Topological graph polynomials and quantum field theories (QFT)

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Plan

- Graph theory: Tutte polynomial
- QFT and Feynman integrals; parametric representation
- Relation Tutte polynomial - parametric representation
- Ribbon graphs & the Bollobás-Riordan polynomial
- Perspectives

Graph theory - some definitions

tadpole line - line which starts and ends on the same vertex *(loop)*

1PR (1 particle reducible) line - a line whose removal increases by 1 the number of connected components of the graph *(bridge)*

regular line - line which is neither 1PR nor a tadpole line

semi-regular line - line which is not a tadpole line

spanning tree - connected subgraph with no loops *(cycle)*

2 natural operations for an edge e in a graph G:

1. deletion $\rightarrow G - e$
2. contraction $\rightarrow G/e$

$\hookrightarrow$ associated to these operations - the Tutte polynomial

(combinatorial object encoding the topological information of a graph)

Tutte polynomial

(W. T. Tutte, *Graph Theory*, '84, H. H. Crapo, *Aequationes Mathematicae,*, '69)

a 1st definition - deletion/contraction:

e - regular line

$$T_G(x,y) := T_{G/e}(x,y) + T_{G-e}(x,y)$$

$\rightarrow$ terminal forms - m 1PR lines and n tadpole lines

$$T_G(x,y) := x^m y^n.$$

rank

$$r(A) := |V_G| - k(A),$$

$r(A)$ - the rank of the subgraph A
V_G - number of vertices
$k(A)$ - number of connected components

2nd definition - sum over subgraphs:

$$T_G(x,y) := \sum_A (x-1)^{r(E)-r(A)}(y-1)^{n(A)}.$$

$n(A)$ - number of loops of the subgraph A
(nulity - cyclomatic number)

the two definitions are equivalent

Multivariate Tutte polynomial

(A. Sokal, *London Math. Soc. Lecture Note Ser.*, 2005)

β_e, $e = 1, \ldots, E$ (different variable for each edge)
E - the total number of edges
q - variable associated to the vertices

1st definition - deletion/contraction:

$$Z_G(q, \{\beta\}) := Z_{G/e}(q, \{\beta - \{\beta_e\}\}) + \beta_e Z_{G-e}(q, \{\beta - \{\beta_e\}\}),$$

e - not necessary regular

$\rightarrow$ terminal forms with v vertices and without edges

$$Z_G(q, \beta) := q^v.$$

Multivariate Tutte polynomial - 2nd definition

2nd definition - sum over subgraphs:

$$Z_G(q, \beta) := \sum_{A \subset E} q^{k(A)} \prod_{e \in A} \beta_e .$$

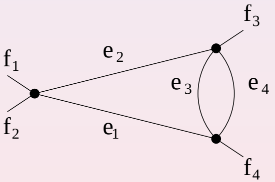
the two definitions are equivalent

Quantum field theory (QFT)

QFT - graph theory

<u>Φ^4 model</u> - 4-*valent vertices*

$\Phi(x)$ - a field, $\quad x \in \mathbb{R}^4$ (the space-time)



propagator (associated to each edge of the graph):

$$C(p_\ell, m) = \frac{1}{p_\ell^2 + m^2},\ p_\ell \in \mathbb{R}^4,\ i = 1, \ldots, E = 3,\ \ m \in \mathbb{R} \text{ the mass}$$

QFT - Feynman integrals

- integration over each of the E internal momentum of the graph G
- conservation of incoming/outgoing momentae at each vertex of the graph G

$\longrightarrow$ *Feynman integral* $\mathcal{A}_G$

Parametric representation of the Feynman integrals

introduction of some parameters α:

$$\frac{1}{p_\ell^2 + m^2} = \int d\alpha_\ell \, e^{-\alpha(p_\ell^2+m^2)}, \qquad \forall \ell = 1, \ldots, E$$

$\rightarrow$ Gaussian integration over internal momentae p_i

$$\Longrightarrow \mathcal{A}_G(p_{\rm ext}) = \int_0^\infty \frac{e^{-V(p_{\rm ext},\alpha)/U(\alpha)}}{U(\alpha)^{\frac{D}{2}}} \prod_{\ell=1}^{E} (e^{-m^2\alpha_\ell} d\alpha_\ell)$$

U, V - polynomials in the parameters α

$$U \;=\; \sum_{\mathcal{T}} \prod_{\ell \notin \mathcal{T}} \alpha_\ell \;,$$

$\mathcal{T}$ - spanning tree of the graph

$$U_G(\alpha) = \alpha_e U_{G-e}(\alpha) + U_{G/e}(\alpha)$$

terminal forms (graph formed only of tadpole or 1PR edges)

$$U_G(\alpha) = \prod_{e \text{ tadpole}} \alpha_e,$$

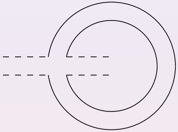
new proof: Grassmannian development of the Pfaffians resulted from the Gaussian integrations over the internal momentae p_i

relation with the multivariate Tutte polynomial - the polynomial U_G satisfies the deletion/contraction relation

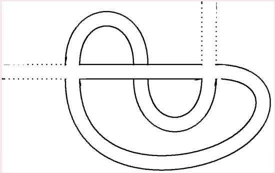
(S. Bloch *et. al.*, Commun. Math. Phys., 2006, F. Brown, arXiv:0804.1660)

it can be obtained as a limit of the multivariate Tutte polynomial

Generalization: ribbon graphs



$bc = 2$



$bc = 1$

bc - number of connected components of the graph's boundary
(if the graph is connected, bc - the number of faces)

Bollobás-Riordan polynomial R_G

(B. Bollobás and O. Riordan, *Proc. London Math. Soc.*, **83** 2001, *Math. Ann.*, **323** (2002)

J. Ellis-Monaghan and C. Merino, arXiv:0803.3079[math.CO], 0806.4699[math.CO])

$\hookrightarrow$ generalization of the Tutte polynomial for ribbon graphs

$$R_G(x, y, z) = \sum_{H \subset G} (x-1)^{r(G)-r(H)} y^{n(H)} z^{k(H)-bc(H)+n(H)}.$$

the additional variable z keeps track of the additional topological information (bc or the graph genus g)

$\hookrightarrow$ some generalizations:

(S. Chumotov, *J. Combinatorial Theory* **99** (2009), F. Vignes-Tourneret, *Discrete Mathematics* **309** (2009)

Deletion/contraction for the Bollobás-Riordan polynomial

$R_G(x,y,z) = R_{G/e}(x,y,z) + R_{G-e}(x,y,z)$, e semi-regular edge

terminal forms (graphs $\mathcal{R}$ with 1 vertex):

$k(\mathcal{R}) = V(\mathcal{R}) = k(H) = V(H) = 1 \to R(x,y,z) = R(y,z)$

$$R_{\mathcal{R}}(y,z) = \sum_{H \subset \mathcal{R}} y^{E(H)} z^{2g(H)}.$$

Multivariate Bollobás-Riordan polynomial

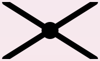
generalization of the Bollobás-Riordan polynomial analogous to the generalization of the Tutte polynomial

$$Z_G(x, \{\beta_e\}, z) = \sum_{H \subset G} x^{k(H)} (\prod_{e \in H} \beta_e) \, z^{bc(H)}.$$

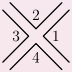
$\hookrightarrow$ satisfies the deletion/contraction relation

Noncommutative quantum field theory (NCQFT)

NCQFTs - ribbon graphs



$\longrightarrow$



Parametric representation for a noncommutative Φ^4 model

$$\mathcal{A}_G^\star(p) = \int_0^\infty \frac{e^{-V^\star(p,\alpha)/U^\star(\alpha)}}{U^\star(\alpha)^{\frac{D}{2}}} \prod_{\ell=1}^{L} (e^{-m^2\alpha_\ell} d\alpha_\ell)$$

Theorem:

$$U^\star = \left(\frac{\theta}{2}\right)^{bc-1+2g} \sum_{\mathcal{T}^\star} \prod_{\ell \notin \mathcal{T}^\star} 2\frac{\alpha_\ell}{\theta}$$

θ - noncommutativity parameter

$\mathcal{T}^\star$ - $\star$-trees (non-trivial generalization of the notion of trees) (quasi-trees)

Relation to the multivariate Bollobás-Riordan polynomial

$$U^{\star}_G(\{\alpha_e\}) = \alpha_e U^{\star}_{G-e} + U^{\star}_{G/e}.$$

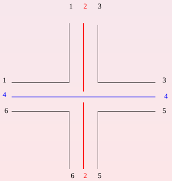
for the sake of completeness ...

$$U^{\star}_G(\alpha,\theta) = (\theta/2)^{E-V+1}\Big(\prod_{e\in E}\alpha_e\Big) \times \lim_{w\to 0} w^{-1} Z_G\Big(\tfrac{\theta}{2\alpha_e}, 1, w\Big).$$

Conclusion et perspectives

relation between combinatorics and QFTs

- other type of topological polynomials related to other QFT models (T. Krajewski *et. al.*, arXiv:0912.5438) - no deletion/contraction property
- generalization to tensor models (appearing in recent approaches for a theory of quantum gravity)

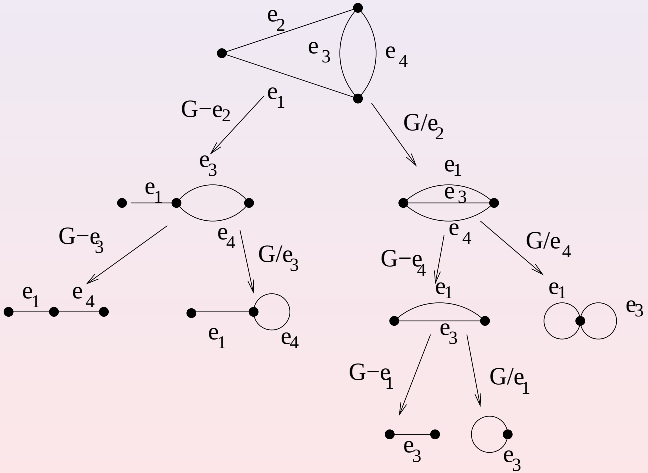


(R. Gurău, *Annales H. Poincaré* **11** (2010), J. Ben Geloun *et. al.*, *Class. Quant. Grav.* **27** (2010))

Vielen Dank fur Ihre Aufmerksamkeit

Thank you for your attention

Exemple



$$T_G(x, y) = x^2 + xy + x + y + y^2.$$

The *Moyal algebra* is the linear space of smooth and rapidly decreasing functions $\mathcal{S}(\mathbb{R}^{\mathcal{D}})$ equipped with the *Moyal product*:

$$(f \star g)(x) = \int \frac{d^D k}{(2\pi)^D} d^D y \, f(x + \frac{1}{2}\Theta \cdot k) g(x + y) e^{ik \cdot y}$$