A module model for Azenhas' bijection

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Overview

Azenhas' bijection = $\rho_3 : \mathcal{LR}(\lambda/\mu, \nu) \longrightarrow \mathcal{LR}(\lambda/\nu, \mu)$ (procedural)

$\mathcal{LR}(\lambda/\mu, \nu)$ = { Littlewood-Richardson tableaux
of outer shape λ,
inner shape μ, weight ν }

$\mathcal{LR}(\lambda/\nu, \mu)$ = { ....
inner shape ν, weight μ }

E.g.

						1	1
					1		
		1	2	2			
1	2	2	3				

$\in \mathcal{LR}\Big((8,6,5,4)/(6,5,2), (5,4,1)\Big)$

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A module model for Azenhas' bijection

Overview

$$\text{Azenhas' bijection} = \quad \begin{array}{ccc} \rho_3 \colon \mathcal{LR}\left({}^{\lambda}\!/_{\mu}, \nu\right) & \longrightarrow & \mathcal{LR}\left({}^{\lambda}\!/_{\nu}, \mu\right) \\ \Uparrow & & \Uparrow \\ T & \longmapsto & T^{\vee} \end{array} \qquad \text{(procedural)}$$

A module model for Azenhas' bijection

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$$\text{Azenhas' bijection} = \rho_3 : \mathcal{LR}(\lambda/\mu, \nu) \longrightarrow \mathcal{LR}(\lambda/\nu, \mu) \quad \text{(procedural)}$$

$$\begin{array}{ccc} \Cup & & \Cup \\ T & \longmapsto & T^{\vee} \end{array}$$

A Steinberg-van Leeuwen-style interpretation of ρ_3

Define $\mathcal{G}^{\lambda}_{\mu\nu}$ — alg. variety, $\mathcal{G}^{\lambda}_{\mu\nu} \xrightarrow{\sim} \mathcal{G}^{\lambda}_{\nu\mu}$ isom. of varieties

$$\rightsquigarrow \{\text{irred. components of } \mathcal{G}^{\lambda}_{\mu\nu}\} \xrightarrow{\sim} \{\text{irred. components of } \mathcal{G}^{\lambda}_{\nu\mu}\}$$

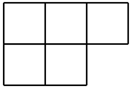
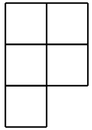
$$\begin{array}{ccc} \Updownarrow & & \Updownarrow \\ \mathcal{LR}(\lambda/\mu, \nu) & \xrightarrow{\quad\sim\quad} & \mathcal{LR}(\lambda/\nu, \mu) \\ \Cup & & \Cup \\ T & \longmapsto & T^{\perp} \end{array}$$

$\Longrightarrow$ $T^{\vee} = T^{\perp}$ (The definition of $\mathcal{G}^{\lambda}_{\mu\nu}$ uses certain modules.)

1. Littlewood-Richardson tableaux, Azenhas' bijection

λ: partition $\overset{\textbf{def}}{\iff}$ $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in \bigcup_{l \geqq 0} \mathbb{N}^l$, $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_l$ λ

as above $\Longrightarrow$ $l(\lambda) \overset{\textbf{def}}{=} l$, $|\lambda| \overset{\textbf{def}}{=} \sum_{i=1}^{l} \lambda_i$, $\lambda' =$ conjugate partition of λ

E.g. $\lambda = (3, 2)$  $\Longrightarrow$ $\lambda' = (2, 2, 1)$ 

partitions λ, μ, ν s.t. $|\lambda| = |\mu| + |\nu| \rightsquigarrow \mathcal{LR}(\lambda/\mu, \nu)$

E.g. $\lambda = (8, 6, 5, 4)$, $\mu = (6, 5, 2)$, $\nu = (5, 4, 1)$ $(23 = 13 + 10)$

$\mathcal{LR}(\lambda/\mu, \nu) = \Bigg\{$

						1	1
					1		
		1	1	2			
2	2	2	3				

						1	1
					1		
		1	2	2			
1	2	2	3				

						1	1
					2		
		1	1	1			
2	2	2	3				

						1	1
					2		
		1	1	2			
1	2	2	3				

						1	1
					2		
		1	1	3			
1	2	2	2				

$\Bigg\}$

E.g. $\lambda = (8,6,5,4)$, $\mu = (6,5,2)$, $\nu = (5,4,1)$ $\quad (23 = 13 + 10)$

$$\mathcal{LR}(\lambda/\mu, \nu) = \left\{ \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & 1 & 1 \\ \hline & & & & & 1 \\ \hline & & 1 & 1 & 2 \\ \hline 2 & 2 & 2 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & 1 & 1 \\ \hline & & & & & 1 \\ \hline & & 1 & 2 & 2 \\ \hline 1 & 2 & 2 & 3 \\ \hline \end{array} \right.$$

$$\left. \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & 1 & 1 \\ \hline & & & & & 2 \\ \hline & & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & 1 & 1 \\ \hline & & & & & 2 \\ \hline & & 1 & 1 & 2 \\ \hline 1 & 2 & 2 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & 1 & 1 \\ \hline & & & & & 2 \\ \hline & & 1 & 1 & 3 \\ \hline 1 & 2 & 2 & 2 \\ \hline \end{array} \right\}$$

outer shape $=$ (# cells in ith row)$_{i=1,2,\ldots}$ (left-aligned)
inner shape $=$ (# empty cells in ith row)$_{i=1,2,\ldots}$ (left-packed)
weight $=$ (# $\boxed{s}$)$_{s=1,2,\ldots}$

$$\boxed{x}\boxed{y} \Rightarrow x \leqq y \qquad \begin{array}{|c|}\hline x \\ \hline y \\ \hline\end{array} \Rightarrow \begin{array}{c} x \\ \wedge \\ y \end{array} \qquad \begin{array}{l} \#\boxed{s} \text{ in} \\ \text{rows } 1 \sim i \end{array} \geqq \begin{array}{l} \#\boxed{s+1} \text{ in} \\ \text{rows } 1 \sim i+1 \end{array}$$

$\#\mathcal{LR}(\lambda/\mu, \nu) = \#\mathcal{LR}(\lambda/\nu, \mu)$ known (called "commutativity")

Several bijections $\mathcal{LR}(\lambda/\mu, \nu) \xrightarrow{\sim} \mathcal{LR}(\lambda/\nu, \mu)$ have been given, shown to be equal to one another as maps (Azenhas, Benkart-Sottile-Stroomer, Danilov-Koshevoy, Pak-Vallejo)

Azenhas' bijection by example

						1	1
					1		
		1	2	2			
1	2	2	3				

$= T^{(4)} = T \longmapsto$

					1	1	1
				1	2		
	1	2	2	2			
1	2	3	3				

$= T^{\vee}$

					1	1	1
				1	2		
1	2	2	2	3			

$= T^{(3)}$

$$\begin{pmatrix} \#\boxed{s} \text{ in} \\ \text{row } i \text{ of } T^{\vee} \end{pmatrix} = \begin{pmatrix} \text{shrink of row } s \\ \text{in } \mu^{(i)} \to \mu^{(i-1)} \end{pmatrix}$$

				1	1	1	1
	1	2	2	2	2		

$= T^{(2)}$

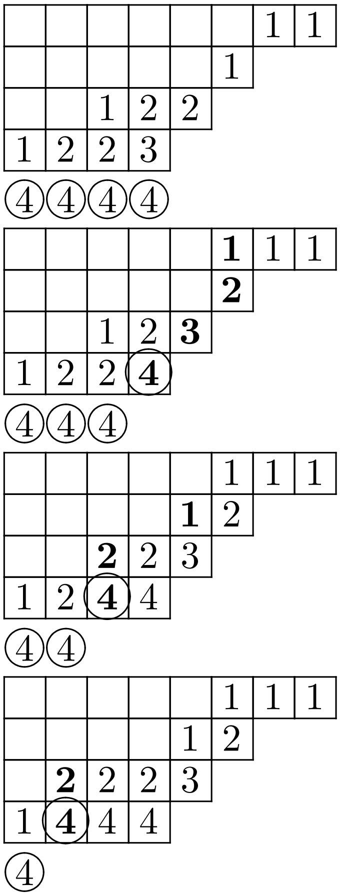
$(\mu^{(i)} = \text{inner shape of } T^{(i)})$

			1	1	1	1	1

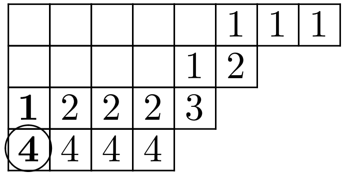
$= T^{(1)}$

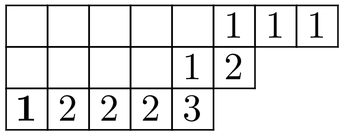
$\varnothing = T^{(0)}$

Obtaining $T^{(i-1)}$ from $T^{(i)}$ by example



$= T^{(4)}$





$= T^{(3)}$

Denote this step by $T \mapsto T^{\flat}$.
$\lambda, \mu, \nu, T^{\flat}$ determines
the last row of $T^{\vee}$.

2. "Hall varieties", "Green-Klein (sub)varieties"

Elementary divisor theory $\Longrightarrow$

$$\left\{\begin{array}{l}\mathbb{C}[[t]]\text{-modules}\\ \text{of length } n\end{array}\right\}\Big/\!\sim \;\; \overset{\sim}{\longleftrightarrow} \;\; \left\{\begin{array}{l}\text{partitions } \lambda\\ \quad\text{s.t. } |\lambda| = n\end{array}\right\}$$

$$\begin{array}{ccc} \Cup & & \Cup \\ \bigoplus_{j=1}^{l} \mathbb{C}[[t]]\big/(t^{\lambda_j}) & \longleftarrow\!\!\!\!\longmapsfrom & \lambda = (\lambda_1, \ldots, \lambda_l) \\ \| & & \end{array}$$

V: n-dim. $\mathbb{C}$-vector sp.

$+\, N \curvearrowright V$, linear, nilpotent

with Jordan blocks of

size $\lambda_1, \ldots, \lambda_l$

Write $M = (V, N)$ for the $\mathbb{C}[[t]]$-module determined by V and N, and write $\lambda = \operatorname{type} M$.

Hereafter, fix a $\mathbb{C}[[t]]$-module M of type λ.

Let μ, ν be partitions s.t. $|\lambda| = |\mu| + |\nu|$, and put

$$\mathcal{G}^M_{\mu\nu} = \{\, N \subset M \ (\mathbb{C}[[t]]\text{-submod.}) \mid \operatorname{type} N = \nu,\ \operatorname{type} {}^{M}\!/\!_{N} = \mu \,\}$$
(or $\mathcal{G}^\lambda_{\mu\nu}$ more abstractly).

$$\mathcal{G}^M_{\mu\nu} \subset (\text{Grassmannian of all } |\nu|\text{-dim. subsp. in } M) \begin{pmatrix} \text{locally closed} \\ \text{subvariety} \end{pmatrix}$$

P. Hall: counterpart of $\mathcal{G}^M_{\mu\nu}$ for DVR $\mathfrak{o}$ with finite residue field $\mathbb{F}_q$

$$\#\mathcal{G}^M_{\mu\nu}(\mathfrak{o}) = (\text{polynomial in } q, \text{ leading coeff. } \#\mathcal{LR}({}^{\lambda}\!/\!_{\mu}, \nu))$$

J. A. Green defined $\mathcal{G}^M_T$, $T \in \mathcal{LR}\left({}^{\lambda'}\!/\!_{\mu'}, \nu'\right)$ s.t. $\mathcal{G}^M_{\mu\nu} = \coprod_T \mathcal{G}^M_T$

$$N \in \mathcal{G}^M_T \overset{\textbf{def}}{\iff} \left(\operatorname{type}\left({}^{M}\!/\!_{t^s N}\right)\right)' = (\text{shape of boxes} \leqq \boxed{s} \text{ in } T),\ {}^\forall s.$$

T. Klein: $\#\mathcal{G}^M_T(\mathfrak{o}) = (\underline{\text{monic}} \text{ polynomial in } q, \text{ degree indep. of } T)$

Klein and I. G. Macdonald's analysis to count $\#\mathcal{G}_T^M(\mathfrak{o})$ gives enough structure on the variety $\mathcal{G}_T^M$:

Th. (van Leeuwen) Fix λ, μ, ν s.t. $|\lambda| = |\mu| + |\nu|$, and M as above.

(1) For each $T \in \mathcal{LR}\left({}^{\lambda'}/_{\mu'}, \nu'\right)$, $\mathcal{G}_T^M$ is a smooth irred. (locally closed) subvariety of $\mathcal{G}_{\mu\nu}^M$ of dim. independent of T.

(2) $\operatorname{Irr} \mathcal{G}_{\mu\nu}^M = \{ \overline{\mathcal{G}_T^M} \mid T \in \mathcal{LR}\left({}^{\lambda'}/_{\mu'}, \nu'\right) \}$

3. (Main result) Azenhas' bijection by "Hall varieties"

$M^* = (V^* = \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C}), {}^tN)$ is also a $\mathbb{C}[[t]]$-module of type λ.

$$\begin{array}{ccc} \mathcal{G}_{\mu\nu}^M & \xrightarrow{\sim} & \mathcal{G}_{\nu\mu}^{M^*} \\ \uplus & & \uplus \\ N & \longmapsto & N^\perp \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} \operatorname{Irr} \mathcal{G}_{\mu\nu}^M & \xrightarrow{\sim} & \operatorname{Irr} \mathcal{G}_{\nu\mu}^{M^*} \\ \updownarrow & & \updownarrow \\ \mathcal{LR}\left({}^{\lambda'}/_{\mu'}, \nu'\right) & \xrightarrow{\sim} & \mathcal{LR}\left({}^{\lambda'}/_{\nu'}, \mu'\right) \\ T & \longmapsto & T^\perp \end{array}$$

$N^\perp = \{ f \in M^* \mid f(N) = 0 \}$
$N^\perp \cong \left({}^M/_N\right)^* \cong {}^M/_N, \quad {}^{M^*}/_{N^\perp} \cong N^* \cong N$

Main Theorem. $T^\vee = T^\perp$.

$LR(N, M) :=$ the Littlewood-Richardson tableau attached to a submodule $N \subset M$
(as used by Green in defining the partition $\mathcal{G}^M_{\mu\nu} = \coprod_T \mathcal{G}^M_T$)

Lem.

$$\underset{\text{sequence of partitions}}{\left(\text{type}\left(\ker t^i_M / N \cap \ker t^i_M\right)\right)^r_{i=0}} \longmapsto \underset{\text{Littlewood-Richardson tableau}}{LR(N^\perp, M^*)}$$

in the same way as "$(\mu^{(i)})^r_{i=0}$" $\longmapsto T^\vee$ in Azenhas' procedure.

$(r = l(\lambda') = \min\{\, r' \mid t^{r'} M = 0 \,\})$

Hence $T^\vee = T^\perp$ will follow if one shows:

Claim. For "almost all" $N \in \mathcal{G}^M_T$,
$T^{(i)}$ in Azenhas' procedure $= LR(N \cap \ker t^i_M, \ker t^i_M)$ $({}^\forall i)$.

Showing Claim = "1st step ($i = r - 1$)" + "induction"

Showing "1st step" uses

- reduction to the case where (largest symbol in T) $< l(\lambda') = r$
  (namely $t^{r-1}N = 0$, namely $N \subset \ker t_M^{r-1}$)

- a construction of an open covering $\mathcal{G}_T^M = \bigcup_{\Xi \in \mathcal{X}(T)} U_\Xi$
  with explicit isom. $U_\Xi \cong$ (open subset of $\mathbb{A}^{\text{dim. independent of } T}$)
  ("coordinates")

- an explicit construction of generators of N from the "coordinates"

- In one particular U_Ξ, all $N \in U_\Xi$ also lie in $\mathcal{G}_{T^{(r-1)}}^{\ker t_M^{r-1}}$
  Showing this uses generators explicitly constructed above,
  and exhibits the "pull up" nature of the process $T \mapsto T^{(r-1)}$.

"Induction" requires a technique similar to one used by Steinberg
in interpreting the Robinson-Schensted correspondence
using the irreducible components of "the Steinberg variety".

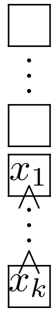
4. Explicit coordinates of $\mathcal{G}_T^M$ by example

$$\mathcal{G}_T^M = \bigcup_{\Xi \in \mathcal{X}(T)} U_\Xi \qquad \text{open covering with index set } \mathcal{X}(T)$$

$$\mathcal{X}\left(\begin{array}{|c|c|c|c|}\hline \; & \; & \; & 1 \\ \hline \; & 1 & 1 & 2 \\ \hline 2 & 2 \\ \cline{1-2}\end{array}\right) = \left\{ \begin{array}{|c|c|c|c|}\hline \; & \; & \; & 1 \\ \hline \; & 1 & 1 & 2 \\ \hline 2 & 2 \\ \cline{1-2}\end{array} \; \begin{array}{|c|c|c|c|}\hline \; & \; & \; & 1 \\ \hline 1 & \; & 1 & 2 \\ \hline 2 & 2 \\ \cline{1-2}\end{array} \; \begin{array}{|c|c|c|c|}\hline \; & \; & \; & 1 \\ \hline 1 & 1 & \; & 2 \\ \hline 2 & 2 \\ \cline{1-2}\end{array} \right.$$

$$\left. \begin{array}{|c|c|c|c|}\hline \; & \; & 1 & \; \\ \hline \; & 1 & 2 & 1 \\ \hline 2 & 2 \\ \cline{1-2}\end{array} \; \begin{array}{|c|c|c|c|}\hline \; & \; & 1 & \; \\ \hline 1 & \; & 2 & 1 \\ \hline 2 & 2 \\ \cline{1-2}\end{array} \; \begin{array}{|c|c|c|c|}\hline \; & \; & 1 & \; \\ \hline 1 & 1 & 2 & \; \\ \hline 2 & 2 \\ \cline{1-2}\end{array} \right\}$$

$\#(\boxed{s}$ in row i) same as in T, $^\forall i$, $^\forall s$



For each Ξ, the coordinate space = an open set of
$\mathbb{A}^{D_1} \times \mathbb{A}^{D_2} \times \cdots \times \mathbb{A}^{D_u}$ $\quad$ ($u = l(\nu')$, the largest symbol in T),

$$D_s = \left\{ \begin{array}{l} (j', j) \\ \text{(pair of col. indices)} \end{array} \middle| \begin{array}{l} s \in (\text{col. } j \text{ of } \Xi),\ s \notin (\text{col. } j' \text{ of } \Xi), \\ \xi_{j'}^{(s)} \geqq \xi_j^{(s)} \end{array} \right\},$$

$\xi_j^{(s)} = \#$(boxes in col. j of Ξ, either empty or w/ content $\leqq s$).

E.g. if $\Xi = \begin{array}{|c|c|c|c|}\hline \; & \; & \; & 1 \\ \hline 1 & 1 & \; & 2 \\ \hline 2 & 2 \\ \cline{1-2}\end{array}$, $D_1 = \{(3,1),(3,2),(3,4)\}$, $D_2 = \{(3,4)\}$.

In this example, $U_{\Xi} \cong$ (the entire $\mathbb{A}^{D_1} \times \mathbb{A}^{D_2}$).

For $(a_{31}^{(1)}, a_{32}^{(1)}, a_{34}^{(1)}, a_{34}^{(2)}) \in \mathbb{A}^{D_1} \times \mathbb{A}^{D_2} \cong \mathbb{A}^4$,
corresponding $N \in U_{\Xi}$ is generated by column vectors of

$$\begin{pmatrix} t^3 & & & \\ & t^3 & & \\ & & t^2 & \\ & & & t^2 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & a_{34}^{(2)} \\ & & & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & & & \\ & t^{-1} & & \\ & & 1 & \\ & & & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ a_{31}^{(1)} & a_{32}^{(1)} & 1 & a_{34}^{(1)} \\ & & & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} t^{-1} & & & \\ & t^{-1} & & \\ & & 1 & \\ & & & t^{-1} \end{pmatrix} = \begin{pmatrix} t & & & \\ & t & & \\ a_{31}^{(1)} t & a_{32}^{(1)} t & t^2 & a_{34}^{(1)} t + a_{34}^{(2)} \\ & & & 1 \end{pmatrix}$$

(modulo L, where $N \subset M = \mathbb{C}[[t]]^{\oplus 4} / \underbrace{(t^3) \oplus (t^3) \oplus (t^2) \oplus (t^2)}_{L}$).