On the arithmetical nature of Euler's constant

Hoang Ngoc Minh,
Université Lille 2-LIPN,
1, Place Déliot, 59024 Lille Cédex, France.

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G.H.E. Duchamp, S. Goodenough, C. Tollu, L. Poinsot
Université Paris-Nord-LIPN)

Summary

1. Introduction
   1.1 From infinitesimal braid relations to *KZ* system to structure of polylogarithms,
   1.2 Polynomial relations among polyzêtas by computer,
   1.3 Irreducible polyzêtas by computer.
2. Sociological approach : noncommutative generating series technology
   2.1 Results *à la* Abel,
   2.2 Generalized Euler constants,
   2.3 Group of associators and relations among their coefficients,
3. Consequences
   3.1 Kernel of ζ,
   3.2 Structure of polyzêtas,
   3.3 Arithmetical nature of γ.

INFINITESIMAL BRAID RELATIONS
&
STRUCTURE OF POLYLOGARITHMS

Infinitesimal braid relation & Knizhnik-Zamolodchikov syst.

In 1986, Drinfel'd introduced the differential system associated to the Lie algebra of pure braid groups $\mathcal{T}_n$:

$$(KZ_n) \qquad dF(z_1,\ldots,z_n) = \Omega_n(z_1,\ldots,z_n)F(z_1,\ldots,z_n),$$

where

$$\Omega_n(z_1,\ldots,z_n) = \sum_{1\le i<j\le n} \frac{t_{i,j}}{2\mathrm{i}\pi}\,\omega_{i,j}(z_1,\ldots,z_n) = \sum_{1\le i<j\le n} \frac{t_{i,j}}{2\mathrm{i}\pi}\frac{d(z_i-z_j)}{z_i-z_j}.$$

The flatness condition, $d\Omega_n - \Omega_n\wedge\Omega_n = 0$, is equivalent to the infinitesimal braid relations generated by $\{t_{i,j}\}_{1\le i,j\le n}$:

$$\begin{aligned} t_{i,j} &= 0 && \text{for} && i=j,\\ t_{i,j} &= t_{j,i} && \text{for} && i\neq j,\\ [t_{i,j}, t_{i,k}+t_{j,k}] &= 0 && \text{for distinct} && i,j,k,\\ [t_{i,j}, t_{k,l}] &= 0 && \text{for distinct} && i,j,k,l. \end{aligned}$$

Hence, the differential system (KZ_n) is completely integrable over $\{(z_1,\ldots,z_n)\in\mathbb{C}^n | z_i\neq z_j \text{ for } i\neq j\}$.

Examples

- $\mathcal{T}_2 = \{t_{1,2}\}$.

$$\Omega_2(z) = \frac{t_{1,2}}{2\mathrm{i}\pi}\frac{d(z_1 - z_2)}{z_1 - z_2}, \qquad F(z_1, z_2) = (z_1 - z_2)^{t_{1,2}/2\mathrm{i}\pi}.$$

- $\mathcal{T}_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\},\ [t_{1,3}, t_{1,2} + t_{2,3}] = [t_{2,3}, t_{1,2} + t_{1,3}] = 0.$

$$\Omega_3(z) = \frac{1}{2\mathrm{i}\pi}\left[t_{1,2}\frac{d(z_1 - z_2)}{z_1 - z_2} + t_{1,3}\frac{d(z_1 - z_3)}{z_1 - z_3} + t_{2,3}\frac{d(z_2 - z_3)}{z_2 - z_3}\right].$$

$$F(z_1, z_2, z_3) = G\left(\frac{z_1 - z_2}{z_1 - z_3}\right)(z_1 - z_3)^{(t_{1,2}+t_{1,3}+t_{2,3})/2\mathrm{i}\pi},$$

where G satisfies the following differential equation

$$(DE) \qquad dG = [x_0\,\omega_0(z) + x_1\,\omega_1(z)]G,$$

$$\text{with} \qquad x_0 := \frac{t_{1,2}}{2\mathrm{i}\pi}, \qquad \omega_0(z) := \frac{dz}{z}$$

$$\text{and} \qquad x_1 := -\frac{t_{2,3}}{2\mathrm{i}\pi}, \qquad \omega_1(z) := \frac{dz}{1 - z}.$$

Iterated integral along a path and dilogarithms

Let $\omega_0(z) = dz/z$ and $\omega_1(z) = dz/(1-z)$. The iterated integral over ω_0, ω_1 associated to $w = x_{i_1} \cdots x_{i_k} \in X^*$ is defined by

$$\alpha_{z_0}^{z}(\varepsilon) = 1 \quad \text{and} \quad \alpha_{z_0}^{z}(x_{i_1} \ldots x_{i_k}) = \int_{z_0}^{z} \ldots \int_{z_0}^{z_{k-1}} \omega_{i_1}(z_1) \ldots \omega_{i_k}(z_k).$$

Example

$$\begin{aligned}
\alpha_0^z(x_0 x_1) &= \int_0^z \int_0^s \omega_0(s)\omega_1(t) \\
&= \int_0^z \int_0^s \frac{ds}{s} \frac{dt}{1-t} \\
&= \int_0^z \frac{ds}{s} \int_0^s dt \sum_{k\geq 0} t^k \\
&= \sum_{k\geq 1} \int_0^z ds \frac{s^{k-1}}{k} \\
&= \sum_{k\geq 1} \frac{z^k}{k^2} \\
&= \mathrm{Li}_2(z).
\end{aligned}$$

Polylogarithms, harmonic sums and polyzêtas

Classical cases ($N \in \mathbb{N}_+, r > 0$ and $|z| < 1$) :

$$\mathrm{Li}_r(z) = \alpha_0^z(x_0^{r-1}x_1) = \sum_{n\geq 1} \frac{z^n}{n^r} \quad \text{and} \quad \mathrm{H}_r(N) = \sum_{n=1}^{N} \frac{1}{n^r}.$$

Generalization to multi-indices $\mathbf{s} = (s_1, \ldots, s_r)$:

$$\alpha_0^z(x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1) = \mathrm{Li}_{\mathbf{s}}(z) = \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_r^{s_r}},$$

$$\mathrm{P}_{\mathbf{s}}(z) = \frac{\mathrm{Li}_{\mathbf{s}}(z)}{1-z} = \sum_{N \geq 0} \mathrm{H}_{\mathbf{s}}(N)\, z^N, \text{ where } \mathrm{H}_{\mathbf{s}}(N) = \sum_{n_1 > \ldots > n_r = 1}^{N} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}.$$

If $s_1 > 1$, by Abel's theorem, one has

$$\lim_{z \to 1} \mathrm{Li}_{\mathbf{s}}(z) = \lim_{N \to \infty} \mathrm{H}_{\mathbf{s}}(N) = \zeta(\mathbf{s}) = \sum_{n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}$$

else ?

Starting with Euler-Maclaurin summation formula

$$\sum_{N\geq n\geq 1}\frac{1}{n}=\log N+\gamma-\sum_{j=1}^{k-1}\frac{B_j}{j}\frac{1}{N^j}+O\left(\frac{1}{N^k}\right),$$

$$\sum_{N\geq n\geq 1}\frac{1}{n^r}=\zeta(r)-\frac{N^{1-r}}{(r-1)}-\sum_{j=r}^{k-1}\frac{B_{j-r+1}}{j-r+1}\binom{k-1}{j-1}\frac{1}{N^j}+O\left(\frac{1}{N^k}\right).$$

Theorem (Costermans, Enjalbert & HNM, 2005)

For any multi-indices $\mathbf{s}=(s_1,\ldots,s_r)$*, there exists algorithmically computable* $c_j\in\mathcal{Z},\alpha_j\in\mathbb{Z},\beta_j\in\mathbb{N}$ *and* $b_i\in\mathcal{Z}',\kappa_i\in\mathbb{N},\eta_i\in\mathbb{Z}$ *such that*

$$\mathrm{Li}_{\mathbf{s}}(z)\quad\widetilde{z\rightarrow 1}\quad\sum_{j=0}^{+\infty}c_j(1-z)^{\alpha_j}\log^{\beta_j}(1-z),$$

$$\mathrm{H}_{\mathbf{s}}(N)\quad\widetilde{N\rightarrow +\infty}\quad\sum_{i=0}^{+\infty}b_iN^{\eta_i}\log^{\kappa_i}(N),$$

where $\mathcal{Z}$ *is the* $\mathbb{Q}$*-algebra generated by convergent polyzêtas and* $\mathcal{Z}'$ *is the* $\mathbb{Q}[\gamma]$*-algebra generated by convergent polyzêtas.*

Examples

Example (convergent case)

$$\begin{aligned}
\mathrm{Li}_{2,1}(z) &= \zeta(3) + (1-z)\log(1-z) - 1 - \frac{1}{2}(1-z)\log^2(1-z) \\
&+ (1-z)^2\left(-\frac{1}{4}\log^2(1-z) + \frac{1}{4}\log(1-z)\right) + \dots, \\
\mathrm{H}_{2,1}(N) &= \zeta(3) - \frac{\log(N) + 1 + \gamma}{N} + \frac{\log(N)}{2N} + \dots.
\end{aligned}$$

Example (divergent case)

$$\begin{aligned}
\mathrm{Li}_{1,2}(z) &= 2 - 2\zeta(3) - \zeta(2)\log(1-z) - 2(1-z)\log(1-z) \\
&+ (1-z)\log^2(1-z) + (1-z)^2\left(\frac{\log^2(1-z)}{2} - \frac{\log(1-z)}{2}\right) + \dots, \\
\mathrm{H}_{1,2}(N) &= \zeta(2)\gamma - 2\zeta(3) + \zeta(2)\log(N) + \frac{\zeta(2)+2}{2N} + \dots,
\end{aligned}$$

$$\zeta(2)\gamma = .94948171111498152454556410223170493364000594947366\dots!$$

Encoding the multi-indices by words

$Y = \{y_k | k \in \mathbb{N}_+\}$ $(y_1 < y_2 < \ldots)$ and $X = \{x_0, x_1\}$ $(x_0 < x_1)$.
Y^* (resp. X^*) : monoid generated by Y (resp. X).

$$\mathbf{s} = (s_1, \ldots, s_r) \leftrightarrow w = y_{s_1} \ldots y_{s_r} \leftrightarrow w = x_0^{s_1 - 1} x_1 \ldots x_0^{s_r - 1} x_1.$$

u and v are convergent if $s_1 > 1$. A divergent word is of the form

$$(\{1\}^k, s_{k+1}, \ldots, s_r) \leftrightarrow y_1^k y_{s_{k+1}} \ldots y_{s_r} \leftrightarrow x_1^k x_0^{s_{k+1}-1} x_1 \ldots x_0^{s_r - 1} x_1, \quad \text{for} \quad k \geq 1.$$

$$\begin{aligned}
\mathrm{Li}_w : w &\mapsto \mathrm{Li}_w(z) = \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_r^{s_r}}, \\
\zeta_w : w &\mapsto \zeta(w) = \sum_{n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}, \\
\mathrm{H}_w : w &\mapsto \mathrm{H}_w(N) = \sum_{N \geq n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}, \\
\mathrm{P}_w : w &\mapsto \mathrm{P}_w(z) = \sum_{N \geq 0} \mathrm{H}_w(N) z^N = \frac{\mathrm{Li}_w(z)}{1 - z}.
\end{aligned}$$

Let $\Pi_X : \mathbb{C}\langle\langle Y \rangle\rangle \longrightarrow \mathbb{C}\langle\langle X \rangle\rangle$ and $\Pi_Y : \mathbb{C}\langle\langle X \rangle\rangle \longrightarrow \mathbb{C}\langle\langle Y \rangle\rangle$ denote the "change" of alphabets over $\mathbb{C}\langle\langle X \rangle\rangle$ and $\mathbb{C}\langle\langle Y \rangle\rangle$ respectively.

Structure of polylogarithms

Theorem (HNM, van der Hoeven & Petitot, 1998)

Putting $\mathrm{Li}_{x_0}(z) = \log z$, $\mathrm{Li} : w \mapsto \mathrm{Li}_w$ *becomes an isomorphism from* $(\mathbb{C}\langle X\rangle, ш)$ *to* $(\mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*}, .)$.

Theorem (HNM, 2003)

$(\mathbb{C}\{\mathrm{P}_w\}_{w\in Y^*}, \odot) \cong (\mathbb{C}\langle Y\rangle, ⧢)$.

Extended to $\mathcal{C} = \mathbb{C}[z, z^{-1}, (1-z)^{-1}]$, we also get as consequence

- $\{\mathrm{Li}_w\}_{w\in X^*}$ (resp. $\{\mathrm{P}_w\}_{w\in Y^*}$) are $\mathcal{C}$-linearly independent.
  Then $\{\mathrm{H}_w\}_{w\in Y^*}$, are linearly independent.
- $\{\mathrm{Li}_l\}_{l\in\mathcal{L}ynX}$ (resp. $\{\mathrm{P}_l\}_{l\in\mathcal{L}ynY}$) are $\mathcal{C}$-algebraically independent.
  Then $\{\mathrm{H}_l\}_{l\in\mathcal{L}ynY}$, are algebraically independent.
- $\{\zeta(l)\}_{l\in\mathcal{L}ynX\setminus\{x_0,x_1\}(\text{resp. } \mathcal{L}ynY\setminus\{y_1\})}$, are generators of $\mathcal{Z}$.

Towards the structure of polyzêtas

Corollary

$\forall u, v \in X^*, \mathrm{Li}_u \mathrm{Li}_v = \mathrm{Li}_{u ш v} \Rightarrow \forall u, v \in x_0 X^* x_1, \zeta(u)\zeta(v) = \zeta(u ш v).$

Example

$$\begin{aligned} x_0x_1 ш x_0^2x_1 &= x_0x_1x_0^2x_1 + 3x_0^2x_1x_0x_1 + 6x_0^3x_1^2, \\ \mathrm{Li}_2 \mathrm{Li}_3 &= \mathrm{Li}_{2,3} + 3\,\mathrm{Li}_{3,2} + 6\,\mathrm{Li}_{4,1}, \\ \zeta(2)\zeta(3) &= \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1). \end{aligned}$$

Corollary

$\forall u, v \in Y^*, \mathrm{H}_u\mathrm{H}_v = \mathrm{H}_{u ⧢ v} \Rightarrow \forall u, v \in Y^* \setminus y_1 Y^*, \zeta(u)\zeta(v) = \zeta(u ⧢ v).$

Example

$$\begin{aligned} y_2 ⧢ y_3 &= y_2y_3 + y_3y_2 + y_5, \\ \mathrm{P}_{y_2} \odot \mathrm{P}_{y_3} &= \mathrm{P}_{y_2y_3} + \mathrm{P}_{y_3y_2} + \mathrm{P}_{y_5}, \\ \mathrm{H}_2\mathrm{H}_3 &= \mathrm{H}_{2,3} + \mathrm{H}_{3,2} + \mathrm{H}_5, \\ \zeta(2)\zeta(3) &= \zeta(2,3) + \zeta(3,2) + \zeta(5). \end{aligned}$$

$$\left.\begin{aligned} \zeta(2)\zeta(3) &= \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1) \\ \zeta(2)\zeta(3) &= \zeta(2,3) + \zeta(3,2) + \zeta(5) \end{aligned}\right\} \Rightarrow \zeta(5) = 2\zeta(3,2) + 6\zeta(4,1).$$

SOCIOLOGICAL APPROACH : NONCOMMUTATIVE GENERATING SERIES TECHNOLOGY

Noncommutative generating series

Definition

$$\mathrm{L}(z) := \sum_{w \in X^*} \mathrm{Li}_w(z)\, w \quad \text{and} \quad \mathrm{H}(N) := \sum_{w \in Y^*} \mathrm{H}_w(N)\, w.$$

Let $\mathcal{L}ynX$ (resp. $\mathcal{L}ynX$) be the transcendence basis of $(\mathbb{C}\langle X \rangle, \shuffle)$ (resp. $(\mathbb{C}\langle Y \rangle, \stuffle)$) and let $\{\hat{l}\}_{l \in \mathcal{L}ynX}$ (resp. $\{\hat{l}\}_{l \in \mathcal{L}ynY}$) be its dual basis. Then

Theorem (HNM, 2009)

L and H are group-like and

$$\mathrm{L}(z) = e^{x_1 \log \frac{1}{1-z}} \mathrm{L}_{\mathrm{reg}}(z) e^{x_0 \log z} \quad \textit{and} \quad \mathrm{H}(N) = e^{\mathrm{H}_1(N)\, y_1} \mathrm{H}_{\mathrm{reg}}(N),$$

$$\textit{where} \quad \mathrm{L}_{\mathrm{reg}}(z) = \prod_{\substack{l \in \mathcal{L}ynX \\ l \neq x_0, x_1}}^{\searrow} e^{\mathrm{Li}_l(z)\, \hat{l}} \quad \textit{and} \quad \mathrm{H}_{\mathrm{reg}}(N) = \prod_{\substack{l \in \mathcal{L}ynY \\ l \neq y_1}}^{\nearrow} e^{\mathrm{H}_l(N)\, \hat{l}}.$$

Definition

$$Z_{\shuffle} := \mathrm{L}_{\mathrm{reg}}(1) \quad \text{and} \quad Z_{\stuffle} := \mathrm{H}_{\mathrm{reg}}(\infty).$$

Results *à la* Abel

Theorem (HNM, 2005)

$$\lim_{z\to 1} e^{y_1 \log\frac{1}{1-z}} \Pi_Y \mathrm{L}(z) = \lim_{N\to\infty} \left[\sum_{k\geq 0} \mathrm{H}_{y_1^k}(N)(-y_1)^k\right] \mathrm{H}(N) = \Pi_Y Z_{ш}.$$

$$\Rightarrow \mathrm{H}(N) \underset{N\to\infty}{\widetilde{}} \exp\left[-\sum_{k\geq 1} \mathrm{H}_{y_k}(N) \frac{(-y_1)^k}{k}\right] \Pi_Y Z_{ш}.$$

Let

$$B(y_1) := \exp\left[-\gamma y_1 + \sum_{k\geq 2} \zeta(k) \frac{(-y_1)^k}{k}\right] \quad \text{and} \quad B'(y_1) := e^{\gamma y_1} B(y_1).$$

Corollary

For any $w \in Y^* \setminus \{y_1\}$, let γ_w be the constant associated to H_w. Let

$$Z_\gamma := \sum_{w\in Y^*} \gamma_w \, w.$$

Then Z_γ is group-like and

$$Z_\gamma = B(y_1) \Pi_Y Z_{ш} = e^{\gamma y_1} Z_{⧢}.$$

Generalized Euler constants by computer

By specializing at $t_1 = \gamma$ and for $l \geq 2$, $t_l = (-1)^{l-1}(l-1)!\zeta(l)$ in the Bell polynomials $b_{n,k}(t_1, \ldots, t_k)$, we get

Corollary

$$\gamma_{y_1^k} = \sum_{\substack{s_1, \ldots, s_k > 0 \\ s_1 + \ldots + k s_k = k+1}} \frac{(-1)^k}{s_1! \ldots s_k!} (-\gamma)^{s_1} \left(-\frac{\zeta(2)}{2}\right)^{s_2} \ldots \left(-\frac{\zeta(k)}{k}\right)^{s_k}.$$

$$\gamma_{y_1^k w} = \sum_{i=0}^{k} \frac{\zeta(x_0[(-x_1)^{k-i} ш \Pi_X w])}{i!} \left[\sum_{j=1}^{i} b_{i,j}(\gamma, -\zeta(2), 2\zeta(3), \ldots)\right].$$

Example (by computer)

$$\begin{aligned}
\gamma_{1,1} &= [\gamma^2 - \zeta(2)]/2 = -.65587807152025388107701951514539048127976638047858\ldots \\
\gamma_{1,1,1} &= [\gamma^3 - 3\zeta(2)\gamma + 2\zeta(3)]/6 = .42002635034095235529003934875429818711394500401111e-1\ldots \\
\gamma_{1,1,1,1} &= [80\zeta(3)\gamma - 60\zeta(2)\gamma^2 + 6\zeta(2)^2 + 10\gamma^4]/240 = .16653861138229148950170079510210523571778150224718. \\
\gamma_{1,7} &= \zeta(7)\gamma + \zeta(3)\zeta(5) - 54\zeta(2)^4/175 = -.43069288670051867025983665738799333032580571600387\ldots \\
\gamma_{1,1,6} &= 4\zeta(2)^3\gamma^2/35 + [\zeta(2)\zeta(5) + 2\zeta(3)\zeta(2)^2/5 - 4\zeta(7)]\gamma + \zeta(6,2) + 19\zeta(2)^4/35 + \zeta(2)\zeta(3)^2/2 - 4\zeta(3)\zeta(5) \\
&= -.22822808729950213397309900979654220767940888270983\ldots \\
\gamma_{1,1,1,5} &= 3\zeta(6,2)/4 - 14\zeta(3)\zeta(5)/3 + 3\zeta(2)\zeta(3)^2/4 + 809\zeta(2)^4/1400 \\
&+ \zeta(5)\gamma^3/6 + [\zeta(3)^2/4 - \zeta(2)^3/5]\gamma^2 - [2\zeta(7) - 3\zeta(2)\zeta(5)/2 + \zeta(3)\zeta(2)^2/10]\gamma \\
&= .19197163665101466877320445556293649152891629214906\ldots
\end{aligned}$$

Group of associators and relations among their coefficients

Proposition

Let A be a commutative $\mathbb{Q}$-algebra. Let $\Phi \in A\langle\langle X\rangle\rangle$ and $\Psi, \Psi' \in A\langle\langle Y\rangle\rangle$ be group-like such that

$$\Psi = B(y_1)\Pi_Y\Phi \quad \textit{and} \quad \Psi' = B'(y_1)\Pi_Y\Phi.$$

There exists an unique $C \in \mathcal{L}ie_A\langle\langle X\rangle\rangle$ such that

$$\Phi = Z_{ш}e^{C} \quad \textit{and} \quad \Psi = Z_{\gamma}\Pi_Y e^{C} \quad \textit{and} \quad \Psi' = Z_{\uplus}\Pi_Y e^{C}.$$

Let $dm(A) = \{Z_{ш}e^{C} | C \in \mathcal{L}ie_A\langle\langle X\rangle\rangle\}$.

Theorem (HNM, 2009)

For all $\Phi \in dm(A)$, the identies $\Psi = B(y_1)\Pi_Y\Phi$, or equivalently $\Psi' = B'(y_1)\Pi_Y\Phi$, yield all polynomial relations with coefficients in A among convergent polyzêtas.

Moreover, these relations are algebraically independent on γ. In other words, under the hypothesis $\gamma \notin A$, the constant γ does not verify any polynomial on polyzêtas with coefficients in A.

Corollary

If $\gamma \notin A$ then it is transcendental over the A-algebra generated by the convergent polyzêtas. Or equivalently, if there exists a polynomial relation with coefficients in A among γ and the convergent polyzêtas then $\gamma \in A$.

Polynomial relations among coefficients of associators

For any $\Phi \in dm(A)$, let $\Psi = B'(y_1)\Pi_Y\Phi$. Then

$$\Phi = \sum_{w\in X^*} \phi(w)\; w = \prod^{\searrow}_{l\in\mathcal{L}ynX, l\neq x_0,x_1} e^{\phi(l)\;\hat{l}},$$

$$\Psi = \sum_{w\in Y^*} \psi(w)\; w = \prod^{\nearrow}_{l\in\mathcal{L}ynY, l\neq y_1} e^{\psi(l)\;\hat{l}}.$$

Therefore,

$$\prod^{\searrow}_{l\in\mathcal{L}ynX, l\neq x_0,x_1} e^{\phi(l)\;\hat{l}} = e^{-\sum_{k\geq 2}\zeta(k)\frac{(-x_1)^k}{k}} \Pi_X \prod^{\nearrow}_{l\in\mathcal{L}ynY, l\neq y_1} e^{\psi(l)\;\hat{l}}.$$

In particular, if $\Phi = Z_{ш}$ and $\Psi = Z_{⧢}$ then

$$\prod^{\searrow}_{l\in\mathcal{L}ynX, l\neq x_0,x_1} e^{\zeta(l)\;\hat{l}} = e^{-\sum_{k\geq 2}\zeta(k)\frac{(-x_1)^k}{k}} \Pi_X \prod^{\nearrow}_{l\in\mathcal{L}ynY, l\neq y_1} e^{\zeta(l)\;\hat{l}}.$$

CONSEQUENCES

Kernel of ζ

Since $\forall l \in \mathcal{L}ynY \iff \Pi_X l \in \mathcal{L}ynX \setminus \{x_0\}$ then identifying the local coordinates, we get polynomial relations among the generators which are algebraically independent on γ.

Theorem

For $\ell \in \mathcal{L}ynY - \{y_1\}$, let $P_\ell \in \mathcal{L}ie_{\mathbb{Q}}\langle X\rangle$ be the decomposition of $\Pi_X\hat{\ell}$ in $\{\hat{l}\}_{l\in\mathcal{L}ynX}$ and let $\check{P}_\ell \in \mathbb{Q}[\mathcal{L}ynX - \{x_0, x_1\}]$ be its dual. Then

$$\Pi_X\ell - \check{P}_\ell \quad \in \quad \ker\phi.$$

In particular, for $\phi = \zeta$ one also obtains

$$\Pi_X\ell - \check{P}_\ell \quad \in \quad \ker\zeta.$$

If $\Pi_X l \equiv \check{P}_l$ then $\zeta(l)$ is irreductible.

Moreover, for any $\ell \in \mathcal{L}ynY - \{y_1\}$, $\Pi_Y\ell - \check{P}_\ell \in \mathbb{Q}\langle Y\rangle$ is homogenous of degree equal $|\ell| > 1$.

Structure of polyzêtas

Theorem

The $\mathbb{Q}$-algebra generated by convergent polyzêtas is isomorphic to the graded algebra $(\mathbb{Q} \oplus (Y - y_1)\mathbb{Q}\langle Y \rangle / \ker \zeta, ⧢)$.

Proof.

Since $\ker \zeta$ is an ideal generated by the homogenous polynomials then the quotient $\mathbb{Q} \oplus (Y - y_1)\mathbb{Q}\langle Y \rangle / \ker \zeta$ is graded. □

Corollary

The $\mathbb{Q}$-algebra of polyzêtas is freely generated by irreducible polyzêtas.

Proof.

For any $\lambda \in \mathcal{L}ynY$, if $\lambda = \check{P}_\lambda$ then one gets the conclusion else $\Pi_X \lambda - \check{P}_\lambda \in \ker \zeta$. Since $\check{P}_\lambda \in \mathbb{Q}[\mathcal{L}ynX]$ then $\check{P}_\lambda$ is polynomial on Lyndon words of degree $\leq |\lambda|$. For each Lyndon word does appear in this decomposition of $\check{P}_\lambda$, after applying Π_Y, the same process goes on untill having irreducible polyzêtas. □

Arithmetical nature of γ

Corollary

If $\gamma \notin A$ then $\gamma \notin \bar{A}$.

With $A = \mathbb{Q}$, it follows immediatly that

Corollary

γ is not an algebraic irrational number.

Theorem

γ is a rational number.

Proof.

1- Since γ verifies $t^2 - \gamma^2 = 0$ then γ is algebraic over $A = \mathbb{Q}(\gamma^2)$.
2- If γ is transcendental then $\gamma \notin A = \mathbb{Q}(\gamma^2)$.
Hence, γ is not algebraic over $A = \mathbb{Q}(\gamma^2)$.
It contradicts the previous assersion.
Thus, γ is not transcendental.
3- It remains that γ is rational. □

THANK YOU FOR YOUR ATTENTION