_n}{b_{n-1}} = \frac{\det\left(a_{i,j}\right)_{1\le i,j\le n}}{b_{n-1}} = \sum_{j=1}^{n} a_{n,j} \underbrace{\frac{(-1)^{n+j} M_{n,j}}{b_{n-1}}}_{=:\, c_{n,j}}$$

$$c_{n,n} = \frac{(-1)^{n+n} M_{n,n}}{b_{n-1}}$$

Zeilberger's holonomic ansatz

Laplace expansion w.r.t. the n-th row:

$$\frac{b_n}{b_{n-1}} = \frac{\det\,(a_{i,j})_{1\le i,j\le n}}{b_{n-1}} \qquad = \sum_{j=1}^{n} a_{n,j}\, \underbrace{\frac{(-1)^{n+j} M_{n,j}}{b_{n-1}}}_{=:\, c_{n,j}}$$

$$c_{n,n} = 1$$

Zeilberger's holonomic ansatz

Laplace expansion w.r.t. the n-th row:

$$\frac{b_n}{b_{n-1}} = \frac{\det\left(a_{i,j}\right)_{1\le i,j\le n}}{b_{n-1}} \qquad = \sum_{j=1}^{n} a_{n,j}\, \underbrace{\frac{(-1)^{n+j} M_{n,j}}{b_{n-1}}}_{=:\, c_{n,j}}$$

$$c_{n,n} = 1$$

Now copy the i-th row $(1 \le i < n)$ into the n-th row:

$$0 = \sum_{j=1}^{n} a_{i,j}\, (-1)^{n+j} M_{n,j}$$

Zeilberger's holonomic ansatz

Laplace expansion w.r.t. the n-th row:

$$\frac{b_n}{b_{n-1}} = \frac{\det\left(a_{i,j}\right)_{1\le i,j\le n}}{b_{n-1}} \qquad = \sum_{j=1}^{n} a_{n,j} \underbrace{\frac{(-1)^{n+j} M_{n,j}}{b_{n-1}}}_{=:\, c_{n,j}}$$

$$c_{n,n} = 1$$

Now copy the i-th row $(1 \le i < n)$ into the n-th row:

$$0 = \sum_{j=1}^{n} a_{i,j} \underbrace{\frac{(-1)^{n+j} M_{n,j}}{b_{n-1}}}_{=\, c_{n,j}}$$

Zeilberger's holonomic ansatz

Laplace expansion w.r.t. the n-th row:

$$\frac{b_n}{b_{n-1}} = \frac{\det\left(a_{i,j}\right)_{1\leq i,j\leq n}}{b_{n-1}} \qquad = \sum_{j=1}^{n} a_{n,j} c_{n,j}$$

$$c_{n,n} = 1$$

Now copy the i-th row $(1 \leq i < n)$ into the n-th row:

$$0 = \sum_{j=1}^{n} a_{i,j} c_{n,j}$$

Zeilberger's holonomic ansatz

Laplace expansion w.r.t. the n-th row:

$$\frac{b_n}{b_{n-1}} = \frac{(q^{2n};q)_n^2}{(q^n;q^2)_n^2} \qquad\qquad = \sum_{j=1}^{n} a_{n,j} c_{n,j}$$

$$c_{n,n} = 1$$

Now copy the i-th row $(1 \leq i < n)$ into the n-th row:

$$0 = \sum_{j=1}^{n} a_{i,j} c_{n,j}$$

Zeilberger's holonomic ansatz

Problem: Given $a_{i,j}$ and $b_n \neq 0$. Show $\det(a_{i,j})_{1\le i,j\le n} = b_n$.

Method: "Pull out of the hat" a function $c_{n,j}$ and prove

$$\begin{aligned} c_{n,n} &= 1 && (n \ge 1), \\ \sum_{j=1}^{n} c_{n,j} a_{i,j} &= 0 && (1 \le i < n), \\ \sum_{j=1}^{n} c_{n,j} a_{n,j} &= \frac{b_n}{b_{n-1}} && (n \ge 1). \end{aligned}$$

Then $\det(a_{i,j})_{1\le i,j\le n} = b_n$ holds.

Advocatus Diaboli

What if $\det\,(a_{i,j})_{1\le i,j\le m} = 0$ for some m???

Then $c_{n,j}$ is not uniquely determined!

Proof is wrong!

Advocatus Diaboli

What if $\det(a_{i,j})_{1\le i,j\le m} = 0$ for some m???

Then $c_{n,j}$ is not uniquely determined!

Proof is wrong!

No! Argue by induction on n.

Holonomic systems

Of course, it is unlikely to get a closed-form description for $c_{n,j}$!

Holonomic systems

Of course, it is unlikely to get a closed-form description for $c_{n,j}$!

Instead we aim at some "suitable description", viz. implicitly via linear recurrences ("holonomic system") plus initial values.

Holonomic systems

Of course, it is unlikely to get a closed-form description for $c_{n,j}$!

Instead we aim at some "suitable description", viz. implicitly via linear recurrences ("holonomic system") plus initial values.

Example: The binomial coefficient $f_{n,k} = \binom{n}{k}$ can be described by

$$\begin{aligned} (n-k+1)f_{n+1,k} &= (n+1)f_{n,k} \\ (k+1)f_{n,k+1} &= (n-k)f_{n,k} \\ f_{0,0} &= 1 \end{aligned}$$

Holonomic systems

Of course, it is unlikely to get a closed-form description for $c_{n,j}$!

Instead we aim at some "suitable description", viz. implicitly via linear recurrences ("holonomic system") plus initial values.

Example: The binomial coefficient $f_{n,k} = \binom{n}{k}$ can be described by

$$\begin{aligned} (n-k+1)f_{n+1,k} &= (n+1)f_{n,k} \\ (k+1)f_{n,k+1} &= (n-k)f_{n,k} \\ f_{0,0} &= 1 \end{aligned}$$

Analogously, we get for the q-binomial coefficient $\bar{f}_{n,k} = \begin{bmatrix} n \\ k \end{bmatrix}_q$:

$$\begin{aligned} (q^{n+1}-q^k)\bar{f}_{n+1,k} &= (q^{k+n+1}-q^k)\bar{f}_{n,k} \\ (q^{2k+1}-q^k)\bar{f}_{n,k+1} &= (q^n-q^k)\bar{f}_{n,k} \\ \bar{f}_{0,0} &= 1 \end{aligned}$$

Holonomic systems

Of course, it is unlikely to get a closed-form description for $c_{n,j}$!

Instead we aim at some "suitable description", viz. implicitly via linear recurrences ("holonomic system") plus initial values.

Example: The binomial coefficient $f_{n,k} = \binom{n}{k}$ can be described by

$$\begin{aligned}(n-k+1)f_{n+1,k} &= (n+1)f_{n,k}\\ (k+1)f_{n,k+1} &= (n-k)f_{n,k}\\ f_{0,0} &= 1\end{aligned}$$

All linear combinations of shifts are again valid recurrences:

$$\begin{array}{l}(n-k)f_{n+1,k+1} - (n+1)f_{n,k+1} = 0\\ (k+1)f_{n+1,k+1} - (n-k+1)f_{n+1,k} = 0\\ \hline (n+1)f_{n+1,k+1} - (n-k+1)f_{n+1,k} - (n+1)f_{n,k+1} = 0\end{array}$$

They form a left ideal in some noncommutative operator algebra.

Holonomic systems

Of course, it is unlikely to get a closed-form description for $c_{n,j}$!

Instead we aim at some "suitable description", viz. implicitly via linear recurrences ("holonomic system") plus initial values.

But there is no reason why $c_{n,j}$ should admit such a recursive description.

Guessing

Manuel Kauers guessed some recurrences for $c_{n,j}$.

Guessing

Manuel Kauers guessed some recurrences for $c_{n,j}$.

Their Gröbner basis has the form

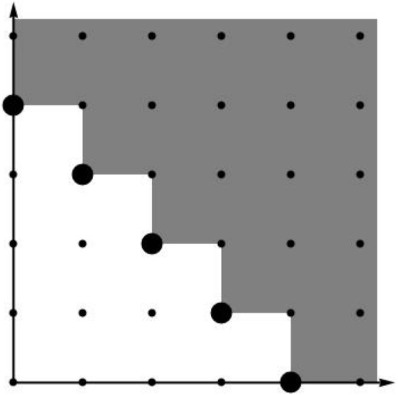
$$\begin{aligned}
\bigcirc c_{n,j+4} &= \bigcirc c_{n,j} + \bigcirc c_{n,j+1} + \bigcirc c_{n,j+2} + \bigcirc c_{n,j+3} \\
&\quad + \bigcirc\, c_{n+2,j} + \bigcirc c_{n+2,j+1} \\
\bigcirc c_{n+1,j+3} &= \bigcirc c_{n,j} + \bigcirc c_{n,j+1} + \bigcirc c_{n,j+2} + \bigcirc c_{n,j+3} \\
&\quad + \bigcirc\, c_{n+1,j} + \bigcirc c_{n+1,j+1} + \bigcirc c_{n+1,j+2} \\
&\quad + \bigcirc\, c_{n+2,j} + \bigcirc c_{n+2,j+1} + \bigcirc c_{n+3,j} \\
\bigcirc c_{n+2,j+2} &= \bigcirc c_{n,j} + \bigcirc c_{n,j+1} + \bigcirc c_{n,j+2} + \bigcirc c_{n,j+3} \\
&\quad + \bigcirc\, c_{n+2,j} + \bigcirc c_{n+2,j+1} \\
\bigcirc c_{n+3,j+1} &= \bigcirc c_{n,j} + \bigcirc c_{n,j+1} + \bigcirc c_{n,j+2} + \bigcirc c_{n,j+3} \\
&\quad + \bigcirc\, c_{n+1,j} + \bigcirc c_{n+1,j+1} + \bigcirc c_{n+1,j+2} \\
&\quad + \bigcirc\, c_{n+2,j} + \bigcirc c_{n+2,j+1} + \bigcirc c_{n+3,j} \\
\bigcirc c_{n+4,j} &= \bigcirc c_{n,j} + \bigcirc c_{n,j+1} + \bigcirc c_{n,j+2} + \bigcirc c_{n,j+3} \\
&\quad + \bigcirc\, c_{n+2,j} + \bigcirc c_{n+2,j+1}
\end{aligned}$$

where each $\bigcirc$ is a polynomial in $\mathbb{Q}[q, q^j, q^n]$ of total degree ≤ 100.

Guessing

Manuel Kauers guessed some recurrences for $c_{n,j}$.

The staircase of the Gröbner basis:



Guessing

Manuel Kauers guessed some recurrences for $c_{n,j}$.

The total size is 244MB (several 1000 pages of paper)!

Great! We found the certificate for the determinant evaluation!

Advocatus Diaboli

The guessed recurrences can be artifacts, that do not describe the true function $c_{n,j}$!!!

Artifacts?

The guessed recurrences are very unlikely to be artifacts for several reasons:

- solutions of a dense overdetermined linear systems
- many polynomial coefficients factor nicely
- recurrences produce correct values for $c_{n,j}$ that were not used for guessing

Advocatus Diaboli

Convincing, but not a proof!

And even if the recursive description of $c_{n,j}$ is correct, this wouldn't prove anything yet!!!

Show:

$$c_{n,n} = 1$$

$$\sum_{j=1}^{n} c_{n,j} a_{i,j} = 0$$

$$\sum_{j=1}^{n} c_{n,j} a_{n,j} = \frac{b_n}{b_{n-1}}$$

The first identity

Prove the identities using the recursive description of $c_{n,j}$.

How to prove $c_{n,n} = 1$ **for all** $n \geq 1$?

The first identity

Prove the identities using the recursive description of $c_{n,j}$.

How to prove $c_{n,n} = 1$ **for all** $n \geq 1$?

- We find an element in the annihilating ideal of $c_{n,j}$ of the form

$$p_v c_{n+v,j+v} = p_{v-1} c_{n+v-1,j+v-1} + \cdots + p_1 c_{n+1,j+1} + p_0 c_{n,j}$$

  with $v \in \mathbb{N}$ and $p_i \in \mathbb{Q}[q, q^j, q^n]$.
- Substituting $j \to n$ yields a recurrence for the diagonal sequence $c_{n,n}$.
- Show that the corresponding operator factors into $P_1 P_2$ where P_2 corresponds to $c_{n+1,n+1} = c_{n,n}$.
- Show that $c_{1,1} = \cdots = c_{v,v} = 1$.

$\longrightarrow$ works with $v = 7$.

Advocatus Diaboli

The leading coefficient p_7 could have singularities!!!

Advocatus Diaboli

The leading coefficient p_7 could have singularities!!!

For $n \geq 7$ we have $p_7(q, q^n) \neq 0$.

The third identity

Recall:

$$\sum_{j=1}^{n} c_{n,j} a_{n,j} = \frac{b_n}{b_{n-1}}$$

with

$$a_{i,j} = q^{i+j-1} \left(\begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_q + q \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_q \right) + (1+q^i)\delta_{i,j} - \delta_{i,j+1}.$$

gives

$$(1+q^n) - c_{n,n-1} + \sum_{j=1}^{n} c'_{n,j} = \frac{b_n}{b_{n-1}}$$

with

$$c'_{n,j} = q^{n+j-1} \left(\begin{bmatrix} n+j-2 \\ n-1 \end{bmatrix}_q + q \begin{bmatrix} n+j-1 \\ n \end{bmatrix}_q \right) c_{n,j}$$

The third identity

How to prove $$(1+q^n) - c_{n,n-1} + \sum_{j=1}^{n} c'_{n,j} = \frac{b_n}{b_{n-1}}\ ?$$

- Compute an annihilating ideal for $c'_{n,j}$ via closure properties.
- Find a relation in this ideal of the form

$$p_v c'_{n+v,j} + \cdots + p_1 c'_{n+1,j} + p_0 c'_{n,j} = t_{n,j+1} - t_{n,j}$$

  where the $p_v, \ldots, p_0$ are rational functions in $\mathbb{Q}(q, q^n)$ and $t_{n,j}$ is a $\mathbb{Q}(q, q^j, q^n)$-linear combination of certain shifts of $c'_{n,j}$.
- Creative telescoping yields a recurrence for the sum.

The third identity

How to prove $\quad (1+q^n) - c_{n,n-1} + \sum_{j=1}^{n} c'_{n,j} = \frac{b_n}{b_{n-1}}$?

- Compute an annihilating ideal for $c'_{n,j}$ via closure properties.
- Find a relation in this ideal of the form

$$p_v c'_{n+v,j} + \cdots + p_1 c'_{n+1,j} + p_0 c'_{n,j} = t_{n,j+1} - t_{n,j}$$

  where the $p_v, \ldots, p_0$ are rational functions in $\mathbb{Q}(q, q^n)$ and $t_{n,j}$ is a $\mathbb{Q}(q, q^j, q^n)$-linear combination of certain shifts of $c'_{n,j}$.
- Creative telescoping yields a recurrence for the sum.
- Closure properties yield a recurrence for the left-hand side.
- Recurrence for right-hand side is a right factor.
- Compare finitely many initial values (again $v = 7$).

A computational challenge

How to find the certificate

$$p_v(q, q^n)c'_{n+v,j} + \cdots + p_0(q, q^n)c'_{n,j} = t_{n,j+1} - t_{n,j}$$

where $t_{n,j} = r_1(q, q^n, q^j)c'_{n+3,j+2} + \cdots + r_{10}(q, q^n, q^j)c'_{n,j}$?

A computational challenge

How to find the certificate

$$p_v(q,q^n)c'_{n+v,j} + \cdots + p_0(q,q^n)c'_{n,j} = t_{n,j+1} - t_{n,j}$$

where $t_{n,j} = r_1(q,q^n,q^j)c'_{n+3,j+2} + \cdots + r_{10}(q,q^n,q^j)c'_{n,j}$?

Zeilberger's slow algorithm: eliminate (e.g. with Gröbner bases) the variable q^j.
Input recurrences have j-degrees between 24 and 30 (in the $q = 1$ case). After 48h, this was reduced to 23.
Estimate: 1677721600 days

A computational challenge

How to find the certificate

$$p_v(q,q^n)c'_{n+v,j} + \cdots + p_0(q,q^n)c'_{n,j} = t_{n,j+1} - t_{n,j}$$

where $t_{n,j} = r_1(q,q^n,q^j)c'_{n+3,j+2} + \cdots + r_{10}(q,q^n,q^j)c'_{n,j}$?

Takayama's algorithm: a faster variant which is also based on elimination.
Estimate: 52428800 days

A computational challenge

How to find the certificate

$$p_v(q,q^n)c'_{n+v,j} + \cdots + p_0(q,q^n)c'_{n,j} = t_{n,j+1} - t_{n,j}$$

where $t_{n,j} = r_1(q,q^n,q^j)c'_{n+3,j+2} + \cdots + r_{10}(q,q^n,q^j)c'_{n,j}$?

Chyzak's algorithm: ansatz with unknown $p_i(q,q^n)$ and $r_k(q,q^n,q^j)$. Leads to a coupled first-order parametrized linear system of q-difference equations.

Estimate: ∞?

A computational challenge

How to find the certificate

$$p_v(q,q^n)c'_{n+v,j} + \cdots + p_0(q,q^n)c'_{n,j} = t_{n,j+1} - t_{n,j}$$

where $t_{n,j} = r_1(q,q^n,q^j)c'_{n+3,j+2} + \cdots + r_{10}(q,q^n,q^j)c'_{n,j}$?

CK's polynomial ansatz: refine

$$r_k(q,q^n,q^j) = \sum_{l=0}^{L} r_{k,l}(q,q^n)(q^j)^l.$$

Leads to a linear system over $\mathbb{Q}(q,q^n)$.
We used this ansatz for proving TSPP (took about 40 days).
Estimate: 4000 days

A computational challenge

How to find the certificate

$$p_v(q,q^n)c'_{n+v,j} + \cdots + p_0(q,q^n)c'_{n,j} = t_{n,j+1} - t_{n,j}$$

where $t_{n,j} = r_1(q,q^n,q^j)c'_{n+3,j+2} + \cdots + r_{10}(q,q^n,q^j)c'_{n,j}$?

CK's rational ansatz: ansatz with

$$r_k(q,q^n,q^j) = \frac{\sum_{l=0}^{L} r_{k,l}(q,q^n)(q^j)^l}{d_k(q,q^n,q^j)}$$

where the denominators d_k can be "guessed" by looking at the leading coefficients of the Gröbner basis.
Leads to a linear system over $\mathbb{Q}(q,q^n)$ with 377 unknowns.

A computational challenge

Even generating this linear system (reducing the ansatz with the Gröbner basis) would already consume too much memory!

A computational challenge

Even generating this linear system (reducing the ansatz with the Gröbner basis) would already consume too much memory!

We use homomorphic images (modular computations):

- do all computations modulo some prime
- plug in concrete integral values for q and q^n
- requires special modular GB reduction
- fixing q and varying q^n (and vice versa) allows to estimate the necessary interpolation points:

A computational challenge

Even generating this linear system (reducing the ansatz with the Gröbner basis) would already consume too much memory!

We use homomorphic images (modular computations):

- do all computations modulo some prime
- plug in concrete integral values for q and q^n
- requires special modular GB reduction
- fixing q and varying q^n (and vice versa) allows to estimate the necessary interpolation points:
- 1167 interpolation points for q
- 363 interpolation points for q^n
- each case takes about a minute (GB reduction, linear solving, for sufficiently many primes)
- estimated computation time: $1167 \cdot 363 \cdot 60s = 294$ days!

Further improvements

- efficient Gröbner basis reduction procedure
- combination of arithmetic in $\mathbb{Q}$ and in $\mathbb{Z}_p$
- rewrite polynomials that have to be evaluated in compact form
- build matrices efficiently
- discard redundant equations
- compute parallel
- normalize w.r.t. a certain component in order to minimize the number of interpolation points
- guess small factors in the components of the solution

Guess small factors

Compute, for example,

- with $q = 19$
- modulo the prime 2147483629

Assume we obtain as solution (after factoring):

$$(q^n + 19)(q^n + 2147483628)(q^{2n} + 381q^n + 2147483610)\ldots$$

Presumably the true solution (for symbolic q and over $\mathbb{Q}$) is

$$(q^n + q)(q^n - 1)(q^{2n} + (q^2 + q + 1)q^n - q)\ldots$$

Many such small factors can be guessed from modular results!

All these optimizations reduced the actual computation to 35 days.

Advocatus Diaboli

The polynomial degrees of the solution are not known: not enough interpolation points???

The size of the integer coefficients is not known: not enough primes???

The guessed small factors can be wrong!!!

Advocatus Diaboli

The polynomial degrees of the solution are not known: not enough interpolation points???

The size of the integer coefficients is not known: not enough primes???

The guessed small factors can be wrong!!!

The final result was reduced once again with the Gröbner basis (non-modular) and yielded 0.

Result

The certificate for the third identity has a size of 7 Gigabytes.

Result

The certificate for the third identity has a size of 7 Gigabytes.

Its principal part confirms the conjectured evaluation

$$b_n = \prod_{1 \le i \le j \le k \le n} \left(\frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}} \right)^2 .$$

The second identity

$$\sum_{j=1}^{n} c_{n,j} a_{i,j} = 0 \qquad (1 \leq i < n).$$

Strategy similar as before, but one variable more. This means:

- linear system over $\mathbb{Q}(q, q^i, q^n)$
- (at least) two creative telescoping relations are necessary

Advocatus Diaboli

Zeros in the denominators of the delta part???

Singularities in the leading coefficients of the principal parts???

Advocatus Diaboli

Zeros in the denominators of the delta part???

Singularities in the leading coefficients of the principal parts???

There are only finitely many which can be checked separately.

Quod erat demonstrandum.

THEOREM (KKZ). Let π/S_3 denote the set of orbits of a plane partition π under the action of the symmetric group S_3.
Then the orbit-counting generating function is given by

$$\sum_{\pi \in T(n)} q^{|\pi/S_3|} = \prod_{1 \le i \le j \le k \le n} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}}$$

where $T(n)$ denotes the set of totally symmetric plane partitions with largest part at most n.

Acknowledgements

Thanks to

- the RISC sysadmins for providing the necessary resources,
- Victor Moll for hosting me in New Orleans,
- Manuel Kauers for guessing and advising,
- Doron Zeilberger for offering a noble prize!

Acknowledgements

Thanks to

- the RISC sysadmins for providing the necessary resources,
- Victor Moll for hosting me in New Orleans,
- Manuel Kauers for guessing and advising,
- Doron Zeilberger for offering a noble prize!

No thanks to

- the cleaning professional who unplugged and messed up my computations!