Linear recurrences for parameter integrals

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Motivation

A trivial example

$$\int_{-1}^{1} \frac{n}{2^n} \left(1 + x\sqrt{5}\right)^{n-1} dx$$

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$$\int_{-1}^{1} \frac{n}{2^n}\left(1+x\sqrt{5}\right)^{n-1} dx \quad = \quad \frac{1}{2^n\sqrt{5}}\left(1+x\sqrt{5}\right)^{n}\Big|_{x=-1}^{1}$$

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$$\int_{-1}^{1} \frac{n}{2^n} \left(1 + x\sqrt{5}\right)^{n-1} dx = \frac{1}{2^n\sqrt{5}} \left(1 + x\sqrt{5}\right)^n \Big|_{x=-1}^{1}$$
$$= \frac{1}{\sqrt{5}} \left(\left(\tfrac{1+\sqrt{5}}{2}\right)^n - \left(\tfrac{1-\sqrt{5}}{2}\right)^n \right)$$

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satisfies the recurrence $F_{n+2} - F_{n+1} - F_n = 0$

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Question

What about the following integral?

$$F(n) := \int_0^{2\pi} \frac{4 - 6\cos(\varphi)}{8\cos(2\varphi) + 12\cos(\varphi) - 21} \frac{2^n \cos(n\varphi)}{\pi} d\varphi$$

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- Bessel's first integral:

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- Hypergeometric function:

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{x^{b-1}(1-x)^{c-b-1}}{(1-zx)^a} dx$$

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Given: integrand $f(x)$

Find: primitive function $g(x) = \int f(x)dx$

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Differential field

(F, D) such that for any $f, g \in F$

$$D(f+g) = Df + Dg \quad \text{and} \quad D(fg) = (Df)g + f(Dg)$$

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Integration in finite terms

Given: differential field (F, D) and $f \in F$

Find: differential field extension $G \supseteq F$ and $g \in G$ s.t.

$$f = Dg$$

Parameter integrals

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Given: integrand $f(x, \vec{y})$, interval (a, b)

Compute: value of $\int_a^b f(x, \vec{y})dx$

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Parametric integration in finite terms

Given: differential field F and $f_0, \ldots, f_m \in F$

Find: coefficients $c_0, \ldots, c_m \in Const(F)^{m+1}$, a differential field extension $G \supseteq F$, and $g \in G$ s.t.

$$c_0 f_0 + \cdots + c_m f_m = Dg$$

Application to definite integrals

Integrals depending on a parameter

- $c_0(n)f(x,n) + \cdots + c_m(n)f(x,n+m) = \frac{d}{dx}g(x,n)$ yields a recurrence for

$$I(n) := \int_a^b f(x,n)dx$$

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- $c_0(y)f(x,y) + \cdots + c_m(y)\frac{\partial^m f}{\partial y^m}(x,y) = \frac{d}{dx}g(x,y)$ yields an ODE for
$$I(y) := \int_{a(y)}^{b(y)} f(x,y)dx$$

Differential fields for the integrand

Liouvillian functions (transcendental case)

A differential field $F = (C(t_1, \ldots, t_n), D)$ is called a regular Liouvillian extension of its constant field $C = Const(F)$, if each t_i is a Liouvillian monomial over $C(t_1, \ldots, t_{i-1})$

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- t_i is transcendental over F_i,
- $Const(F_i(t_i)) = Const(F_i)$, and
- t_i is either
  - primitive over F_i, i.e. $Dt_i = f \in F_i$ ("$t_i = \int f$"), or
  - hyperexponential over F_i, i.e. $\frac{Dt_i}{t_i} = f \in F_i$ ("$t_i = \exp(\int f)$")

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Examples

log, exp, trigonometric/hyperbolic functions, arctan, artanh, logarithmic/exponential integrals, polylogarithms, error function, Fresnel functions, ...

Field extension for the integral

Elementary extensions

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Elementary integral

We say that $f \in F$ has an elementary integral over F if there exists an elementary extension E of F and $g \in E$ s.t.

$$f = Dg$$

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Find: all $(c_0, \ldots, c_m) \in Const(F)^{m+1}$ s.t. $c_0 f_0 + \cdots + c_m f_m$ has an elementary integral over F and corresponding g's from some elementary extension of F with

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$$W(n) = \begin{cases} \frac{(1/2)_k \pi}{2(k!)} & \text{if } n = 2k \\ \frac{k!}{(3/2)_k} & \text{if } n = 2k+1 \end{cases}$$

Gamma function

$$\Gamma^{(n)}(z) = \int_0^\infty \underbrace{x^{z-1} \exp(-x) \ln(x)^n}_{=:f(n,z,x)} dx \quad \text{for } z > 0, n \in \mathbb{N}_0$$

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After integrating this relation we obtain

$$\Gamma^{(n)}(z+1) = z\Gamma^{(n)}(z) + n\Gamma^{(n-1)}(z)$$

Fibonacci cosine series

$$F(n) := \int_0^{2\pi} \frac{4 - 6\cos(\varphi)}{8\cos(2\varphi) + 12\cos(\varphi) - 21} \frac{2^n \cos(n\varphi)}{\pi} d\varphi \quad \text{for } n \in \mathbb{N}_0$$

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Using the algorithm we obtain the recurrence

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with certificate $\frac{2^{n+2}}{\pi}\left(-3\frac{\sin((n+1)\varphi)}{n+1}+4\frac{\sin((n+2)\varphi)}{n+2}-3\frac{\sin((n+3)\varphi)}{n+3}\right)$

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Compare this with Fibonacci numbers F_n

$$F_{n+4} \quad -F_{n+3} \qquad -F_{n+2} \qquad\qquad = \quad 0$$

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F_{n+4} & +3F_{n+3} & -21F_{n+2} & +12F_{n+1} & +16F_n & = & 0
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Also initial values match, hence $F(n) = F_n$.

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Applying the algorithm to $f(n,x), f(n+1,x), f(n+2,x)$ yields

$$-\frac{16n^2-1}{16(a^2-1)n(n+1)}f(n,x) - \frac{(a^2-2)(2n+1)}{2(a^2-1)(n+1)}f(n+1,x) + f(n+2,x) =$$

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$$= \frac{d}{dx}\frac{(4n+1)x^5+2a(6n+1)x^3+(8a^2n+1)x}{16(a^2-1)n(n+1)(x^4+2ax^2+1)^{n+1}}$$

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After integrating this relation we obtain

$$Q(n+2) = \tfrac{(a^2-2)(2n+1)}{2(a^2-1)(n+1)} Q(n+1) + \tfrac{16n^2-1}{16(a^2-1)n(n+1)} Q(n)$$

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Applying the algorithm to $f(n,x), f(n+1,x), f(n+2,x)$ yields

$$-\tfrac{16n^2-1}{16(a^2-1)n(n+1)} f(n,x) - \tfrac{(a^2-2)(2n+1)}{2(a^2-1)(n+1)} f(n+1,x) + f(n+2,x) = \\ = \tfrac{d}{dx} \tfrac{(4n+1)x^5+2a(6n+1)x^3+(8a^2n+1)x}{16(a^2-1)n(n+1)(x^4+2ax^2+1)^{n+1}}$$

After integrating this relation we obtain

$$Q(n+2) = \tfrac{(a^2-2)(2n+1)}{2(a^2-1)(n+1)} Q(n+1) + \tfrac{16n^2-1}{16(a^2-1)n(n+1)} Q(n)$$

Solution can be written as

$$Q(n) = \frac{\pi}{2} \frac{P_{n-1}^{(n-\frac{1}{2},\frac{1}{2}-n)}(a)}{(2(a+1))^{n-1/2}}$$

Quartic integral

$$Q(n) := \int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^n} \quad \text{for } n \in \mathbb{N}^+, a > -1$$

Applying the algorithm to $f(n,x), f(n+1,x), f(n+2,x)$ yields

$$-\tfrac{16n^2-1}{16(a^2-1)n(n+1)} f(n,x) - \tfrac{(a^2-2)(2n+1)}{2(a^2-1)(n+1)} f(n+1,x) + f(n+2,x) =$$
$$= \tfrac{d}{dx} \tfrac{(4n+1)x^5+2a(6n+1)x^3+(8a^2n+1)x}{16(a^2-1)n(n+1)(x^4+2ax^2+1)^{n+1}}$$

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Solution can be written as

$$Q(n) = \frac{\pi}{2} \frac{P_{n-1}^{(n-\frac{1}{2},\frac{1}{2}-n)}(a)}{(2(a+1))^{n-1/2}} = \tfrac{\sqrt{\pi}\Gamma(2n-\frac{1}{2})}{2^{2n-1/2}(a+1)^{n-1/2}\Gamma(n)^2} \int\limits_0^1 \tfrac{\left((a-1)x^2+2x\right)^{n-1}}{\sqrt{1-x}} dx$$

Thank you
for your attention!