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VECTOR VALUED MACDONALD POLYNOMIALS

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Abstract. This paper defines and investigates nonsymmetric Macdonald polynomials with values in an irreducible module of the Hecke algebra of type A_{N-1}. These polynomials appear as simultaneous eigenfunctions of Cherednik operators. Several objects and properties are analyzed, such as the canonical bilinear form which pairs polynomials with those arising from reciprocals of the original parameters, and the symmetrization of the Macdonald polynomials. The main tool of the study is the Yang–Baxter graph. We show that these Macdonald polynomials can be easily computed following this graph. We give also an interpretation of the symmetrization and the bilinear forms applied to the Macdonald polynomials in terms of the Yang–Baxter graph.

1. Introduction

For each partition λ of N there is an irreducible module of the Hecke algebra of type A_{N-1} whose basis is labeled by standard tableaux of shape λ. This paper defines and analyzes nonsymmetric Macdonald polynomials with values in such modules. The double affine Hecke algebra generated by multiplication by coordinate functions, q-type Dunkl operators, the Hecke algebra and a q-shift acts on these polynomials. They appear as simultaneous eigenfunctions of the associated Cherednik operators. There is a canonical bilinear form which pairs these polynomials with those arising from the reciprocals of the original parameters. The Macdonald polynomials and their reciprocal-parameter versions constitute a biorthogonal set of the form. The values of the form are found explicitly.

There are symmetric Macdonald polynomials in this structure. They are labeled by column-strict tableaux of shape λ (non-decreasing entries in each row, strictly increasing in each column). Formulae for these polynomials in terms of nonsymmetric Macdonald polynomials are derived and the values of the bilinear form are obtained in this case. There are analogous results for antisymmetric Macdonald polynomials, which are labeled by row-strict tableaux. There is a hook-length type formula for the bilinear form evaluated at the minimal symmetric polynomial associated with λ.

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In the study of one-variable orthogonal polynomials the very simple graph $0 \to 1 \to 2 \to \cdots$ symbolizes the Gram–Schmidt process used to produce the polynomials. In the present multi-variable setting, the Yang–Baxter graph displays how each Macdonald polynomial is produced. Each arrow corresponds to either an adjacent transposition or an affine step $(u_1, \ldots, u_N) \to (u_2, \ldots, u_N, u_1 + 1)$. This idea is developed in Section 4.

In Section 2 we give the basic definitions of the Hecke algebra, its modules, and the machinery necessary to describe the leading terms of Macdonald polynomials. Section 3 begins with the simplest two-dimensional module associated to the partition $(2, 1)$ of $N = 3$. We describe how the basic operations arise in this situation and thus motivate our general definitions. The rest of the section gives the definitions and proves the fundamental relations, notably the braid relations, for the vector-valued situation. A key part is played by the triangularity property of the Cherednik operators with respect to a natural partial order on monomials.

Section 4 contains the description of the simultaneous eigenfunctions, the spectral vectors, the transformation rules for the action of the generators of the Hecke algebra on the polynomials, and the Yang–Baxter graph.

Section 5 concerns the connected components of the Yang–Baxter graph modified by the removal of the affine edges. Here we find the conditions under which the component contains a unique symmetric or antisymmetric polynomial.

The bilinear form is defined and evaluated in Section 6. The method of evaluation relies on relatively simple calculations of the effects of a single arrow in the Yang–Baxter graph. The minimal symmetric polynomials are studied in this section. The hook-length formula for the bilinear form gives some information about aspherical modules of the double affine Hecke algebra, a topic to be pursued in future work.

The paper concludes with a symbol index and a list of basic relations for quick reference.

2. Double affine Hecke algebra

2.1. **Definitions and basic properties.** Consider the elements T_i and w verifying the following four relations:

(1) $(T_i + t_1)(T_i + t_2) = 0$,
(2) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$,
(3) $T_i T_j = T_j T_i$ for $|i - j| > 1$,
(4) $T_i w = w T_{i-1}$.

These operators act on the right on $\mathbb{C}(t_1, t_2, q)[x_1, \ldots, x_N]$ by

(1) $T_i := \overline{\pi}_i (t_1 + t_2) - t_2 s_i$,
(2) $w := \tau_1 s_1 \cdots s_{N-1}$,

where $\overline{\pi}_i = \partial_i x_{i+1}$, ∂_i is the divided difference defined by

$$\partial_i = (1 - s_i) \frac{1}{x_i - x_{i+1}},$$

s_i denotes the transposition $(i, i+1)$ and

$$f(x_1, \ldots, x_N)\tau_i = f(x_1, \ldots, x_{i-1}, q x_i, x_{i+1}, \ldots, x_N).$$

Note that the parameter t_1 could be omitted, since dividing each T_i by t_1 we obtain

$$\frac{1}{t_1}T_i^{t_1,t_2,q} = T_i^{1,\frac{t_2}{t_1},q}.$$

For simplicity, we will use the parameters $t_1 = 1$ and $t_2 = -s$. Then the quadratic relation is $(T_i+1)(T_i-s) = 0$ and $T_i := \overline{\pi}_i(1-s) + ss_i$. Note that these operators have interesting commutation properties with respect to the multiplications by x_i:

$$x_iT_i - T_ix_{i+1} - (1-s)x_{i+1} = 0 \tag{2.1}$$

$$x_{i+1}T_i - T_ix_i + (1-s)x_{i+1} = 0. \tag{2.2}$$

The double affine Hecke algebra is defined as

$$\mathcal{H}_N(q,s) := \mathbb{C}(s,q)[T_1,\dots,T_{N-1},w^{\pm 1},x_1^{\pm 1},\dots,x_N^{\pm 1}].$$

The double affine Hecke algebra admits a maximal commutative subalgebra generated by the Cherednik elements:

$$\xi_i := s^{i-N}T_{i-1}^{-1}\cdots T_1^{-1}wT_{N-1}\cdots T_i.$$

The (nonsymmetric) Macdonald polynomials are the simultaneous eigenfunctions of the Cherednik operators. This implies that one can compute them using the Yang–Baxter graphs: the spectral vector of 1 is

$$\zeta = \left[\left(\frac{1}{s}\right)^{i-1}\right]_{1\le i\le N}.$$

The nonaffine edges act by s_i on the spectral vector and by $T_i - \frac{1-s}{\frac{\zeta[i+1]}{\zeta[i]}-1}$ on the polynomials. The affine edges act by w on the spectral vector and by $\Phi_q := T_1^{-1}\cdots T_{N-1}^{-1}x_N$ on the polynomial. Note that there exists a shifted version. All of that is contained in the papers [10, 1].

From [1], we define a (q,s)-version of the Dunkl operator:

(1) $D_N := (1-s^{N-1}\xi_N)x_N^{-1}$,
(2) $D_i := \frac{1}{s}T_iD_{i+1}T_i$.

These operators generalize the Dunkl operator for the double affine Hecke algebra. For instance, one has

$$\begin{gathered} D_{i+1}T_i = -sT_i^{-1}D_i,\ -T_iD_{i+1} + (1-s)D_i + D_iT_i = 0, \\ -D_{i+1}T_i^{-1} - (1-\frac{1}{s})D_{i+1} + T_i^{-1}D_i = 0, \\ [D_i,T_j] = 0 \text{ when } |i-j|>1. \end{gathered} \tag{2.3}$$

The (q,s)-Dunkl operators have also interesting commutation properties with respect to the operator w, namely

$$D_{i+1}w = wD_i,\quad 1\le i\le N-1, \tag{2.4}$$

$$qD_1w = wD_N \tag{2.5}$$

Note also that the operators D_i commute with each other:

$$[D_i, D_j] = 0,\ 1 \le i, j \le N. \tag{2.6}$$

2.2. **Modules for the Hecke algebra.**

Definition 2.1. A **tableau** of shape λ is a filling with integers which is weakly increasing in each row and in each column. In the sequel **row-strict** means (strictly) increasing in each row and **column-strict** means (strictly) increasing in each column.

A **reverse standard tableau** (RST) is obtained by filling the shape λ with integers $1, \dots, N$ subject to the condition that entries along rows and columns are strictly decreasing. We denote by Tab_λ, the set of the RST with shape λ.

Let $\mathbb{T}$ be an RST. We define the **vector of contents** of τ as the vector $\mathrm{CT}_\mathbb{T}$ with the property that $\mathrm{CT}_\mathbb{T}[i]$ is the **content** of i in $\mathbb{T}$. (The coordinates of the cell are $(\mathrm{ROW}_\mathbb{T}[i], \mathrm{COL}_\mathbb{T}[i])$, row and column; $\mathrm{CT}_\mathbb{T}[i] = \mathrm{COL}_\mathbb{T}[i] - \mathrm{ROW}_\mathbb{T}[i]$.)

Example 2.2.

$$\mathrm{CT}\ \begin{array}{ccc} 2 & & \\ 5 & 4 & \\ 6 & 3 & 1 \end{array} = [2, -2, 1, 0, -1, 0].$$

As in [3, 4] (see also [9]), let us introduce the pairwise commuting Murphy elements

$$L_N := 0,$$
$$L_i := T_i + \frac{1}{s} T_i L_{i+1} T_i, \quad 1 \le i < N.$$

Let V_λ be the vector space spanned by (independent) $\{\mathbb{T} : \mathbb{T} \in \mathrm{Tab}\,(\lambda)\}$. The action of $\mathcal{H}_N\,(q, s)$ on V_λ satisfies

$$\mathbb{T} L_i = s \frac{1 - s^{\mathrm{CT}_\mathbb{T}[i]}}{1 - s} \mathbb{T}, 1 \le i \le N.$$

These equations determine $\{\mathbb{T}\}$ up to scalar multiplication. There is a modification of the Murphy elements which is actually more useful for our applications.

Definition 2.3. For $1 \le i \le N$ let $\phi_i := s^{i-N} T_i T_{i+1} \cdots T_{N-1} T_{N-1} \cdots T_i$, or, equivalently, $\phi_N = 1$ and $\phi_i = \frac{1}{s} T_i \phi_{i+1} T_i$ for $1 \le i < N$.

Proposition 2.4. *We have* $\phi_i = 1 + \frac{s-1}{s} L_i$ *for* $1 \le i \le N$, *and if* $\mathbb{T} \in \mathrm{Tab}\,(\lambda)$ *then* $\mathbb{T}\phi_i = s^{\mathrm{CT}_\mathbb{T}[i]}\mathbb{T}$.

Proof. Use downward induction. The statement is true for $i = N$. Suppose the statement is true for ϕ_{i+1}, then

$$\begin{aligned}
\phi_i &= \frac{1}{s} T_i \left(1 + \frac{s-1}{s} L_{i+1}\right) T_i \\
&= \frac{1}{s}\left(T_i^2 + \frac{s-1}{s} T_i L_{i+1} T_i\right) = \frac{1}{s}\left((s-1)\,T_i + s + \frac{s-1}{s} T_i L_{i+1} T_i\right) \\
&= 1 + \frac{s-1}{s} L_i.
\end{aligned}$$

Thus $\mathbb{T}\phi_i = \left(1 + \frac{s-1}{s}\frac{s\left(1-s^{\mathrm{CT}_{\mathbb{T}}[i]}\right)}{1-s}\right)\mathbb{T} = s^{\mathrm{CT}_{\mathbb{T}}[i]}\mathbb{T}$. □

There is an important commutation relation.

Lemma 2.5. *Suppose* $1 \le i, j \le N-1$ *and* $i \ne j, j+1$. *Then* $T_j^{-1}\phi_i T_j = \phi_i$.

Proof. If $j < i-1$ the result follows from $T_kT_j = T_jT_k$ for $|i-j| \ge 2$. If $j > i$ then (note that $T_j^{-1}T_{j-1}T_j = T_{j-1}T_jT_{j-1}^{-1}$)

$$\begin{aligned} s^{N-i}T_j^{-1}\phi_iT_j &= T_j^{-1}T_i\cdots T_{N-1}T_{N-1}\cdots T_iT_j \\ &= T_i\cdots T_{j-2}T_j^{-1}T_{j-1}T_j\cdots T_jT_{j-1}T_j\cdots T_i \\ &= T_i\cdots T_{j-2}T_{j-1}T_jT_{j-1}^{-1}T_{j+1}\cdots T_{j+1}T_jT_{j-1}T_j\cdots T_i \\ &= T_i\cdots T_{j-2}T_{j-1}T_jT_{j+1}\cdots T_{j+1}T_{j-1}^{-1}T_{j-1}T_jT_{j-1}\cdots T_i \\ &= s^{N-i}\phi_i. \end{aligned}$$

□

We describe the action of T_i on $\mathbb{T}$. There are two special cases:

$$\begin{aligned} &\text{if } \mathrm{ROW}_{\mathbb{T}}[i]) = \mathrm{ROW}_{\mathbb{T}}[i+1] \text{ then } \mathbb{T}T_i = s\mathbb{T}, \\ &\text{if } \quad \mathrm{COL}_{\mathbb{T}}[i] = \mathrm{COL}_{\mathbb{T}}[i+1] \ \text{ then } \mathbb{T}T_i = -\mathbb{T}. \end{aligned}$$

Otherwise, if we denote by $\mathbb{T}^{(i,j)}$ the tableau $\mathbb{T}$ where the entries i and j have been interchanged, the tableaux $\mathbb{T}^{(i,i+1)}$ is an RST. If $\mathrm{ROW}_{\mathbb{T}}[i] < \mathrm{ROW}_{\mathbb{T}}[i+1]$ (implying $\mathrm{COL}_{\mathbb{T}}[i] > \mathrm{COL}_{\mathbb{T}}[i+1]$) then

$$\mathbb{T}T_i = \mathbb{T}^{(i,i+1)} - \frac{1-s}{1-s^{\mathrm{COL}_{\mathbb{T}}[i+1]-\mathrm{COL}_{\mathbb{T}}[i]}}\mathbb{T}. \tag{2.7}$$

Note that this is a formula for $\mathbb{T}^{(i,i+1)}$. If $\mathrm{ROW}_{\mathbb{T}}[i] > \mathrm{ROW}_{\mathbb{T}}[i+1]$ (implying $\mathrm{COL}_{\mathbb{T}}[i] < \mathrm{COL}_{\mathbb{T}}[i+1]$) then set $m := \mathrm{CT}_{\mathbb{T}}[i+1] - \mathrm{CT}_{\mathbb{T}}[i]$ (which is > 0 by the hypothesis). It follows that

$$\mathbb{T}T_i = \frac{s-1}{1-s^m}\mathbb{T} + \frac{s\left(1-s^{m+1}\right)\left(1-s^{m-1}\right)}{\left(1-s^m\right)^2}\mathbb{T}^{(i,i+1)}. \tag{2.8}$$

Formally this gives also the special cases $m = 1$ when $\mathrm{COL}_{\mathbb{T}}[i] = \mathrm{COL}_{\mathbb{T}}[i+1]$ and $m = -1$ when $\mathrm{ROW}_{\mathbb{T}}[i] = \mathrm{ROW}_{\mathbb{T}}[i+1]$.

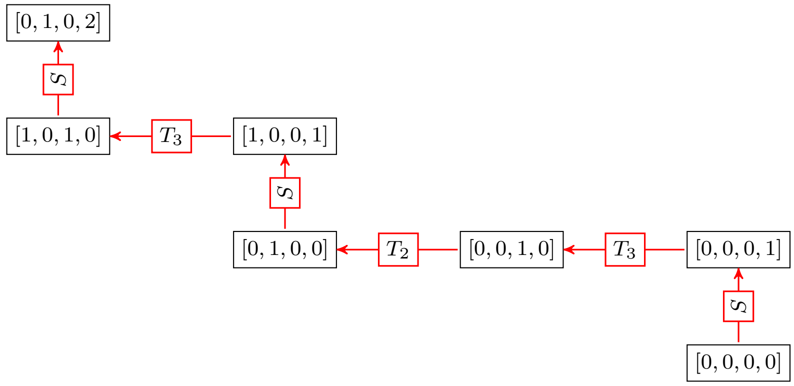
2.3. **Hecke elements associated to a multi-index.** Denote $S := T_1\cdots T_{N-1}$ and $\theta := s_1\cdots s_{N-1}$. Observe that if $i > 1$

$$T_iS = ST_{i-1} \text{ and } s_i\theta = \theta s_{i-1}. \tag{2.9}$$

For each multi-index $u = [u_1, \ldots, u_N]$ we define

$$T_u = \begin{cases} 1 & \text{if } u = [0,\ldots,0], \\ T_{[u_N-1,u_1,\ldots,u_{N-1}]}S & \text{if } u_N > 0, \\ T_{[u_1,\ldots,u_{i-1},0,u_i,0,\ldots,0]}T_i & \text{if } u_i > 0. \end{cases} \tag{2.10}$$

Example 2.6. Let $u = [0,1,0,2]$ then $T_u = ST_3T_2ST_3S$:



Since we use only braid relations and commutations, if $u[j] > u[j+1]$ one has

$$T_u = T_{us_j} T_j. \tag{2.11}$$

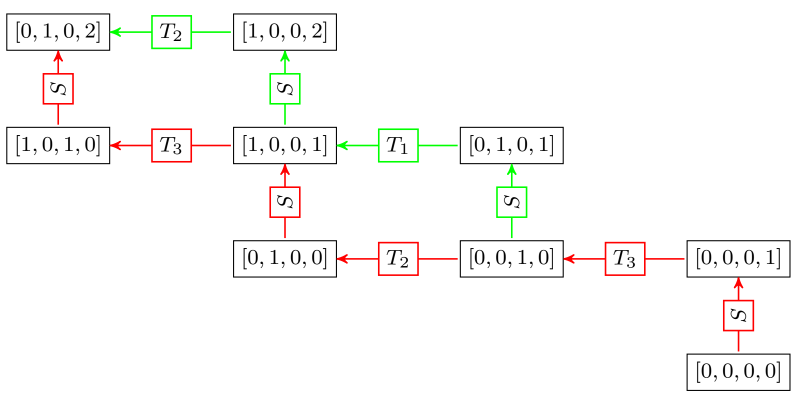
Hence, the vector T_u can be obtained by any product of the type $A_1 \cdots A_k$ where $A_i \in \{S\} \cup \{T_i : i = 1, \dots, N-1\}$ are such that

(1) We obtain u from $[0, \dots, 0]$ by applying $a_1 \cdots a_k$ where $a_i = s_j$ if $A_i = T_j$ and $a_i = \theta$ if $A_i = S$.
(2) If $a_i = s_j$ then $u' := u \cdot a_1 \cdots a_{i-1}$ verifies $u'[j] < u'[j+1]$.

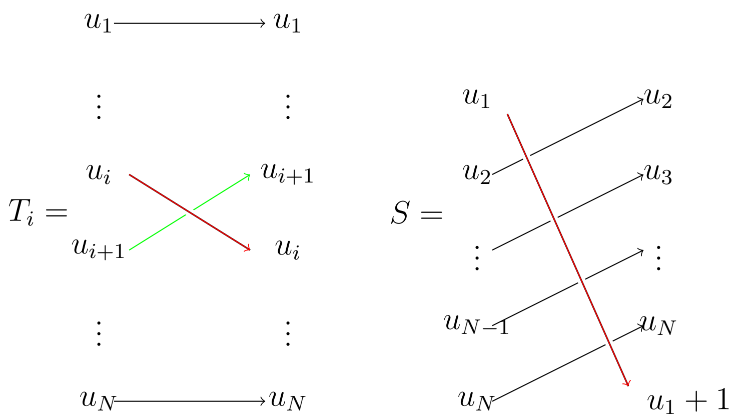
Example 2.7. One has

$$\begin{array}{rcl} T_{[0102]} & = & ST_3T_2ST_3S \\ & = & ST_3T_2T_1T_2T_3T_3T_1T_2T_3 \\ & = & ST_3T_1T_2T_1T_3T_3T_1T_2T_3 \\ & = & ST_3T_1T_2T_3T_1T_1T_3T_2T_3 \\ & = & ST_3T_1T_2T_3T_1T_1T_2T_3T_2 \\ & = & ST_3ST_1ST_2 \end{array}$$

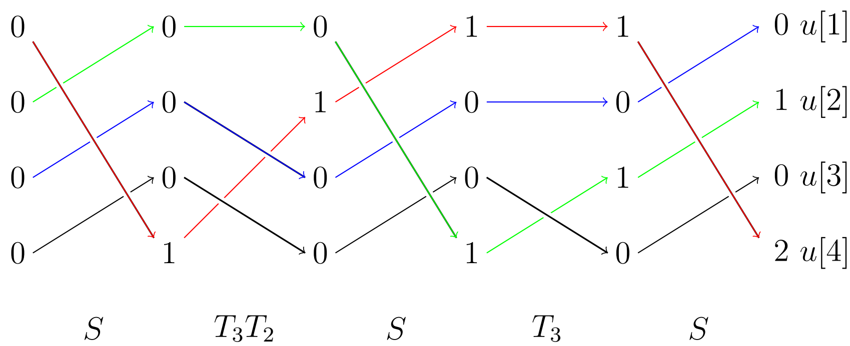
Graphically this is



Remark 2.8. The construction of T_u can be illustrated in terms of braids. The generators T_i and S are interpreted as



For instance for $u = [0, 1, 0, 2]$ one obtains the braid:



We introduce the creation operator

$$\mathfrak{C}_i := (ST_{N-1}\cdots T_i)^i$$

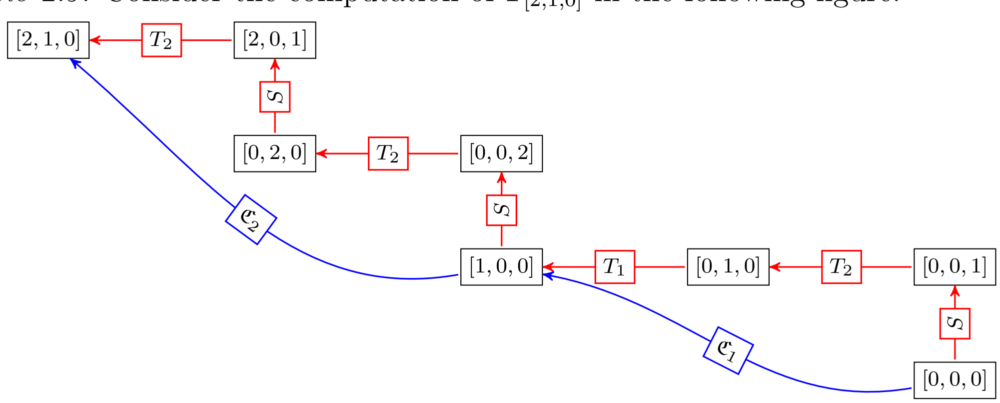
This operator has the property that, if $v = [v[1], \ldots, v[N]]$ is a partition, then

$$T_v\mathfrak{C}_i = T_{[v[1]+1,\ldots,v[i]+1,v[i+1],\ldots,v[N]]}$$

is the partition obtained from v by adding 1 to the i first entries. As a consequence, the element associated to a partition is a product of creation operators

$$T_{[v_1,\ldots,v_N]} = \mathfrak{C}_1^{v_1-v_2}\cdots\mathfrak{C}_{N-1}^{v_{N-1}-v_N}\mathfrak{C}_N^{v_N}.$$

Example 2.9. Consider the computation of $T_{[2,1,0]}$ in the following figure.



Setting $\widetilde{\phi}_i := s^{N-i}\phi_i = T_i \cdots T_{N-1}T_{N-1}\cdots T_i$, one obtains the following factorization.

Proposition 2.10. *For each i, we have*

$$\mathfrak{C}_i = \widetilde{\phi}_1 \cdots \widetilde{\phi}_i.$$

In order to provide a proof, we need the following auxiliary result.

Lemma 2.11. *Let $i-k>1$. Then we have*

$$(T_{i-k}\cdots T_i)(ST_{N-1}\cdots T_i) = (ST_{N-1}\cdots T_{i+1})(T_{i-k-1}\cdots T_i).$$

Proof. By eq. (2.9), one has

$$\begin{aligned} T_i(ST_{N-1}\cdots T_i) &= ST_{i-1}(T_{N-1}\cdots T_i) \\ &= (ST_{N-1}\cdots T_{i+1})(T_{i-1}T_i). \end{aligned}$$

Hence, using eq. (2.9) iteratively, one obtains

$$\begin{aligned} (T_{i-k}\cdots T_i)(ST_{N-1}\cdots T_i) &= (T_{i-k}\cdots T_{i-1})(ST_{N-1}\cdots T_{i+1})(T_{i-1}T_i) \\ &= S(T_{i-k-1}\cdots T_{i-2})(T_{N-1}\cdots T_{i+1})(T_{i-1}T_i) \\ &= (ST_{N-1}\cdots T_{i+1})(T_{i-k-1}\cdots T_i), \end{aligned}$$

as expected. □

Proof of Proposition 2.10. Applying Lemma 2.11 iteratively, one has

$$\begin{aligned} \widetilde{\phi}_1\widetilde{\phi}_2\cdots\widetilde{\phi}_i &= (ST_{N-1}\cdots T_i)(T_{i-1}\cdots T_2)(ST_{N-1}\cdots T_2)\widetilde{\phi}_3\cdots\widetilde{\phi}_i \\ &= (ST_{N-1}\cdots T_i)^2(T_{i-2}T_{i-1})\cdots(T_1T_2)\widetilde{\phi}_3\cdots\widetilde{\phi}_i \\ &= (ST_{N-1}\cdots T_i)^2(T_{i-2}T_{i-1})\cdots(T_2T_3)ST_{N-1}\cdots T_3\widetilde{\phi}_4\cdots\widetilde{\phi}_i \\ &= (ST_{N-1}\cdots T_i)^3(T_{i-3}T_{i-2}T_{i-1})\cdots(T_1T_2T_3)\widetilde{\phi}_4\cdots\widetilde{\phi}_i \\ &= (ST_{N-1}\cdots T_i)^4(T_{i-4}T_{i-3}T_{i-2}T_{i-1})\cdots(T_1T_2T_3T_4)\widetilde{\phi}_5\cdots\widetilde{\phi}_i \\ &= \cdots \\ &= (ST_{N-1}\cdots T_{N-i})^i. \end{aligned}$$

□

As a consequence, if $\mathbb{T}$ is an RST and v is a partition, one has

$$\mathbb{T}T_v = s^*\mathbb{T}, \tag{2.12}$$

where $*$ denotes an integer which depends only on v and $\mathbb{T}$.

2.4. **Rank function.** There is a unique element of $\mathcal{H}_N(q,s)$ associated to each $\sigma \in \mathfrak{S}_N$. The length of $\sigma \in \mathfrak{S}_N$ is

$$\ell(\sigma) := \#\{(i,j) : 1 \le i < j \le N, i\cdot\sigma > j\cdot\sigma\}.$$

There is a shortest expression $\sigma = s_{i_1}\cdots s_{i_{\ell(\sigma)}}$ and a unique element $\widetilde{T}_\sigma \in \mathcal{H}_N(q,s)$ defined by

$$\widetilde{T}_\sigma = T_{i_1}\cdots T_{i_{\ell(\sigma)}}. \tag{2.13}$$

For any s_i, $\ell(s_i\sigma) = \ell(\sigma) \pm 1$; if $\ell(s_i\sigma) = \ell(\sigma) + 1$ then $\widetilde{T}_{s_i\sigma} = T_i\widetilde{T}_\sigma$, and if $\ell(s_i\sigma) = \ell(\sigma) - 1$ then $\widetilde{T}_{s_i\sigma} = T_i^{-1}\widetilde{T}_\sigma$. Similarly, if $\ell(\sigma s_i) = \ell(\sigma) + 1$ then $\widetilde{T}_{\sigma s_i} = \widetilde{T}_\sigma T_i$, and if $\ell(\sigma s_i) = \ell(\sigma) - 1$ then $\widetilde{T}_{\sigma s_i} = \widetilde{T}_\sigma T_i^{-1}$. The following fact will be used in the analysis of the raising operator for polynomials.

Proposition 2.12. *Let $\sigma \in \mathfrak{S}_N$. Then $\widetilde{T}_\sigma^{-1}\widetilde{T}_\theta\widetilde{T}_{\theta^{-1}\sigma} = s^{N-(1\cdot\sigma)}\phi_{1\cdot\sigma}$.*

Proof. Use induction on $\ell(\sigma)$. The statement is true for $\ell(\sigma) = 0$, $\sigma = 1$, because $\widetilde{T}_\theta\widetilde{T}_{\theta^{-1}} = T_1 \cdots T_{N-1}T_{N-1}\cdots T_1 = s^{N-1}\phi_1$. Suppose the statement is true for all σ' with $\ell(\sigma') \leq n$ and $\ell(\sigma) = n+1$. For some k one has $\ell(\sigma s_k) = \ell(\sigma) - 1$. Set $\sigma' := \sigma s_k$ and $i := 1\cdot\sigma'$. Then we have $\widetilde{T}_\sigma = \widetilde{T}_{\sigma'}T_k$. If $\ell(\theta^{-1}\sigma' s_k) = \ell(\theta^{-1}\sigma') - 1$ then $\widetilde{T}_{\theta^{-1}\sigma} = \widetilde{T}_{\theta^{-1}\sigma'}T_k^{-1}$ and

$$\begin{aligned}\widetilde{T}_\sigma^{-1}\widetilde{T}_\theta\widetilde{T}_{\theta^{-1}\sigma} &= T_k^{-1}\widetilde{T}_{\sigma'}^{-1}\widetilde{T}_\theta\widetilde{T}_{\theta^{-1}\sigma'}T_k^{-1}\\ &= s^{N-i}T_k^{-1}\phi_i T_k^{-1},\end{aligned}$$

by the inductive hypothesis. If $\ell(\theta^{-1}w' s_k) = \ell(\theta^{-1}\sigma') + 1$ then $\widetilde{T}_{\theta^{-1}\sigma} = \widetilde{T}_{\theta^{-1}\sigma'}T_k$ and $\widetilde{T}_\sigma^{-1}\widetilde{T}_\theta\widetilde{T}_{\theta^{-1}\sigma} = s^{N-i}T_k^{-1}\phi_i T_k$ by a similar argument.

Let $i_1 = k\cdot\sigma'^{-1}$ and $i_2 = (k+1)\cdot\sigma'^{-1}$. By hypothesis, we have $i_1 < i_2$. Let $j_1 = k\cdot(\theta^{-1}\sigma')^{-1} = i_1\cdot\theta$ and $j_2 = (k+1)\cdot(\theta^{-1}\sigma')^{-1} = i_2\cdot\theta$. Then $\ell(\theta^{-1}\sigma' s_k) = \ell(\theta^{-1}\sigma') + 1$ if and only if $j_1 < j_2$. (Note that $j\cdot\theta = j-1$ if $j > 1$ and $1\cdot\theta = N$.) Since $i_2 > i_1 \geq 1$, it follows that $j_2 = i_2 - 1$. If $i_1 = 1$ then $j_1 = N > j_2$, and so $\ell(\theta^{-1}\sigma' s_k) = \ell(\theta^{-1}\sigma') - 1$, $k = 1\cdot\sigma' = i$. This implies $1\cdot\sigma = i+1$ and $\widetilde{T}_\sigma^{-1}\widetilde{T}_\theta\widetilde{T}_{\theta^{-1}\sigma} = s^{N-i}T_i^{-1}\phi_i T_i^{-1} = s^{N-i-1}\phi_{i+1}$. If $i_1 > 1$ then $j_1 = i_1 - 1 < j_2$ and $\ell(\theta^{-1}\sigma' s_k) = \ell(\theta^{-1}\sigma') + 1$. In this case $1\cdot\sigma' \neq k, k+1$, and so $s^{N-i}T_k^{-1}\phi_i T_k = s^{N-i}\phi_i$, by Lemma 2.5; also $1\cdot\sigma = 1\cdot\sigma' = i$; and this completes the induction. □

Consider the rank function of a multi-index $v = [v[1], \ldots, v[N]]$ as an element of $\mathfrak{S}_N$,

$$r_v[i] := \#\{j : 1 \leq j \leq i, v[j] \geq v[i]\} + \#\{j : i < j \leq N, v[j] > v[i]\}.$$

Example 2.13. (1) If $v = [4,2,2,3,2,1,4,4]$ then $r_v = [1,5,6,4,7,8,2,3]$.
(2) If v is a (decreasing) partition then $r_v = id$.

The length of r_v is

$$\ell(r_v) := \#\mathrm{inv}(v),$$

with $\mathrm{inv}(v) := \{(i,j) : 1 \leq i < j \leq N, v[i] < v[j]\}$ being the number of inversions in v (note that for $i < j$ we have $r_v[i] > r_v[j]$ if and only if $v[i] < v[j]$). There is a shortest expression $r_v = s_{i_1}\cdots s_{i_{\ell(r_v)}}$ and an element $R_v \in \mathcal{H}_N(q,s)$ defined by

$$R_v := T_{i_{\ell(r\{\alpha\})}}^{-1}\cdots T_{i_1}^{-1} = \widetilde{T}_{r_v}^{-1}.$$

We have the following auxiliary result.

Lemma 2.14. (1) *If $v[i] > v[i+1]$ then $R_{vs_i} = R_v T_i^{-1}$.*
(2) *If $v[i] < v[i+1]$ then $R_{vs_i} = R_v T_i$.*
(3) *If $v[i] = v[i+1]$ then $R_v T_i = T_{r_v[i]}R_v$.*

Proof. (1) If $v[i] > v[i+1]$ then $r_{vs_i} = s_i r_v$and $\#\mathrm{inv}(vs_i) = \#\mathrm{inv}(v) + 1$, so $R_{vs_i} = R_v T_i^{-1}$.

(2) Similarly if $v[i] < v[i+1]$ then $R_{vs_i} = R_v T_i$.

(3) If $v[i] = v[i+1]$ and $k = r_v[i]$ then $s_i r_v = r_v s_k$ and $\ell(s_i r_v) = \ell(r_v) + 1$ (one extra inverted pair $(k+1,k)$); thus $\widetilde{T}_{s_i r_v} = T_i \widetilde{T}_{r_v}$ and $\widetilde{T}_{r_v s_k} = \widetilde{T}_{r_v} T_k$. Hence, $R_v T_i = T_k R_v$.

□

We compare the elements T_v and R_v in terms of $T_v R_v^{-1}$. We need to consider three cases:

(1) If $T_{[0,\dots,0]} = I$ then $r_v = I = T_{[0,\dots,0]}$.

(2) In the case $T_{[v_1,v_2,\dots,v_{i-1},v_i,0,\dots]} = T_{[v_1,v_2,\dots,v_{i-1},0,v_i,0,\dots]} T_i$ ($v_i \geq 1$, $i < N$), we see that $\#\mathrm{inv}(v \cdot s_i) = \#\mathrm{inv}(v) + 1$, hence $r_{v \cdot s_i} = s_i r_v$ (see Lemma 2.14 (1)) and $\widetilde{T}_{r_{v \cdot s_i}} = T_i \widetilde{T}_{r_v}$, $R_{v \cdot s_i} = R_v T_i^{-1}$. So we have

$$T_{v \cdot s_i} R_{v \cdot s_i}^{-1} = T_v R_v^{-1}. \tag{2.14}$$

(3) If $T_{v\Psi} = T_v S$ (with $v\Psi := (v_2, v_3, \dots, v_N, v_1 + 1)$), then we have

$$r_{v\Psi} = s_{N-1} s_{N-2} \cdots s_1 r_v = \theta^{-1} r_v,$$

where $\theta = s_1 s_2 \cdots s_{N-1}$. By Proposition 2.12 (let $k = r_v[1]$), we obtain

$$\widetilde{T}_{r_v}^{-1} \widetilde{T}_\theta \widetilde{T}_{\theta^{-1} r_v} = s^{N-k} \phi_k,$$
$$s^{-N+k} \phi_k^{-1} R_v S = R_{v\Psi},$$

and thus

$$T_{v\Psi} R_{v\Psi}^{-1} = s^{N-k} T_v S S^{-1} R_v^{-1} \phi_k = s^{N-k} T_v R_v^{-1} \phi_k. \tag{2.15}$$

As a consequence, we are able to derive the following result.

Proposition 2.15. *$T_v R_v^{-1}$ is in the commutative algebra generated by*

$$\{\phi_i : 1 \leq i \leq N\}$$

for each v, and it acts by scalar multiplication (by powers of s) on each $\mathbb{T}$ (recall that $\mathbb{T}\phi_i = s^{CT(i,\mathbb{T})}\mathbb{T}$, $1 \leq i \leq N$). Furthermore we have

$$T_v = \prod_{i=1}^{N} \left(s^{N-i} \phi_i\right)^{v_i^+} R_v.$$

Proof. By eq. (2.14), we see that, if the formula is true for v with $v_j = 0$ for $j > i$ and $v_i \geq 1$, then it is also true for $v \cdot s_i$ (note that $(v \cdot s_i)^+ = v^+$). Using induction, suppose the formula is true for all v with $|v| \leq n$, for some $n \geq 0$ (the case $n = 0$ is trivially satisfied). Let $|v| = n + 1$. Using the case 2 step as often as necessary, assume that $v_N \geq 1$. Thus $v = u\psi$ with $|u| = n$, and $r_v = \theta^{-1} r_u$; in particular, let $k = r_v[N] = r_u[1]$. Then $v^+ = \left(u_1^+, \dots, u_k^+ + 1, \dots, u_N^+\right)$ (u has exactly $k - 1$ entries $> u_1$, and thus v has exactly k entries $\geq v_N = u_1 + 1$, including v_N; hence $v_k^+ = v_N = u_1 + 1 = u_k^+ + 1$). By eq. (2.15) and the inductive hypothesis, we obtain

$$T_v R_v^{-1} = \left(s^{N-k}\phi_k\right) T_u R_u^{-1} = \left(s^{N-k}\phi_k\right) \prod_{i=1}^{N} \left(s^{N-i}\phi_i\right)^{u_i^+},$$

and this proves the claim. □

In particular, if v is a partition then $T_v = \prod_{i=1}^{N} \left(s^{N-i}\phi_i\right)^{v_i}$.

3. Vector valued polynomials

3.1. **First Examples.** To motivate our definitions we consider the simplest two-dimensional situation: $N = 3$, isotype $\lambda = (2, 1)$. A basis for the representation of $\{T_1, T_2\}$ is

$$f_1 = s x_1 - \frac{1}{s+1}(x_2 + x_3),$$
$$f_2 = x_2 - \frac{1}{s} x_3.$$

Then $f_1 T_2 = s f_1$, $f_2 T_2 = -f_2$ and

$$f_1 T_1 = -\frac{1}{s+1} f_1 + \frac{s\left(1+s+s^2\right)}{(1+s)^2} f_2,$$
$$f_2 T_1 = f_1 + \frac{s^2}{1+s} f_2.$$

We aim to set up a Macdonald-type structure in $\{p_1(x) f_1 + p_2(x) f_2\}$. Firstly define operators $T_i^{'}$ acting on pairs $[p_1, p_2]$ so that

$$[p_1, p_2]\, T_i^{'} \cdot [f_1, f_2] = (p_1 f_1 + p_2 f_2)\, T_i, \quad i = 1, 2,$$

where $[a_1, a_2] \cdot [b_1, b_2] := a_1 b_1 + a_2 b_2$. Indeed, we have

$$[p_1, p_2]\, T_1^{'} = \left[p_1 T_1 - \frac{1+s+s^2}{1+s} p_1 s_1 + p_2 s_1, p_2 T_1 - \frac{s}{1+s} p_2 s_1 + \frac{s\left(1+s+s^2\right)}{(1+s)^2} p_1 s_1\right],$$
$$[p_1, p_2]\, T_2^{'} = [p_1 T_2, p_2 T_2 - (s+1)\, p_2 s_2].$$

The inverses are obtained from the quadratic relation: $T_i^{'-1} = \frac{1}{s}\left(T_i^{'} + 1 - s\right)$.

Secondly, we need a definition of w (to be generalized in the sequel). The relation $w T_1 = T_2 w$ must be satisfied. The braid relation gives a solution $T_2 (T_1 T_2) = (T_1 T_2) T_1$. Using $w' = T_1 T_2$, let

$$f_1 w' = -\frac{s}{1+s} f_1 - \frac{s\left(1+s+s^2\right)}{(1+s)^2} f_2,$$
$$f_2 w' = s f_1 - \frac{s^2}{1+s} f_2.$$

Then $w'T_1 = T_2 w'$, acting on *span* $\{f_1, f_2\}$. Now define

$$[p_1, p_2]\, w = \left[-\frac{s}{1+s} p_1 w + s p_2 w, -\frac{s\,(1+s+s^2)}{(1+s)^2} p_1 w - \frac{s^2}{1+s} p_2 w\right].$$

Set

$$\begin{aligned}\xi_1 &= s^{-2} w T_2' T_1',\\ \xi_2 &= s^{-1} T_1'^{-1} w T_2',\\ \xi_3 &= T_2'^{-1} T_1'^{-1} w.\end{aligned}$$

These operators commute. Here are the degree 1 simultaneous eigenfunctions:

$$\begin{aligned}&[-(1+s)\,x_3, s x_3]\,,\\ &\left[x_3, \frac{1+s+s^2}{1+s} x_3\right],\\ &\left[(s+1)\,x_2 + \frac{q\,(1-s^2)}{1-qs} x_3, x_2 - \frac{sq\,(1-s)}{1-qs} x_3\right],\\ &\left[x_2 - \frac{q\,(1-s)}{s\,(q-s)} x_3, -\frac{1+s+s^2}{s\,(1+s)}\left\{x_2 + \frac{q\,(1-s)}{q-s} x_3\right\}\right].\\ &\left[\frac{q\,(1-s)}{1-q^2 s}\{s x_2 - x_3\}\,, x_1 + \frac{sq\,(1-s)}{(1+s)\,(1-q^2 s)}\{x_2 + x_3\}\right],\\ &\left[x_1 + \frac{qs\,(1-s)}{(1+s)\,(q-s^2)}\{x_2 + x_3\}\,, -\frac{q\,(1+s+s^2)\,(1-s)}{(1+s)^2\,(q-s^2)}\{x_2 - s x_3\}\right].\end{aligned}$$

To generalize this setup to an arbitrary irreducible module V_λ (basis corresponding to Tab_λ), we need to define w; a necessary condition is that there be an intertwining operator S on V so that $ST_i = T_{i+1}S$ for $1 \le i < N$. The correct definition is $S = T_1 T_2 \cdots T_{N-1}$. Indeed,

$$\begin{aligned}ST_i &= T_1 \cdots T_{i-1} T_i T_{i+1} T_i T_{i+2} \cdots T_{N-1}\\ &= T_1 \cdots T_{i-1} T_{i+1} T_i T_{i+1} T_{i+2} \cdots T_{N-1}\\ &= T_{i+1} S.\end{aligned}$$

Definition 3.1. The space of vector valued polynomials for the isotype λ (a partition of N) will be denoted by $\mathcal{M}_\lambda := \mathbb{C}[x_1, \ldots, x_N] \otimes V_\lambda$.

The elements of $\mathcal{M}_\lambda$ are linear combinations of $x^v \mathbb{T}$, where $x^v := x_1^{v[1]} \cdots x_N^{v[N]}$. We denote by 'normal symbols' (s_i, T_i, w, ξ_i, etc.) the operators acting only on tableaux. The operators acting only on letters will be denoted with superscript ^{x} (s_i^x, T_i^x, w^x, ξ_i^x, etc.). The operators acting on both letters and tableaux will be denoted by bold symbols ($\mathbf{s}_i$, $\mathbf{T}_i$, $\mathbf{w}$, $\boldsymbol{\xi}_i$, etc.).

3.2. **Action of the double Hecke algebra on vectors.** Denote $\delta_i^x := T_i^x - s \cdot s_i^x = \partial_i^x x_{i+1}(1-s)$ and $\mathbf{T}_i := \delta_i^x + s_i^x T_i$. We have the following fact.

Lemma 3.2. *The operator $\mathbf{T}_i$ satisfies the quadratic relation:*

$$(\mathbf{T}_i + 1)(\mathbf{T}_i - s) = 0. \tag{3.1}$$

Proof. From

$$\partial_i^x x_{i+1}\partial_i^x = \partial_i^x\partial_i^x x_{i+1} + \partial_i^x s_i^x(x_{i+1}\partial_i^x) = -\partial_i^x,$$

we deduce

$$\delta_i^{x2} - (1-s)^2\partial_i^x x_{i+1} = -(1-s)\delta_i^x. \tag{3.2}$$

Moreover, from

$$\partial_i^x x_{i+1}s_i^x + s_i^x\partial_i^x x_{i+1} = \partial_i^x(x_i - x_{i+1}) = 1 - s_i^x,$$

one obtains

$$\delta_i^x s_i^x + s_i^x\delta_i^x = (1-s)(1-s_i^x). \tag{3.3}$$

Now, expanding $(\mathbf{T}_i+1)(\mathbf{T}_i-s)$, from eqs. (3.2) and (3.3) we observe that

$$\begin{aligned}(\mathbf{T}_i+1)(\mathbf{T}_i-s) &= (\delta_i^{x2} + (1-s)\delta_i^x) + (\delta_i^x s_i^x + s_i^x\delta_i^x + (1-s)(s_i^x-1))T_i\\ &= 0,\end{aligned}$$

□

We find also commutation relations.

Lemma 3.3. *If $|i-j|>1$, we have*

$$\mathbf{T}_i\mathbf{T}_j = \mathbf{T}_j\mathbf{T}_i. \tag{3.4}$$

Proof. First we expand

$$\mathbf{T}_i\mathbf{T}_j = \delta_i^x\delta_j^x + \delta_i^x s_j^x T_j + s_i^x\delta_j^x T_i + s_i^x s_j^x T_iT_j. \tag{3.5}$$

But since $|i-j|>1$, one has straightforwardly $s_i^x s_j^x = s_j^x s_i^x$, $T_iT_j = T_jT_i$, $\delta_i^x s_j^x = s_j^x\delta_i^x$ and $\delta_i^x\delta_j^x = \delta_j^x\delta_i^x$. Using these relations in eq. (3.5), we find the result. □

To prove the braid relations, we need the following preliminary results.

Lemma 3.4.
(1) $s_i^x s_{i+1}^x s_i^x T_iT_{i+1}T_i = s_{i+1}^x s_i^x s_{i+1}^x T_{i+1}T_iT_{i+1}$,
(2) $\delta_i^x\delta_{i+1}^x\delta_i^x = \delta_{i+1}^x\delta_i^x\delta_{i+1}^x$,
(3) $\delta_{i+1}^x s_i\delta_{i+1}^x = s_i^x\delta_{i+1}^x\delta_i^x + \delta_i^x\delta_{i+1}^x s_i^x + (s-1)s_i^x\delta_{i+1}^x s_i^x$,
(4) $\delta_i^x s_{i+1}^x\delta_i^x = s_{i+1}^x\delta_i^x\delta_{i+1}^x + \delta_{i+1}^x\delta_i^x s_{i+1}^x + (s-1)s_{i+1}^x\delta_i^x s_{i+1}^x$,
(5) $\delta_i^x s_{i+1}^x s_i^x = s_{i+1}^x s_i^x\delta_{i+1}^x$,
(6) $\delta_{i+1}^x s_i^x s_{i+1}^x = s_i^x s_{i+1}^x\delta_i^x$,
(7) $s_i^x\delta_{i+1}^x s_i^x = s_{i+1}^x\delta_i^x s_{i+1}^x$.

Proof. The first identity is trivial, but the others need to be proved. The simplest way to check these formulae is the direct verification of the action on a monomial $x_i^a x_{i+1}^b x_{i+2}^c$. For instance, the second equality follows from

$$\begin{aligned}x_i^a x_{i+1}^b x_{i+2}^c\delta_i^x\delta_{i+1}^x\delta_i^x &= (1-s)^3\frac{x_{i+2}^2x_{i+1}}{V(x_i,x_{i+1},x_{i+2})}\det\begin{pmatrix} x_i^a & x_i^b & x_i^c\\ x_{i+1}^a & x_{i+1}^b & x_{i+1}^c\\ x_{i+2}^a & x_{i+2}^b & x_{i+2}^c\end{pmatrix}\\ &= x_i^a x_{i+1}^b x_{i+2}^c\delta_{i+1}^x\delta_i^x\delta_{i+1}^x,\end{aligned}$$

where $V(x_1,x_2,x_3) := \prod_{0<i<j<4}(x_i-x_j)$ denotes the Vandermonde determinant. The other identities can be verified in a similar fashion (for simplicity we omit the superscript ^{x} on δ and s):

(3)

$$
\begin{aligned}
&x_i^a x_{i+1}^b x_{i+2}^c \delta_i s_{i+1} \delta_i \\
&\quad = (1-s)^2 \frac{x_{i+2}x_{i+1}}{V(x_i,x_{i+1},x_{i+1})} \left[x_{i+2}^a \left(x_{i+2}(x_i^b x_{i+1}^c - x_i^c x_{i+1}^b) - (x_i^b x_{i+1}^{c+1} - x_i^{c+1} x_{i+1}^b) \right) \right. \\
&\qquad\qquad \left. - x_{i+2}^b \left(x_{i+2}(x_i^a x_{i+1}^c - x_i^c x_{i+1}^a) - (x_i^a x_{i+1}^{c+1} - x_i^{c+1} x_{i+1}^a) \right) \right] \\
&\quad = x_i^a x_{i+1}^b x_{i+2}^c (s_{i+1}\delta_i\delta_{i+1} + \delta_{i+1}\delta_i s_{i+1} + (s-1) s_{i+1}\delta_i s_{i+1}).
\end{aligned}
$$

(4)

$$
\begin{aligned}
x_i^a x_{i+1}^b x_{i+2}^c \delta_{i+1} s_i \delta_{i+1} &= (1-s)^2 \frac{x_{i+1}}{V(x_i,x_{i+1},x_{i+1})} \\
&\quad \times \left[x_i^b \left(x_i(x_{i+1}^a x_{i+2}^{c+1} - x_{i+1}^{c+1} x_{i+2}^a) - (x_{i+1}^{a+1} x_{i+2}^{c+1} - x_{i+1}^{c+1} x_{i+2}^{a+2}) \right) \right. \\
&\qquad \left. - x_i^c \left(x_i(x_{i+1}^a x_{i+2}^{b+1} - x_{i+1}^{b+1} x_{i+2}^a) - (x_{i+1}^{a+1} x_{i+2}^{b+1} - x_{i+1}^{b+1} x_{i+2}^{a+1}) \right) \right] \\
&= x_i^a x_{i+1}^b x_{i+2}^c (s_i\delta_{i+1}\delta_i + \delta_i\delta_{i+1} s_i + (s-1) s_i\delta_{i+1} s_i).
\end{aligned}
$$

(5)

$$
\begin{aligned}
x_i^a x_{i+1}^b x_{i+2}^c (s_{i+1} s_i \delta_{i+1}) &= (1-s) x_{i+1} x_{i+2}^a \frac{x_i^b x_{i+1}^c - x_i^c x_{i+1}^b}{x_i - x_{i+1}} \\
&= x_i^a x_{i+1}^b x_{i+2}^c (\delta_i s_{i+1} s_i).
\end{aligned}
$$

(6)

$$
\begin{aligned}
x_i^a x_{i+1}^b x_{i+2}^c (s_i s_{i+1} \delta_i) &= (1-s) x_{i+2} x_i^c \frac{x_{i+1}^a x_{i+2}^b - x_{i+1}^b x_{i+2}^a}{x_{i+1} - x_{i+2}} \\
&= x_i^a x_{i+1}^b x_{i+2}^c (\delta_{i+1} s_i s_{i+1}).
\end{aligned}
$$

(7)

$$
\begin{aligned}
x_i^a x_{i+1}^b x_{i+2}^c (s_i \delta_{i+1} s_i) &= (1-s) x_{i+2} x_{i+1}^b \frac{x_i^a x_{i+2}^c - x_i^c x_{i+2}^a}{x_i - x_{i+2}} \\
&= x_i^a x_{i+1}^b x_{i+2}^c (s_{i+1} \delta_i s_{i+1}).
\end{aligned}
$$

□

Next we show that the operators $\{\mathbf{T}_i\}$ satisfy the braid relations.

Proposition 3.5. *For each $i < N-1$, one has*

$$\mathbf{T}_i\mathbf{T}_{i+1}\mathbf{T}_i = \mathbf{T}_{i+1}\mathbf{T}_i\mathbf{T}_{i+1}. \tag{3.6}$$

Proof. Expanding the braid $\mathbf{T}_i\mathbf{T}_{i+1}\mathbf{T}_i$, we obtain

$$
\begin{aligned}
\mathbf{T}_i\mathbf{T}_{i+1}\mathbf{T}_i &= \delta_i^x \delta_{i+1}^x \delta_i^x + \delta_i^x s_{i+1}^x \delta_i^x T_{i+1} \\
&\quad + \left(s_i^x \delta_{i+1}^x \delta_i^x + \delta_i^x \delta_{i+1}^x s_i^x \right) T_i + s_i^x s_{i+1}^x s_i^x T_{i+1} T_i + s_i^x s_{i+1}^x \delta_i^x T_i T_{i+1} \\
&\quad + s_i^x \delta_{i+1}^x s_i^x T_i^2 + s_i^x s_{i+1}^x s_i^x T_i T_{i+1} T_i.
\end{aligned}
$$

Using the fact that $T_i^2 = (s-1)T_i + s$, we obtain

$$\begin{aligned}\mathbf{T}_i\mathbf{T}_{i+1}\mathbf{T}_i = {} & \delta_i^x\delta_{i+1}^x\delta_i^x + \delta_i^x s_{i+1}^x\delta_i^x T_{i+1} \\ & + \left(s_i^x\delta_{i+1}^x\delta_i^x + \delta_i^x\delta_{i+1}^x s_i^x + (s-1)s_i^x\delta_{i+1}^x s_i^x\right) T_i + s_i^x s x_{i+1} s_i^x T_{i+1}T_i \\ & + s_i^x s_{i+1}^x\delta_i^x T_i T_{i+1} + s s_i^x\delta_{i+1}^x s_i^x + s_i^x s_{i+1}^x s_i^x T_i T_{i+1} T_i.\end{aligned}$$

Now applying Lemma 3.4, we arrive at the desired result. □

Next we examine the relation between the generators $\mathbf{T}_i$ and the multiplication by an indeterminate x_i. There are three identities satisfied by these operations.

Proposition 3.6. (1) $x_i\mathbf{T}_i - \mathbf{T}_i x_{i+1} - (1-s)x_{i+1} = 0$;
(2) $x_{i+1}\mathbf{T}_i - \mathbf{T}_i x_i + (1-s)x_{i+1} = 0$;
(3) $x_i\mathbf{T}_j = \mathbf{T}_j x_i$ *when* $|i-j| > 1$.

Proof. (1) One has

$$\begin{aligned}x_i\delta_i^x &= (1-s)x_i\partial_i^x x_{i+1} \\ &= (1-s)\partial_i^x x_{i+1}^2 + (1-s)x_{i+1} \\ &= \delta_i^x x_{i+1} + (1-s)x_{i+1}.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}x_i\mathbf{T}_i &= [\delta_i^x + s_i^x T_i]\, x_{i+1} + (1-s)x_{i+1} \\ &= \mathbf{T}_i x_{i+1} + (1-s)x_{i+1},\end{aligned}$$

as expected.

(2) The second equality is proved in the same way, by observing that

$$\begin{aligned}x_{i+1}\delta_i^x &= (1-s)x_{i+1}\partial_i^x x_{i+1} \\ &= (1-s)\partial_i^x x_{i+1}x_i - (1-s)x_{i+1} \\ &= \delta_i^x x_i - (1-s)x_{i+1}.\end{aligned}$$

(3) The third equality is straightforward.

□

Now, we examine the affine action and set

$$\mathbf{w} = \tau_1^x\theta^x S,$$

where $\theta^x = s_1^x \cdots s_{N-1}^x$ and $S = T_1 \cdots T_{N-1}$. When $i < N-1$, one has

$$\mathbf{w}\mathbf{T}_i = (\tau_1^x\theta^x S)(\delta_i^x + s_i^x T_i).$$

But since $i < N-1$, one has

$$\tau_1^x\theta^x\partial_i^x x_{i+1} = \tau_1^x\partial_{i+1}^x x_{i+2}\theta^x,$$

and $i+1 > 1$ implies $\tau_1^x\partial_{i+1}^x x_{i+2} = \partial_{i+1}^x x_{i+2}\tau_1^x$. Hence,

$$\tau_1^x\theta^x\delta_i^x = \delta_{i+1}^x\tau_1^x\theta^x.$$

One easily obtains $\tau_1^x\theta^x s_i^x = s_{i+1}^x\tau_1^x\theta^x$ and $ST_i = T_{i+1}S$. From this, we deduce the following commutation relation.

Lemma 3.7. *For each i, we have $\mathbf{w}\mathbf{T}_i = \mathbf{T}_{i+1}\mathbf{w}$.*

From Lemmas 3.2, 3.3, 3.7, Propositions 3.5 and 3.6, we obtain the following result.

Theorem 3.8. *The algebra $\mathbb{C}(s,q)[x_1^{\pm1},\dots,x_N^{\pm1},\mathbf{T}_1,\dots,\mathbf{T}_{N-1},\mathbf{w}^{\pm1}]$ is isomorphic to $\mathcal{H}_N(s,q)$. More precisely, the morphism sends T_i to $\mathbf{T}_i$, w to $\mathbf{w}$, and x_i to x_i.*

3.3. **Cherednik and Dunkl operators.**

Definition 3.9. In this context, the (vector valued) Cherednik operators are defined by

$$\boldsymbol{\xi}_i = s^{i-N}\mathbf{T}_{i-1}^{-1}\cdots\mathbf{T}_1^{-1}\mathbf{w}\mathbf{T}_{N-1}\cdots\mathbf{T}_i,$$

where

$$\mathbf{T}_i^{-1} = \frac{1}{s}(\mathbf{T}_i + (1-s)) = \frac{1}{s}((1-s)(\partial_i^x x_{i+1}+1) + s_i^x T_i).$$

It follows immediately that

$$[\boldsymbol{\xi}_i,\boldsymbol{\xi}_j] = 0, \tag{3.7}$$

since, by Theorem 3.8, the operators $\boldsymbol{\xi}_i$ are the image of the Cherednik operators ξ_i. Furthermore, the tableaux are simultaneous eigenfunctions of the Cherednik elements, and the associated spectral vectors can be expressed in terms of contents.

Proposition 3.10. *For each tableau $\mathbb{T}$, one has*

$$\mathbb{T}\boldsymbol{\xi}_i = s^{\mathbf{CT}_{\mathbb{T}}[i]}\mathbb{T}.$$

Proof. Since,

1. $\mathbb{T}\mathbf{T_i} = \mathbb{T}T_i$,
2. $\mathbb{T}\mathbf{T_i}^{-1} = \mathbb{T}T_i^{-1}$,
3. $\mathbb{T}\mathbf{w} = \mathbb{T}S$,

one has $\mathbb{T}\boldsymbol{\xi}_i = \mathbb{T}\phi_i$. Hence, the result follows from Proposition 2.4. □

In the aim to define the Dunkl–Cherednik operators, we set $\mathbf{F}_N = 1 - \boldsymbol{\xi}_N$.

Proposition 3.11. *The operator $\mathbf{F}_N$ is divisible by x_N, that is, for each $P \in \mathbb{C}[x_1,\dots,x_N]\otimes V_\lambda$, $P\mathbf{F}_N = x_N Q$ with $Q \in \mathbb{C}[x_1,\dots,x_N]\otimes V_\lambda$.*

Proof. We prove the result by induction on N. Suppose first that $N=2$. Then our operator is

$$\begin{aligned}\mathbf{F}_2 &= 1 - \frac{1}{s}((1-s)(\partial_1^x x_2+1) + s_1^x T_1)(\tau_1^x s_1^x T_1)\\ &= 1 - \frac{1}{s}((1-s)(\partial_1^x x_2+1)\tau_1^x s_1^x + s_1^x\tau_1^x s_1^x T_1^2).\end{aligned}$$

From $T_1^2 = (s-1)T_1 + s$ and $s_1\tau_1 s_1 = \tau_2$, one obtains

$$\mathbf{F}_2 = 1 - \frac{1}{s}\left((1-s)(\partial_1^x x_2 + 1 - s_1)\tau_1^x s_1^x T_1 + s\tau_2^x\right).$$

Note that

$$\partial_1^x x_2 + 1 - s_1^x = \partial_1^x x_1$$

implies

$$\mathbf{F}_2 = \frac{s-1}{s} q\partial_1^x \tau_1^x s_1^x T_1 x_2 + 1 - \tau_2^x.$$

But, for any polynomial P, one has

$$P(x_1)x_2^b(1-\tau_2^x) = \begin{cases} 0 & \text{if } b=0, \\ P(x_1)x_2^b(1-q^b) & \text{if } b>0. \end{cases}$$

This proves the result for $N=2$.

Now suppose that $N>2$. Then we have

$$\mathbf{F}_N = 1 - \mathbf{T}_{N-1}^{-1}\cdots\mathbf{T}_1^{-1}(\tau_1^x s_1^x \cdots s_{N-1}^x T_1 \cdots T_{N-1}).$$

Similarly to the case $N=2$, one obtains

$$\mathbf{F}_N = 1 - \frac{1}{s}\mathbf{T}_{N-1}^{-1}\cdots\mathbf{T}_2^{-1}\left((sq\partial_1^x\tau_1^x s_1^x \cdots s_{N-1}^x T_1\cdots T_{N-1})x_{N-1}\right.$$
$$\left. + s\tau_2^x s_2^x \cdots s_{N-1}^x T_2 \cdots T_{N-1}\right).$$

So it suffices to prove that the operator $1-\mathbf{T}_{N-1}^{-1}\cdots\mathbf{T}_2^{-1}s_2^x\cdots s_{N-1}^x T_2\cdots T_{N-1}$ is divisible by x_N. Observing that

$$1 - \mathbf{T}_{N-1}^{-1}\cdots\mathbf{T}_2^{-1}(s_2^x\cdots s_{N-1}^x T_2 \cdots T_{N-1}) = \theta^{-1}\mathbf{F}_{N-1}\theta,$$

the result follows by induction. □

Definition 3.12. The vector valued Dunkl operators are defined by $\mathbf{D}_N := \mathbf{F}_N x_N^{-1}$ and $\mathbf{D}_i := \frac{1}{s}\mathbf{T}_i\mathbf{D}_{i+1}\mathbf{T}_i$.

As for the Cherednik operators, Theorem 3.8 implies that the classical relations hold. For instance, one has

$$[\mathbf{D}_i, \mathbf{D}_j] = 0,$$

and the relations with respect to the generators $\mathbf{T}_i$ are

$$\mathbf{D}_{i+1}\mathbf{T}_i = -s\mathbf{T}_i^{-1}\mathbf{D}_i, \quad -\mathbf{T}_i\mathbf{D}_{i+1} + (1-s)\mathbf{D}_i + \mathbf{D}_i\mathbf{T}_i = 0, \tag{3.8}$$

$$-\mathbf{D}_{i+1}\mathbf{T}_i^{-1} - (1-\frac{1}{s})\mathbf{D}_{i+1} + \mathbf{T}_i^{-1}\mathbf{D}_i = 0 \tag{3.9}$$

$$[\mathbf{D}_i, \mathbf{T}_j] = 0 \text{ if } |i-j|>1.$$

Note that identities (1) and (2) of Proposition 3.6 are equivalent to $x_i\mathbf{T}_i = sx_{i+1}\mathbf{T}_i^{-1}$ or $sx_{i+1} = \mathbf{T}_i x_i \mathbf{T}_i$ (these are dual to the $\mathbf{D}_i$ relations $\mathbf{D}_i = (1/s)\mathbf{T}_i\mathbf{D}_{i+1}T_i$).

3.4. **Triangularity of the Cherednik operators.** Let v be a vector. In the sequel we will denote by v^+ (respectively v^R) the unique decreasing (respectively increasing) partition whose entries are obtained by permuting those of v.

Let $\overline{\pi}_i^x = \partial_i^x x_{i+1} = \frac{1}{1-s}\delta_i^x$, $\pi_i^x = \partial_i^x x_{i+1} + 1$, and, more generally, $\pi_{ij}^x = \partial_{ij}^x x_j + 1$.

Observe that, if $i<j$, then one has

$$x^v \pi_{ij}^x = \sum_{v' \trianglelefteq v} (*) x^{v'}, \tag{3.10}$$

where $(*)$ denotes a coefficient and $\trianglelefteq$ is the dominance order on vectors defined by

$$v \trianglelefteq v' \quad \text{if and only if} \quad \begin{cases} v^+ \prec v'^+ & \text{if } v^+ \neq v'^+, \\ v \prec v' & \text{if } v^+ = v'^+. \end{cases}$$

Here, $\prec$ denotes the (classical) dominance order on partitions given by

$$v \prec v' \text{ if and only if, for each } i,\ v[1] + \cdots + v[i] \leq v'[1] + \cdots + v'[i].$$

Indeed, it suffices to understand the computation of $x_1^a x_2^b \pi_1$. We have three cases to consider:

(1) if $a < b$, then

$$x_1^a x_2^b \pi_1^x = -\sum_{i=1}^{b-a-1} x_i^{a+i} x_2^{b-i}.$$

In this case, one has $x_1^a x_2^b \pi_1^x = \sum_{v'^+ \prec [b,a]} (*) x^{v'}$.

(2) if $a = b$, then

$$x_1^a x_2^b \pi_1^x = x_1^a x_2^b.$$

(3) if $a > b$, then

$$x_1^a x_2^b \pi_1^x = \sum_{i=0}^{a-b} x_i^{a-i} x_2^{b+i},$$

and the leading term in this expression is $x^{[a,b]}$.

Similarly,

$$x^v \overline{\pi}_{ij}^x = \sum_{v' \trianglelefteq v} (*) x^{v'}. \tag{3.11}$$

With these notations, write

$$\mathbf{T}_i = (*)\overline{\pi}_i^x + (*) s_i^x T_i$$

and

$$\mathbf{T}_i^{-1} = (*)\pi_i^x + (*) s_i^x T_i.$$

Here, $(*)$ denotes a certain coefficient (we need not know it to follow the computation). Observe that, for each j, we have

$$\mathbf{T}_1^{-1} s_1^x \cdots s_{j-1}^x = [(*)\pi_1^x + (*) s_1^x T_1] s_1^x \cdots s_{j-1}^x = [(*)\pi_1^x + (*) T_1] s_2^x \cdots s_{j-1}^x$$

since $\pi_1^x s_1^x = \pi_1^x$. But $\pi_1^x s_2^x \cdots s_{j-1}^x = s_2^x \cdots s_{j-1}^x \pi_{1,j}^x$, and hence

$$\mathbf{T}_1^{-1} s_1^x \cdots s_{j-1}^x = s_2^x \cdots s_{j-1}^x [(*)\overline{\pi}_{1j}^x + (*) T_1].ya$$

Iterating the process, one finds

$$\mathbf{T}_{j-1}^{-1} \cdots \mathbf{T}_1^{-1} s_1^x \cdots s_{j-1}^x = [(*)\overline{\pi}_{j-1j}^x + (*) T_{j-1}] \cdots [(*)\overline{\pi}_{1j}^x + (*) T_1]. \tag{3.12}$$

One has also

$$s_j^x \cdots s_{N-1}^x \mathbf{T}_{N-1} = s_j^x \cdots s_{N-1}^x [(*)\overline{\pi}_{N-1}^x + (*) s_{N-1}^x T_{N-1}];$$

but $s_{N-1}^x \partial_{N-1}^x = -\partial_{N-1}^x$, and hence

$$s_j^x \cdots s_{N-1}^x \mathbf{T}_{N-1} = s_j^x \cdots s_{N-2}^x [(*)\overline{\pi}_{N-1}^x + (*) T_{N-1}].$$

Since $s_j^x \cdots s_{N-2}^x \overline{\pi}_{N-1}^x = \overline{\pi}_{j,N}^x s_j^x \cdots s_{N-2}^x$, one obtains

$$s_j^x \cdots s_{N-1}^x \mathbf{T}_{N-1} = [(*)\overline{\pi}_{j,N}^x + (*)T_{N-1}] s_j^x \cdots s_{N-2}^x.$$

Iterating this process, one finds

$$s_j^x \cdots s_{N-1}^x \mathbf{T}_{N-1} \cdots \mathbf{T}_j = [(*)\overline{\pi}_{j,N}^x + (*)T_{N-1}] \cdots [(*)\overline{\pi}_{j,j+1}^x + (*)T_j]. \tag{3.13}$$

Now, with these notations, the Cherednik operator reads

$$\boldsymbol{\xi}_j = \left[(*)\pi_{j-1}^x + (*)s_{j-1}^x T_{j-1}\right] \cdots \left[(*)\pi_1^x + (*)s_1^x T_1\right] \tau_1^x s_1^x \cdots s_{N-1}^x S$$
$$\left[(*)\overline{\pi}_{N-1}^x + (*)s_{N-1}^x T_{N-1}\right] \cdots \left[(*)\overline{\pi}_j^x + (*)s_j^x T_j\right].$$

Now apply eq. (3.12) and (3.13) to obtain

$$\boldsymbol{\xi}_i = (*)[(*)\pi_{j-1,N}^x + (*)T_{j-1}] \cdots [(*)\pi_{1,N}^x + (*)T_1](\tau_j^x S)$$
$$\left[(*)\overline{\pi}_{j,N-1}^x + (*)T_{N-1}\right] \cdots \left[(*)\overline{\pi}_{j,j+1}^x + (*)T_j\right],$$

where $x_i \tau_j^x = x_i$ if $i \neq j$ and $x_j \tau_j^x = q x_j$.

From (3.10) and (3.11), we obtain

$$\mathbb{T} x^v \boldsymbol{\xi}_i = \mathbb{T}\left[x^v H_v + \sum_{v' \lhd v} x^{v'} H_{v'}\right], \tag{3.14}$$

with $H_u \in \mathcal{H}_N(q,s)$ (we apply an algebraic combination of π^x and $\overline{\pi}^x$ to x^v, and the operator τ_j^x does not change the exponents). Finally, we arrive at the following theorem.

Theorem 3.13. *We have*

$$x^v \mathbb{T} \boldsymbol{\xi}_j = x^v(\mathbb{T} \cdot H_v) + \sum_{v' \lhd v} x^{v'}(\mathbb{T} \cdot H_{v'}),$$

where $H_u \in \mathcal{H}_N(q,s)$.

Proof. Eq. 3.14 gives

$$\begin{aligned} x^v \mathbb{T} \boldsymbol{\xi}_j &= \mathbb{T} x^v \boldsymbol{\xi}_j \\ &= x^v(\mathbb{T} H_v) + \sum_{v' \lhd v} x^{v'}(\mathbb{T} \cdot H_{v'}). \end{aligned}$$

□

4. Eigenfunctions of Cherednik operators

4.1. **Yang–Baxter graph.** As in [6], we construct a Yang–Baxter-type graph with vertices labeled by 4-tuples $(\mathbb{T}, \zeta, v, \sigma)$, where $\mathbb{T}$ is an RST, ζ is a vector of length N (ζ will be called the **spectral vector**), $v \in \mathbb{N}^N$, and $\sigma \in \mathfrak{S}_N$. First, consider an RST of shape λ and write a vertex labeled by the 4-tuple $(\mathbb{T}, \mathrm{CT}_{\mathbb{T}}^s, 0^N, [1, \ldots, N])$, where

$\mathrm{CT}^s_{\mathbb{T}}[i] = s^{\mathrm{CT}_{\mathbb{T}}[i]}$. Now, we consider the action of the elementary transposition of $\mathfrak{S}_N$ on the 4-tuple given by

$$(\mathbb{T},\zeta,v,\sigma)s_i := \begin{cases} (\mathbb{T},\zeta s_i, v s_i, \sigma s_i) & \text{if } v[i+1] \neq v[i], \\ (\mathbb{T}^{(\sigma[i],\sigma[i+1])},\zeta s_i, v, \sigma) & \text{if } v[i] = v[i+1] \\ & \text{and } \mathbb{T}^{(\sigma[i],\sigma[i+1])} \in \mathrm{Tab}_\lambda, \\ (\mathbb{T},\zeta,v,\sigma) & \text{otherwise,} \end{cases}$$

where $\mathbb{T}^{(i,j)}$ denotes the filling obtained by permuting the values i and j in $\mathbb{T}$. Consider also the affine action given by

$$(\mathbb{T},\zeta,v,\sigma)\Psi := (\mathbb{T},[\zeta[2],\dots,\zeta[N],q\zeta[1]],[v[2],\dots,v[N],v[1]+1],[\sigma_2,\dots,\sigma_N,\sigma_1]).$$

In the sequel we will use the notation $v\Psi^q = [v_2,\dots,v_N,qv_1]$.

Example 4.1.

(1) $\left(\substack{31\\542}, [s,1,q^2,qs^2,qs^{-1}],[0,0,2,1,1],[45123]\right) s_2$
$$= \left(\substack{31\\542}, [s,q^2,1,qs^2,qs^{-1}],[0,2,0,1,1],[41523]\right).$$
(2) $\left(\substack{31\\542}, [s,1,q^2,qs^2,qs^{-1}],[0,0,2,1,1],[45123]\right) s_4$
$$= \left(\substack{21\\543}, [s,1,q^2,qs^{-1},qs^2],[0,0,2,1,1],[45123]\right).$$
(3) $\left(\substack{31\\542}, [s,1,q^2,q^2s^2,qs^{-1}],[0,0,2,1,1],[45123]\right) s_1$
$$= \left(\substack{31\\542}, [s,1,q^2,qs^2,qs^{-1}],[0,0,2,1,1],[45123]\right).$$
(4) $\left(\substack{31\\542}, [s,1,q^2,qs^2,qs^{-1}],[0,0,2,1,1],[45123]\right) \Psi$
$$= \left(\substack{31\\542}, [1,q^2,qs^2,qs^{-1},qs],[0,2,1,1,1],[51234]\right).$$

Definition 4.2. If λ is a partition, denote by $\mathbb{T}_\lambda$ the tableau obtained by filling the shape λ from bottom to top and left to right by the integers $\{1,\dots,N\}$ in decreasing order.

The graph $G^{q,s}_\lambda$ is the infinite directed graph constructed from the 4-tuple

$$(\mathbb{T}_\lambda, \mathrm{CT}^s_{\mathbb{T}_\lambda}, [0^N], [1,2,\dots,N]),$$

called the **root**, adding vertices and edges according to the following rules:

(1) We add an arrow labeled by s_i from the vertex $(\mathbb{T},\zeta,v,\sigma)$ to $(\mathbb{T}',\zeta',v',\sigma')$ if $(\mathbb{T},\zeta,v,\sigma)s_i = (\mathbb{T}',\zeta',v',\sigma')$ and $v[i] < v[i+1]$ or $v[i] = v[i+1]$, and τ is obtained from τ' by interchanging the position of two integers $k < \ell$ such that k is in the south-east of ℓ (i.e., $\mathrm{CT}_{\mathbb{T}}(k) \geq \mathrm{CT}_{\mathbb{T}}(\ell) + 2$).
(2) We add an arrow labeled by Ψ from the vertex $(\mathbb{T},\zeta,v,\sigma)$ to $(\mathbb{T}',\zeta',v',\sigma')$ if $(\mathbb{T},\zeta,v,\sigma)\Psi = (\mathbb{T}',\zeta',v',\sigma')$.
(3) We add an arrow s_i from the vertex (τ,ζ,v,σ) to $\emptyset$ if $(\mathbb{T},\zeta,v,\sigma)s_i = (\mathbb{T},\zeta,v,\sigma)$.

An arrow of the form

$$\boxed{(\mathbb{T},\zeta,v,\sigma)} \longrightarrow \boxed{s_i \text{ or } \Psi} \longrightarrow \boxed{(\mathbb{T},\zeta',v',\sigma')}$$

will be called a **step**. The other arrows will be called **jumps**, and in particular an arrow

$$\boxed{(\mathbb{T},\zeta,v,\sigma)} \longrightarrow \boxed{s_i} \longrightarrow \boxed{\emptyset}$$

will be called a **fall**; the other jumps will be called **correct jumps**.

As usual, a **path** is a finite sequence of consecutive arrows in G_λ starting from the root, and it is denoted by the vector of the labels of its arrows. Two paths $\mathfrak{P}_1 = (a_1, \dots, a_k)$ and $\mathfrak{P}_2 = (b_1, \dots, b_\ell)$ are said to be **equivalent** (denoted by $\mathfrak{P}_1 \equiv \mathfrak{P}_2$) if they lead to the same vertex.

We note that, when $v[i] = v[i+1]$, part (1) of Definition 4.2 is equivalent to the following statement: $\mathbb{T}'$ is obtained from $\mathbb{T}$ by interchanging $\sigma_v[i]$ and $\sigma_v[i+1] = \sigma_v[i]+1$, where $\sigma_v[i]$ is to the south-east of $\sigma_v[i]+1$, that is, $\mathrm{CT}_{\mathbb{T}}[\sigma_v[i]] - \mathrm{CT}_{\mathbb{T}}[\sigma_v[i]+1] \geq 2$.

Example 4.3. The arrow below is a correct jump,

$$\boxed{\begin{array}{c}{}^{\mathbf{3}1}_{54\mathbf{2}},[s,1,q^2,\mathbf{qs^2},\mathbf{qs^{-1}}]\\ [0,0,2,1,1],[45123]\end{array}} \xrightarrow{\boxed{s_4}} \boxed{\begin{array}{c}{}^{\mathbf{2}1}_{54\mathbf{3}},[s,1,q^2,\mathbf{qs^{-1}},\mathbf{qs^2}]\\ [0,0,2,1,1],[45123]\end{array}},$$

whilst

$$\boxed{\begin{array}{c}{}^{31}_{542},[q,\mathbf{1},\mathbf{q^2},qs^2,qs^{-1}]\\ [0,\mathbf{0},\mathbf{2},1,1],[4\mathbf{51}23]\end{array}} \xrightarrow{\boxed{s_2}} \boxed{\begin{array}{c}{}^{31}_{542},[s,\mathbf{q^2},\mathbf{1},qs^{-1},qs^2]\\ [0,\mathbf{2},\mathbf{0},1,1],[4\mathbf{15}23]\end{array}}$$

is a step. The arrows

$$\boxed{\begin{array}{c}{}^{31}_{542},[s,1,q^2,qs^2,qs^{-1}]\\ [0,0,2,1,1],[45123]\end{array}} \xleftarrow{\boxed{s_4}} \boxed{\begin{array}{c}{}^{21}_{543},[s,1,q^2,qs^{-1},qs^2]\\ [0,0,2,1,1],[45123]\end{array}}$$

and

$$\boxed{\begin{array}{c}{}^{31}_{542},[s,1,q^2,qs^2,qs^{-1}]\\ [0,0,2,1,1],[45123]\end{array}} \xleftarrow{\boxed{s_2}} \boxed{\begin{array}{c}{}^{31}_{542},[s,q^2,1,qs^{-1},qs^2]\\ [0,2,0,1,1],[41523]\end{array}}$$

are not allowed.

The graph $G_\lambda^{q,s}$ is very similar to the Yang–Baxter graph G_λ described in [6]: only the spectral vectors change. Indeed, these are the same graphs, but with different labels: the spectral vector of $G_\lambda^{q,s}$ is obtained from G_λ by sending $a\alpha + b$ to $q^a s^b$. Hence, many properties still apply. An example is given in the next proposition.

Proposition 4.4. *All the paths joining two given vertices in G_λ have the same length.*

For a given 4-tuple $(\mathbb{T}, \zeta, v, \sigma)$, the values of ζ and σ are determined by those of $\mathbb{T}$ and v, as provided by the following proposition.

Proposition 4.5. *If $(\mathbb{T}, \zeta, v, \sigma)$ is a vertex in G_λ, then $\sigma = r_v$ and $\zeta[i] = q^{v[i]} s^{CT_{\mathbb{T}}[\sigma[i]]}$.*

We will set $\zeta_{v,\mathbb{T}} := \zeta$.

Example 4.6. Consider the RST

$$\tau = \begin{array}{llll} 3 & & & \\ 7 & 4 & 1 & \\ 8 & 6 & 5 & 2 \end{array}$$

and the vector $v = [6,2,4,2,2,3,1,4]$. One has $r_v = [1,5,2,6,7,4,8,3]$ and $\mathrm{CT}_{\mathbb{T}} = [1,3,-2,0,2,1,-1,0]$. Consequently,

$$\zeta_{v,\tau} = [q^6 s, q^2 s^2, q^4 s^3, q^2 s^1, q^2 s^{-1}, q^3, q, q^4 s^{-2}].$$

Hence, the 4-tuple

$$\left(\begin{array}{cccc}3&&&\\7&4&1&\\8&6&5&2\end{array}, [q^6,q^2s^2,q^4s^3,q^2s^1,q^2s^{-1},q^3,q,q^4s^{-2}], [6,2,4,2,2,3,1,4], [1,5,2,6,7,4,8,3]\right)$$

labels a vertex of $G^{q,s}_{431}$.

As a consequence, we obtain the following result.

Corollary 4.7. *Let $(\mathbb{T}, v)$ be a pair consisting of $\mathbb{T} \in \mathrm{Tab}(\lambda)$ (where λ is a partition of N) and a multi-index $v \in \mathbb{N}^N$. Then there exists a unique vertex in $G^{q,s}_\lambda$ labeled by a 4-tuple of the form $(\mathbb{T}, \zeta, v, \sigma)$.*

We will write $\mathfrak{V}_{\mathbb{T},\zeta,v,\sigma} := (\mathbb{T}, v)$.

Conversely, all the information can be retrieved from the spectral vector ζ — the exponents of q give v, the rank function of v gives σ, and the exponent of s in the spectral vector gives the content vector which does uniquely determine the RST $\mathbb{T}$.

For simplicity, when needed, we will label the vertices by pairs $(\mathbb{T}, v)$ or by the associated spectral vector $\zeta_{v,\mathbb{T}}$.

Example 4.8. In Figure 1, the first several vertices are labeled by pairs $(\mathbb{T}, v)$ of the graph $G^{q,s}_{21}$, while, in Figure 2, the vertices are labeled by spectral vectors.

Definition 4.9. We define the subgraph $G^{q,s}_{\mathbb{T}}$ as the graph obtained from $G^{q,s}_\lambda$ by erasing all the vertices labeled by RST's other than $\mathbb{T}$ and the associated arrows. Such a graph is connected.

The graph $G^{q,s}_\lambda$ is the union of the graphs $G^{q,s}_{\mathbb{T}}$ connected by jumps. Furthermore, if $G^{q,s}_{\mathbb{T}}$ and $G^{q,s}_{\mathbb{T}'}$ are connected by a succession of jumps, then there is no step from $G^{q,s}_{\mathbb{T}'}$ to $G^{q,s}_{\mathbb{T}}$. Since the graphs $G^{q,s}_{\mathbb{T}}$ are connected graphs, we infer the following result.

Proposition 4.10. *Each vertex $(\mathbb{T}, v)$ is obtained from $(\mathbb{T}, 0^N)$ by a sequence of steps.*

Example 4.11. In Figures 1 and 2, the graph $G^{q,s}_{21}$ is constituted by the two graphs $G^{q,s}_{\begin{smallmatrix}1&\\3&2\end{smallmatrix}}$ and $G^{q,s}_{\begin{smallmatrix}2&\\3&1\end{smallmatrix}}$ connected by jumps (in blue).

4.2. **Macdonald polynomials from scratch.** Following [1], we define the operator

$$\mathbf{\Phi} = \mathbf{T}_1^{-1} \cdots \mathbf{T}_{N-1}^{-1} x_N,$$

which satisfies

$$\begin{aligned}\mathbf{\Phi}\boldsymbol{\xi}_j &= \boldsymbol{\xi}_{j+1}\mathbf{\Phi}, \quad 1 \leq j < N,\\ \mathbf{\Phi}\boldsymbol{\xi}_N &= q\boldsymbol{\xi}_1\mathbf{\Phi}.\end{aligned}$$

The operator $\mathbf{\Phi}$ is injective (its kernel is $\{0\}$).

Let λ be a partition and $G^{q,s}_\lambda$ be the associated graph. We construct the set of polynomials $(P_{\mathfrak{P}})_{\mathfrak{P} \text{ path in } G_\lambda}$ using the following recurrence rules:

(1) $P_{[]} := (\mathbb{T}_\lambda)$.

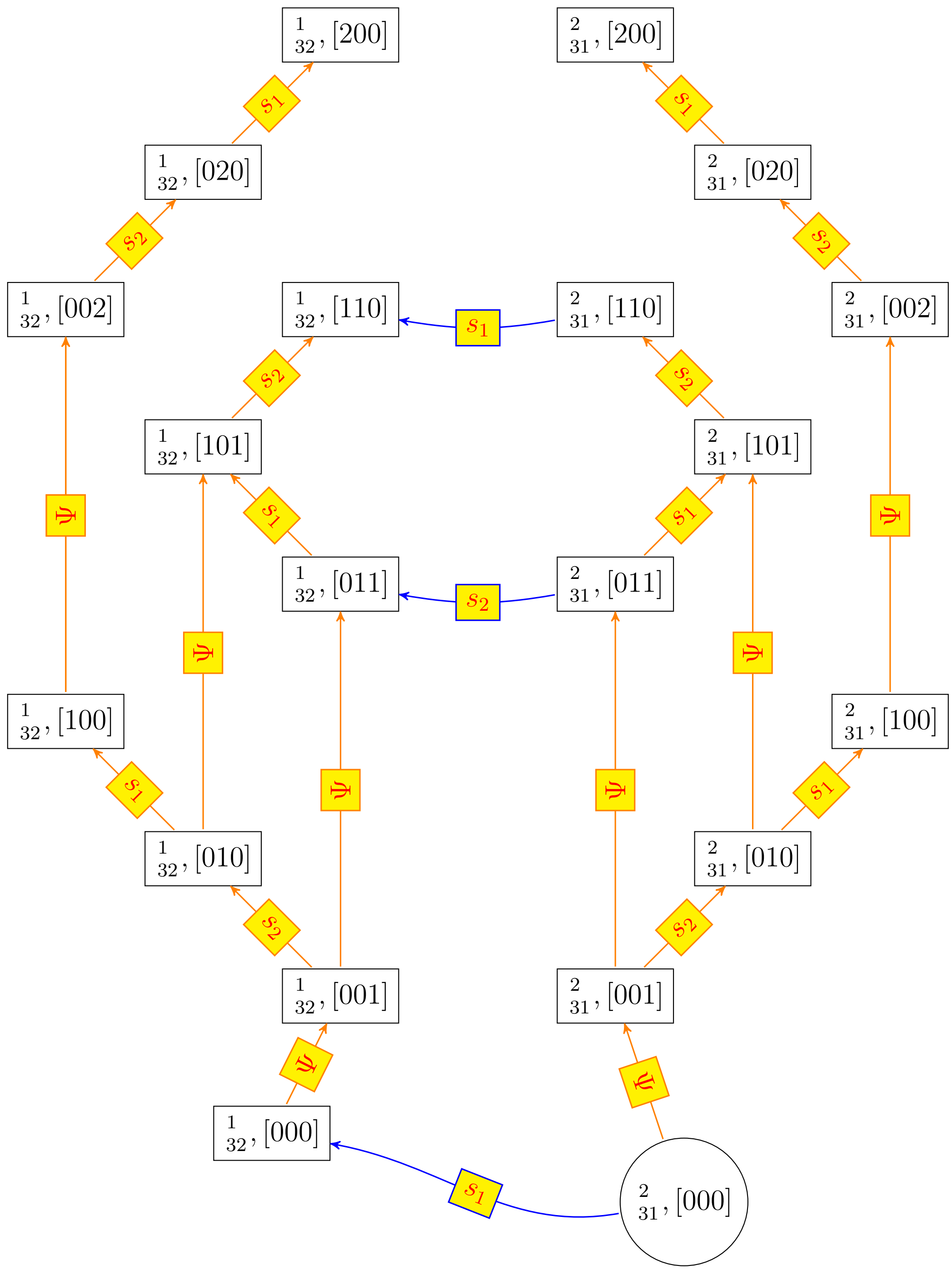


FIGURE 1. The first vertices labeled by pairs $(\mathbb{T}, v)$ of the graph $G_{21}^{q,s}$ where we omit to write the vertex $\emptyset$ and the associated arrows.

(2) If $\mathfrak{P} = [a_1, \dots, a_{k-1}, s_i]$, then

$$P_{\mathfrak{P}} := P_{[a_1,\dots,a_{k-1}]} \left(\mathbf{T}_i + \frac{1-s}{1-\frac{\zeta[i+1]}{\zeta[i]}} \right),$$

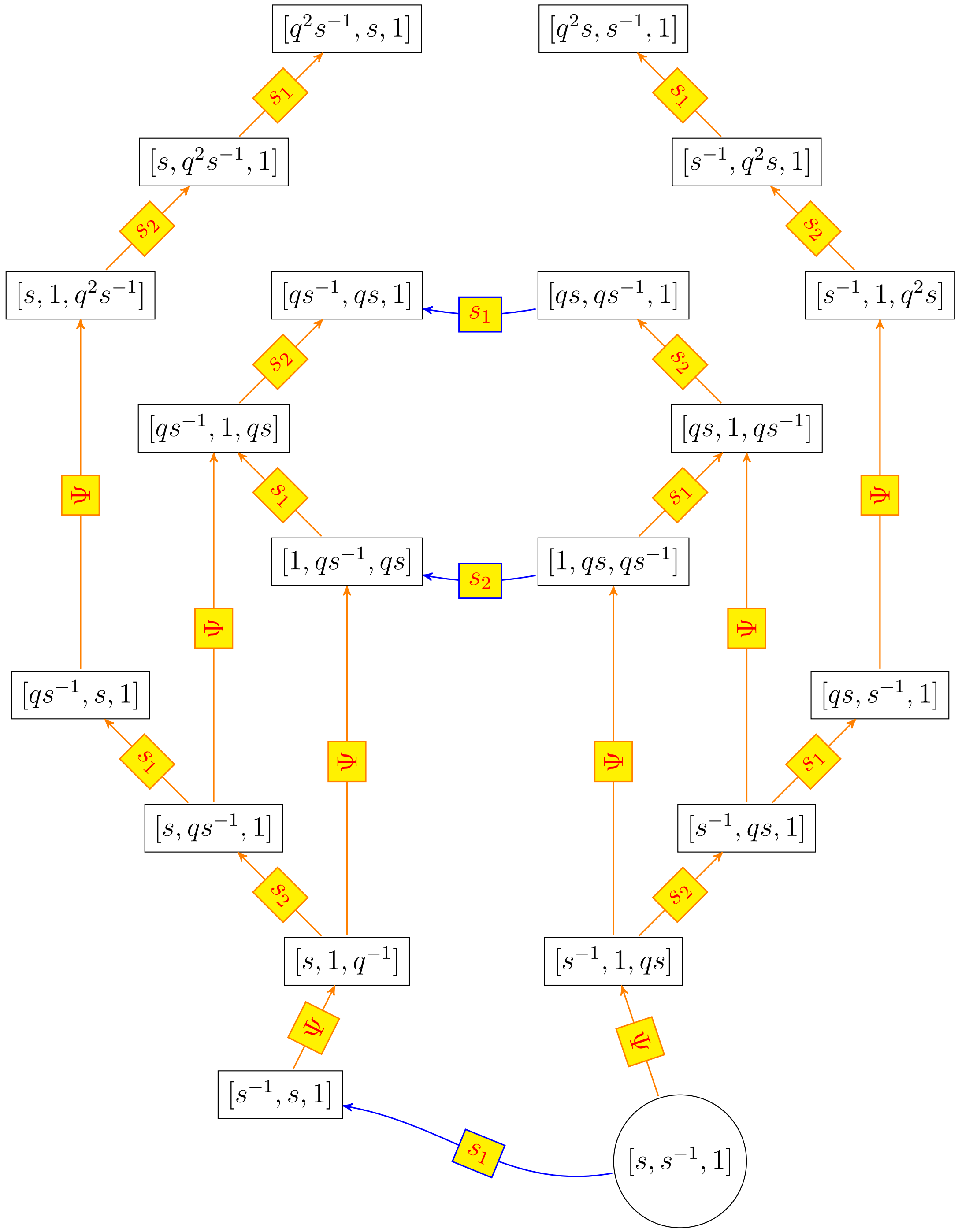


FIGURE 2. The first vertices labeled by spectral vector of the graph $G_{21}^{q,s}$.

where the vector ζ is defined by

$$(\mathbb{T}_\lambda, \mathrm{CT}^s_{\mathbb{T}_\lambda}, 0^N, [1, 2, \ldots, N])a_1 \cdots a_{k-1} = (\mathbb{T}, \zeta, v, \sigma).$$

(3) If $\mathfrak{P} = [a_1, \dots, a_{k-1}, \Psi]$, then

$$P_{\mathfrak{P}} = P_{[a_1,\dots,a_{k-1}]}\mathbf{\Phi}.$$

One has the following theorem.

Theorem 4.12. *Let $\mathfrak{P} = [a_0, \dots, a_k]$ be a path in $G_\lambda^{q,s}$ from the root to $(\mathbb{T}, \zeta, v, \sigma)$ with no fall. The polynomial $P_{\mathfrak{P}}$ is a simultaneous eigenfunction of the operators $\boldsymbol{\xi}_i$. Furthermore, the eigenvalues of $\boldsymbol{\xi}_i$ associated to $P_{\mathfrak{P}}$ are equal to $\zeta[i]$.*

Consequently, $P_{\mathfrak{P}}$ does not depend on the path, but only on the end point $(\mathbb{T}, \zeta, v, \sigma)$, and it will be denoted by $P_{v,\mathbb{T}}$ or alternatively by P_ζ. The family $(P_{v,\mathbb{T}})_{v,\mathbb{T}}$ forms a basis of $\mathcal{M}_\lambda$ of simultaneous eigenfunctions of the Cherednik operators.

Proof. We prove the result by induction on the length k. If $k = 0$ then the result follows from Proposition 3.10.

Suppose now that $k > 0$ and let

$$(\mathbb{T}', \zeta', v', r_{v'}) = (\mathbb{T}_\lambda, \mathrm{CT}^{q,s}_{\mathbb{T}_\lambda}, 0^N, [1, \dots, N])a_1 \cdots a_{k-1}.$$

By induction, $P_{[a_1,\dots,a_{k-1}]}$ is a simultaneous eigenfunction of the operators $\boldsymbol{\xi}_i$ such that the associated vector of eigenvalues is given by

$$P_{[a_1,\dots,a_{k-1}]}\boldsymbol{\xi}_i = \zeta'[i]P_{[a_1,\dots,a_{k-1}]}.$$

The argument depends on the value of the last operator a_k.

(1) If $a_k = \Psi$ is an affine arrow, then $\mathbb{T} = \mathbb{T}'$, $\zeta = [\zeta'[2], \dots, \zeta'[N], q\zeta'[1]]$, $v = v'\Psi$, $r_v = r_{v'}[2, \dots, N, 1]$ and $P_{\mathfrak{P}} = J_{[a_1,\dots,a_{k-1}]}\mathbf{\Phi}$.

If $i \neq N$, then

$$\begin{aligned} P_{\mathfrak{P}}\boldsymbol{\xi}_i &= P_{[a_1,\dots,a_{k-1}]}\mathbf{\Phi}\boldsymbol{\xi}_i \\ &= P_{[a_1,\dots,a_{k-1}]}\boldsymbol{\xi}_{i+1}\mathbf{\Phi} \\ &= \zeta'[i+1]P_{\mathfrak{P}} \\ &= \zeta[i]P_{\mathfrak{P}}. \end{aligned}$$

If $i = N$, then

$$\begin{aligned} P_{\mathfrak{P}}\boldsymbol{\xi}_N &= P_{[a_1,\dots,a_{k-1}]}\mathbf{\Phi}\boldsymbol{\xi}_N \\ &= P_{[a_1,\dots,a_{k-1}]}q\boldsymbol{\xi}_1\mathbf{\Phi} \\ &= (\zeta'[1]q)P_{\mathfrak{P}} \\ &= \zeta[N]P_{\mathfrak{P}}. \end{aligned}$$

(2) Suppose now that $a_k = s_i$ is a non-affine arrow. Then we have $\zeta = \zeta' s_i$, $v = v' s_i$, and

$$P_{\mathfrak{P}} = P_{[a_1,\dots,a_{k-1}]}\left(\mathbf{T}_i + \frac{1-s}{1-\frac{\zeta'[i+1]}{\zeta'[i]}}\right).$$

If $j \neq i,\ i+1$, then

$$\begin{aligned}
P_{\mathfrak{P}}\boldsymbol{\xi}_j &= P_{[a_1,\dots,a_{k-1}]}\left(\mathbf{T}_i + \frac{1-s}{1-\frac{\zeta'[i+1]}{\zeta'[i]}}\right)\boldsymbol{\xi}_j \\
&= P_{[a_1,\dots,a_{k-1}]}\boldsymbol{\xi}_j\left(\mathbf{T}_i + \frac{1-s}{1-\frac{\zeta'[i+1]}{\zeta'[i]}}\right) \\
&= \zeta'[j]P_{\mathfrak{P}} \\
&= \zeta[j]P_{\mathfrak{P}}.
\end{aligned}$$

If $j = i$, then

$$\begin{aligned}
P_{\mathfrak{P}}\boldsymbol{\xi}_i &= P_{[a_1,\dots,a_{k-1}]}\left(\mathbf{T}_i + \frac{1-s}{1-\frac{\zeta'[i+1]}{\zeta'[i]}}\right)\boldsymbol{\xi}_i \\
&= P_{[a_1,\dots,a_{k-1}]}\left(\boldsymbol{\xi}_{i+1}\mathbf{T}_i + (1-s)\left(-1+\frac{1}{1-\frac{\zeta'[i+1]}{\zeta'[i]}}\right)\boldsymbol{\xi}_i\right) \\
&= P_{[a_1,\dots,a_{k-1}]}\left(\zeta'[i+1]\mathbf{T}_i + (1-s)\left(-1+\frac{1}{1-\frac{\zeta'[i+1]}{\zeta'[i]}}\right)\zeta'[i]\right) \\
&= \zeta'[i+1]P_{[a_1,\dots,a_{k-1}]}\left(\mathbf{T}_i + \frac{1-s}{1-\frac{\zeta'[i+1]}{\zeta'[i]}}\right) \\
&= \zeta[i]P_{\mathfrak{P}}.
\end{aligned}$$

If $j = i+1$, then

$$\begin{aligned}
P_{\mathfrak{P}}\boldsymbol{\xi}_{i+1} &= P_{[a_1,\dots,a_{k-1}]}\left(\mathbb{T}_i + \frac{1-s}{1-\frac{\zeta'[i+1]}{\zeta'[i]}}\right)\boldsymbol{\xi}_{i+1} \\
&= P_{[a_1,\dots,a_{k-1}]}\left(s\boldsymbol{\xi}_i\mathbf{T}_i^{-1} + \boldsymbol{\xi}_{i+1}\frac{1-s}{1-\frac{\zeta'[i+1]}{\zeta'[i]}}\right) \\
&= P_{[a_1,\dots,a_{k-1}]}\left(\zeta'[i]\mathbf{T_i} + \zeta'[i](1-s) + \zeta'[i+1]\frac{1-s}{1-\frac{\zeta'[i+1]}{\zeta'[i]}}\right) \\
&= \zeta'[i]P_{[a_1,\dots,a_{k-1}]}\left(\mathbf{T}_i + \frac{1-s}{1-\frac{\zeta'[i+1]}{\zeta'[i]}}\right) \\
&= \zeta[i+1]P_{\mathfrak{P}}.
\end{aligned}$$

□

Example 4.13. Figure 3 illustrates how to obtain the first values of the polynomial P_ζ for isotype $(2,1)$.

Besides $\boldsymbol{\Phi} = T_1^{-1}\cdots T_{N-1}^{-1}x_N$, there is another raising operator, namely $\boldsymbol{\Phi'} := \mathbf{w}x_N$.

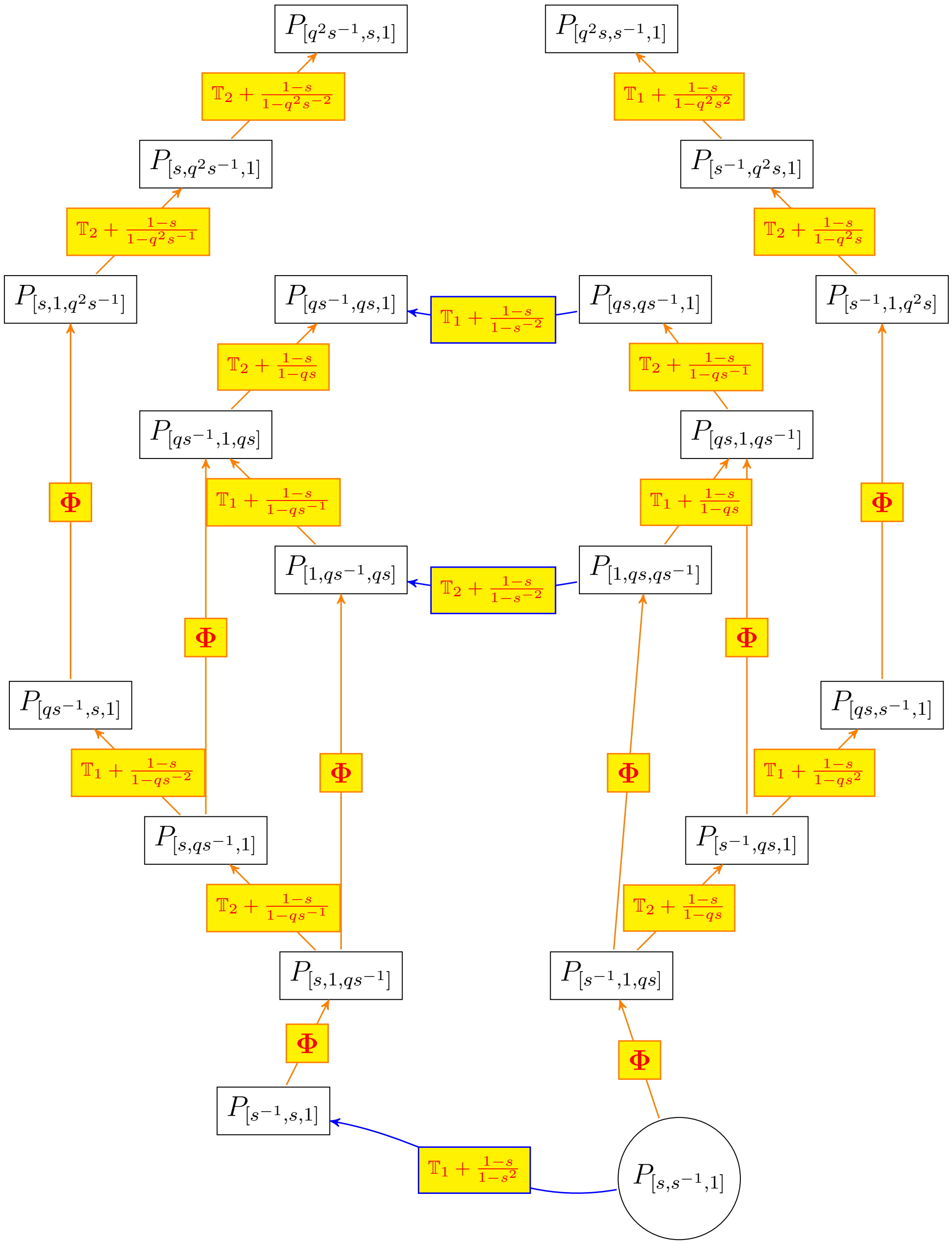


FIGURE 3. The first Macdonald polynomials for isotype (21).

Proposition 4.14. *We have*

$$\mathbf{\Phi}' = s^{N-1}\boldsymbol{\xi}_1\mathbf{\Phi},$$

and, if $v \in \mathbb{N}_0^N, \mathbb{T} \in \mathrm{Tab}_\lambda$, *then*

$$P_{v,T}\mathbf{\Phi}' = s^{N-1+\mathrm{CT}_{\mathbb{T}}[r_v[1]]}q^{v[1]}P_{v,T}\mathbf{\Phi}.$$

Proof. From $\boldsymbol{\xi}_1 = s^{1-N}\mathbf{w}T_{N-1}\cdots T_1$, it follows that

$$\boldsymbol{\xi}_1 T_1^{-1}T_2^{-1}\cdots T_{N-1}^{-1} = s^{1-N}\mathbf{w},$$
$$\boldsymbol{\xi}_1 \boldsymbol{\Phi} = s^{1-N}\boldsymbol{\Phi}'.$$

Moreover, we have $P_{v,\mathbb{T}}\boldsymbol{\xi}_1 = q^{v[1]}s^{N-1+\mathrm{CT}_{\mathbb{T}}[r_v[1]]}P_{v,\mathbb{T}}$. □

Note that it is easier to compute $P\boldsymbol{\Phi}'$ for a polynomial P.

4.3. **Leading terms.** We will write $x^{v,\mathbb{T}} := x^v\mathbb{T}R_v$. By abuse of language, $x^{v,\mathbb{T}}$ will be referred to as a monomial. Note that the space $\mathcal{M}_\lambda$ is spanned by the set of polynomials

$$M_\lambda := \{x^{v,\mathbb{T}} : v \in \mathbb{N}^N, \mathbb{T} \in \mathrm{Tab}_\lambda\},$$

which can be naturally endowed with the order $\lhd$ defined by

$$x^{v,\mathbb{T}} \lhd x^{v',\mathbb{T}'} \quad \text{if and only if} \quad v \lhd v'.$$

Theorem 4.15. *The leading term (up to constant multiple) of $P_{v,\mathbb{T}}$ is $x^{v,\mathbb{T}}$.*

Proof. Theorem 3.13 shows that the leading term of $P_{v,\mathbb{T}}$ is $x^v\mathbb{T}H_v$ for some $H_v \in \mathcal{H}_N(q,s)$ (because the eigenvalues determine $q^{v[i]}$).

Use induction on $\#\mathrm{inv}(v) = \#\{(i,j) : 1 \le i < j \le N, v[i] < v[j]\}$. The claim is true for partitions v, that is, in the case where $\#\mathrm{inv}(v) = 0$. Suppose the claim is true for all u with $\#\mathrm{inv}(u) \le k$ and $\#\mathrm{inv}(v) = k+1$. There is some i for which $v[i] < v[i+1]$. By Theorem 4.12, we know that $p := P_{v,\mathbb{T}}\left(\mathbf{T}_i + \frac{(1-s)\zeta[i]}{\zeta[i]-\zeta[i+1]}\right)$ is a $\boldsymbol{\xi}$-eigenvector with eigenvalues $[\zeta[1],\ldots,\zeta[i+1],\zeta[i],\ldots]$, where $\zeta[j] = \zeta_{v,\mathbb{T}}[j]$. The list of eigenvalues implies that the leading term of p is $x^{v\cdot s_i}\mathbb{T}'$ for some $\mathbb{T}' \in V_\lambda$. In fact, $p\boldsymbol{\xi}_j = q^{vs_i[j]}s^{\mathrm{CT}_{\mathbb{T}}[r_{vs_i}]}p$ for all j, and so the inductive hypothesis ($\#\mathrm{inv}(vs_i) = \#\mathrm{inv}(v) - 1$) implies that p is a scalar multiple of $P_{vs_i,\mathbb{T}}$ and has leading term $x^{vs_i}\mathbb{T}T_{r_{vs_i}}$. The only appearance of x^{vs_i} in p comes from $x^v\mathbb{T}H_vT_i$ (by dominance, x^{vs_i} does not appear in $P_{v,\mathbb{T}}$).

However, if $v[i] < v[i+1]$ and $\mathbb{T} \in V_\lambda$, then

$$\begin{aligned} x^v\mathbb{T}\mathbf{T}_i &= x^v\delta_i\mathbb{T} + x^{vs_i}(\mathbb{T}T_i) \\ &= -(1-s)x^v\mathbb{T} + x^{vs_i}(\mathbb{T}T_i) + \sum_{\substack{v' \lhd v \\ \mathbb{P}_{v'} \in V_\lambda}} x^{v'}\mathbb{P}_{v'}. \end{aligned} \tag{4.1}$$

Hence, by (4.1), we have

$$x^v\mathbb{T}H_v\mathbf{T}_i = -(1-s)x^v\mathbb{T}H_v + x^{vs_i}(\mathbb{T}H_vT_i) + \sum_{\substack{v' \lhd v \\ \mathbb{P}_{v'} \in V_\lambda}} x^{v'}\mathbb{P}_{v'}.$$

Thus $\mathbb{T}\mathcal{H}_vT_i = \mathbb{T}R_{r_{vs_i}}$ and

$$\mathbb{T}H_v = \mathbb{T}R_{r_{vs_i}}T_i^{-1} = \mathbb{T}R_{r_v},$$

by Lemma 2.14. This completes the proof of the theorem. □

The last theorem has the following consequence.

Corollary 4.16. *Let $\mathfrak{P} = [a_1,\ldots,a_k]$ such that a_k is a fall. Then we have $P_{\mathfrak{P}} = 0$.*

Proof. Without loss of generality, we can suppose that $[a_1, \ldots, a_{k-1}]$ is a path without fall. By Theorem 4.12, there exists a pair $(v, \mathbb{T})$ such that $P_{v,\mathbb{T}} = P_{[a_1,\ldots,a_{k-1}]}$. On the other hand, by Theorem 4.15, one has

$$P_{v,\mathbb{T}} = x^{v,\mathbb{T}} + \sum_{\substack{v' \lhd v \\ \mathbb{P}_{v'} \in V_\lambda}} x^{v'} \mathbb{P}_{v'}.$$

Since a_k is a fall, one has

$$P_{\mathfrak{P}} = x^v \mathbb{P} + \sum_{\substack{v' \lhd v \\ \mathbb{P}_{v'} \in V_\lambda}} x^{v'} \mathbb{P}_{v'},$$

with $\mathbb{P} \in V_\lambda$. Since $P_{\mathfrak{P}}$ is a simultaneous eigenfunction of the Cherednik operators, it is proportional to $P_{v,\mathbb{T}}$. Noting that the associated eigenvectors are uniquely determined, one obtains $P_{\mathfrak{P}} = 0$. ☐

4.4. **Action of $\mathbf{T}_i$.** We have more formulae than those exhibited in the proof of Theorem 4.12. Examples are given by the following theorem.

Proposition 4.17. *Suppose that $v \in \mathbb{N}_0^N, \mathbb{T} \in \mathrm{Tab}_\lambda$, $v[i] = v[i+1]$ for some i, and $k := r_v[i]$, $m := \mathrm{CT}_\mathbb{T}[k+1] - \mathrm{CT}_\mathbb{T}[k]$. Then we have:*
(1) *if* $\mathrm{CT}_\mathbb{T}[k+1] = \mathrm{CT}_\mathbb{T}[k] - 1$ *then* $P_{v,\mathbb{T}}\mathbf{T}_i = sP_{v,\mathbb{T}}$;
(2) *if* $\mathrm{CT}_\mathbb{T}[k+1] = \mathrm{CT}_\mathbb{T}[k] + 1$ *then* $P_{v,\mathbb{T}}\mathbf{T}_i = -P_{v,\mathbb{T}}$;
(3) *if* $\mathrm{CT}_\mathbb{T}[k+1] \le \mathrm{CT}_\mathbb{T}[k] - 2$ *then* $P_{v,\mathbb{T}}\mathbf{T}_i = P_{v,\mathbb{T}^{(k,k+1)}} - \frac{1-s}{1-s^m} P_{v,\mathbb{T}}$;
(4) *if* $\mathrm{CT}_\mathbb{T}[k+1] \ge \mathrm{CT}_\mathbb{T}[k] + 2$ *then* $P_{v,\mathbb{T}}\mathbf{T}_i = \frac{s\left(1-s^{m+1}\right)\left(1-s^{m-1}\right)}{(1-s^m)^2} P_{v,\mathbb{T}^{(k,k+1)}} - \frac{1-s}{1-s^m} P_{v,\mathbb{T}}$.

We introduce a partial order which will be used to compare eigenvalues, that is, the spectral vectors.

Definition 4.18. For integers n_1, m_1, n_2, m_2 define

$$\begin{aligned} q^{n_1}s^{m_1} \succ q^{n_2}s^{m_2} &\quad \text{if and only if} \quad n_1 > n_2 \text{ or } n_1 = n_2, m_1 \le m_2 - 2; \\ q^{n_1}s^{m_1} \nsim q^{n_2}s^{m_2} &\quad \text{if and only if} \quad n_1 = n_2, |m_1 - m_2| = 1. \end{aligned}$$

We will also write $q^{n_1}s^{m_1} > q^{n_2}s^{m_2}$ if $n_1 > n_2$.

This formulation is used to unify the various recursion relations. Note that, if $\zeta = \zeta_{v,\mathbb{T}}$ is a spectral vector, then we have necessarily $\zeta[i] \ne \zeta[i+1]$ for each i. Indeed, either $v[i] <> v[i+1]$ or $v[i] = v[i+1]$, and the contents are different (since an RST can not have adjacent entries on a diagonal).

Here is a unified transformation formula. Theorem 4.12 is implicitly used.

Proposition 4.19. *Suppose that $v \in \mathbb{N}_0^N, \mathbb{T} \in \mathrm{Tab}_\lambda$, and $1 \le i < N$. Then*

$$P_\zeta\left(\mathbf{T}_i + \frac{(1-s)\,\zeta[i]}{\zeta[i] - \zeta[i+1]}\right) = \begin{cases} P_{\zeta s_i} \ \text{if } \zeta[i+1] \succ \zeta[i], \\ \frac{(\zeta[i] - s\zeta[i+1])(s\zeta[i] - \zeta[i+1])}{(\zeta[i] - \zeta[i+1])^2} P_{\zeta s_i} \ \text{if } \zeta[i] \succ \zeta[i+1], \\ 0 \ \text{if } \zeta[i] \nsim \zeta[i+1]. \end{cases} \tag{4.2}$$

and

$$P_\zeta \mathbf{\Phi} = P_{\zeta \Psi^q}. \tag{4.3}$$

This proposition shows that we can easily use the spectral vector ζ instead of the pair $(v,\mathbb{T})$ for labeling the Macdonald polynomials (assuming that $\zeta=\zeta_{v,\mathbb{T}}$ for a given vector v and a given tableau $\mathbb{T}$).

Indeed, we showed that, if ζ is a spectral vector and $\zeta[i]\succ\zeta[i{+}1]$ or $\zeta[i]\prec\zeta[i{+}1]$, then ζs_i is also a spectral vector. Such an action is called a *permissible transposition.* If $\zeta[i]\not\sim\zeta[i+1]$ then $\zeta\cdot s_i$ is not a spectral vector. We use some of the ideas developed by [14], see Theorem 5.8, p. 22 there. Let μ be a decreasing partition. Suppose that $\mu[i]=\mu[j]$, $i<j$, and $\mathrm{CT}_{\mathbb{T}}[i])=\mathrm{CT}_{\mathbb{T}}[j]=a$. Then $\{a-1,a+1\}\subset\{\mathrm{CT}_{\mathbb{T}}[i+1],\ldots,\mathrm{CT}_{\mathbb{T}}[j-1]\}$. That is, there exists k with $i<k<j$ such that $\mathrm{CT}_{\mathbb{T}}[k]=a+1$, and $\mu[k]=\mu[i]$ (because of the partition property). Thus the spectral vector ζ contains a substring $\left(q^{\mu[i]}s^a,q^{\mu[i]}s^{a+1},q^{\mu[i]}s^a\right)$ (preserving the order from ζ; it is impossible to move $q^{\mu[i]}s^a$ past $q^{\mu[i]}s^{a+1}$ with a permissible transposition), and adjacent entries of a spectral vector can not be equal.

One description of permissible permutations is as the set of permutations of ζ in which each pair $(\zeta[i],\zeta[j])$ with $\zeta[i]\not\sim\zeta[j]$ maintains its order, that is, if $i<j$ and $\zeta[i]\not\sim\zeta[j]$ and $(\zeta[i\cdot\sigma])_{i=1}^N$ is a spectral vector then $i\cdot\sigma<j\cdot\sigma$. The structure of permissible permutations is analyzed in Section 5.1.

For example, take $\lambda=(3,2)\,,\mu=(1,1,1,1,0)$,

$$\begin{aligned}\mathbb{T}&=\begin{matrix}2&1&\\5&4&3\end{matrix},\\ \zeta&=\left(q,qs^{-1},qs^2,qs,1\right),\\ \zeta[1]\not\sim\zeta[2]&\succ\zeta[3]\not\sim\zeta[4]\succ\zeta[5].\end{aligned}$$

However, also $\zeta[1]\not\sim\zeta[4]$, so the order of the pairs $(\zeta[1],\zeta[2])\,,(\zeta[3],\zeta[4])\,,(\zeta[1],\zeta[4])$ must be preserved in the permissible permutations (of which there are 25). Observe that ζ is a maximal element, in the sense that only $\succ$ and $\not\sim$ occur in the comparisons of adjacent elements. Clearly there must be a minimal element (if $\zeta[i]\succ\zeta[i+1]$, then apply s_i to ζ). In the example this is

$$\begin{aligned}\zeta&=\left(1,qs^2,q,qs,qs^{-1}\right)\\ &=\zeta_{(0,1,1,1,1),\mathbb{T}_1},\\ \mathbb{T}_1&=\begin{matrix}4&2&\\5&3&1\end{matrix}.\end{aligned}$$

To finish this discussion, we show that the maximal and minimal elements are unique. By the definition of $\succ$, we need only consider the possible arrangements of $\zeta[i],\zeta[i+1],\ldots,\zeta[j]$ where $\mu[i-1]>\mu[i]=\cdots=\mu[j]>\mu[j+1]$ (or $i=1$, or $j=N$ and $\mu[N]>0$). Let

$$\mathrm{inv}\,(\mu,\mathbb{T})=\{(i,j):\mu[i]=\mu[j]\,,i<j,\zeta_{\mu,\mathbb{T}}[i]\prec\zeta_{\mu,\mathbb{T}}[j]\}\,.$$

We showed that there is a unique RST $\mathbb{T}_0$, where $(\zeta_{\mu,\mathbb{T}_0}[i])_{i=1}^N$ is a permissible permutation of ζ and $\#\mathrm{inv}\,(\mu,\mathbb{T}_0)=0$. By a similar argument there is a unique RST $\mathbb{T}_1$ which maximizes $\mathrm{inv}\,(\mu,\mathbb{T})$. The minimum spectral vector is $\left(\zeta_{\mu^R,\mathbb{T}_1}[i]\right)_{i=1}^N$, where $\mu^R[i]=\mu[N+1-i]$, $1\le i\le N$.

According to the previous remark, we will use the following notations.

Definition 4.20. If $\zeta = \zeta_{v,\mathbb{T}}$, we write

$$\mathrm{inv}_{\triangleleft}(\zeta) := \{(i,j) : 1 \le i < j \le N, \zeta[i] \triangleleft \zeta[j]\},$$

for $\triangleleft \in \{<, >, \prec, \succ\}$. If $\zeta = \zeta_{v,\mathbb{T}}$, then we write $\zeta^+ = \zeta_{v^+,\mathbb{T}}$. Note that $\zeta^+[1] \ge \zeta^+[2] \ge \cdots \ge \zeta^+[N]$. We set

$$\mathrm{inv}(\zeta) := \mathrm{inv}_{<}(\zeta) = \mathrm{inv}(v).$$

The action of the symmetric group $\mathfrak{S}_N$ on the spectral vector is defined by

$$\zeta s_i = \begin{cases} [\zeta[1], \dots, \zeta[i-1], \zeta[i+1], \zeta[i], \zeta[i+2], \dots, \zeta[N]] & \text{if } \zeta[i] \prec \zeta[i+1], \\ & \quad\text{or } \zeta[i] \succ \zeta[i+1], \\ \zeta & \text{otherwise.} \end{cases} \tag{4.4}$$

Say $\zeta' \prec \zeta$ if and only if there exists a sequence of elementary transpositions $(s_{i_1}, \dots, s_{i_k})$ such that

$$\zeta_0 = \zeta,\ \zeta_1 = \zeta_0 s_{i_1}, \dots,\ \zeta_k = \zeta s_{i_1} \cdots s_{i_k} = \zeta'$$

and, for each $j < k$, $\zeta_j[i_{j+1}] \prec \zeta_j[i_{j+1} + 1]$.

5. Stable subspaces

5.1. **Connected components.** We denote by $H_\lambda^{q,s}$ the graph obtained from $G_\lambda^{q,s}$ by removing the affine edges, all the falls, and the vertex $\emptyset$.

Recall that v^+ is the unique decreasing partition obtained by permuting the entries of v.

Definition 5.1. Let $v \in \mathbb{N}^N$ and $\mathbb{T} \in \mathrm{Tab}_\lambda$ (where λ a partition). We define the filling $T(\mathbb{T}, v)$ as the one obtained from $\mathbb{T}$ by replacing i by $v^+[i]$ for each i..

As in [6], the following fact holds true.

Proposition 5.2. *Two 4-tuples $(\mathbb{T}, \zeta, v, \sigma)$ and $(\mathbb{T}', \zeta', v', \sigma')$ are in the same connected component of $H_\lambda^{q,s}$ if and only if $T(\mathbb{T}, v) = T(\mathbb{T}, v')$.*

This shows that the connected components of $H_\lambda^{q,s}$ are indexed by the $T(\mathbb{T}, \mu)$, where μ is a partition.

Definition 5.3. We denote by $H_T^{q,s}$ the connected component associated with T in $H_\lambda^{q,s}$. The component $H_T^{q,s}$ will be said to be **1-compatible** if T is a column-strict tableau. The component $H_T^{q,s}$ will be said to be (-1)**-compatible** if T is a row-strict tableau.

Note that each connected component has a unique minimal element (i.e., an element without antecedent) called its root and denoted by

$$\mathrm{root}(T) := (\mathbb{T}_{\mathrm{root}(T)}, \zeta_{\mathrm{root}(T)}, v_{\mathrm{root}(T)}, r_{\mathrm{root}(T)}),$$

and a unique maximal element called its sink and denoted by

$$\mathrm{sink}(T) := (\mathbb{T}_{\mathrm{sink}(T)}, \zeta_{\mathrm{sink}(T)}, v_{\mathrm{sink}(T)}, r_{\mathrm{sink}(T)}).$$

.

With the notations of the previous section, we have $v_{\mathrm{sink}(T)} = v^+$ and $\mathbb{T}_{\mathrm{sink}(T)} = \mathbb{T}_0$ for any pair $(v, \mathbb{T}) \in T$. In the same way, $v_{\mathrm{root}(T)} = v^R_{\mathrm{sink}(T)}$ and $\mathbb{T}_{\mathrm{root}(T)} = \mathbb{T}_1$.

Example 5.4. Let $\mu = [2, 1, 1, 0, 0]$ and $\lambda = [3, 2]$. There are four connected components with vertices labeled by permutations of μ in $H^{q,s}_\lambda$. The possible values of $T(\mathbb{T}, \mu)$ are

$$\begin{smallmatrix}12\\001\end{smallmatrix},\ \begin{smallmatrix}02\\011\end{smallmatrix},\ \begin{smallmatrix}01\\012\end{smallmatrix},\ \text{and}\ \begin{smallmatrix}11\\002\end{smallmatrix}.$$

The 1-compatible components are $H^{q,s}_{\substack{12\\001}}$ and $H^{q,s}_{\substack{11\\002}}$, while there is only one (-1)-compatible component $H^{q,s}_{\substack{01\\012}}$. The component $H^{q,s}_{\substack{02\\011}}$ is neither 1-compatible nor (-1)-compatible.

The component $H^{q,s}_{\substack{12\\001}}$ contains vertices of $G^{q,s}_{\substack{31\\542}}$ and $G^{q,s}_{\substack{21\\543}}$ connected by jumps. In Figure 4 we have drawn the components $H^{q,s}_{\substack{11\\002}}$ and $H^{q,s}_{\substack{01\\012}}$.

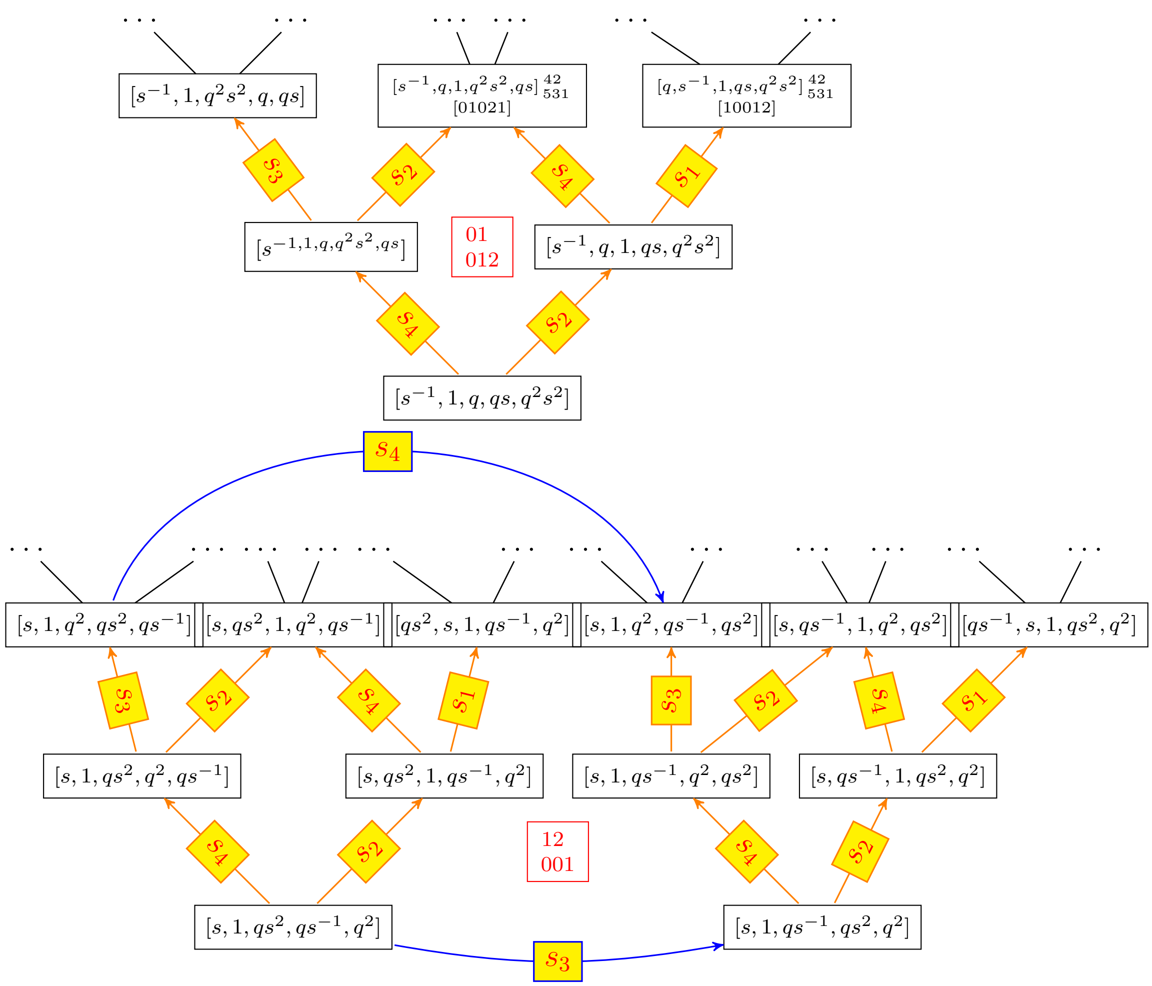
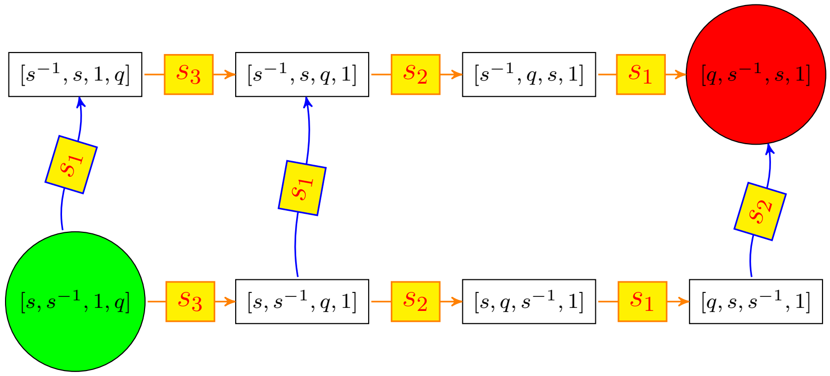


FIGURE 4. Two connected components of $H^{q,s}_{32}$

Example 5.5. Consider the tableau $T = \begin{smallmatrix}01\\00\end{smallmatrix}$. The graph $H^{q,s}_T$ is given by:



The sink is indicated by a red disk and the root by a green disk.

By abuse of language, we write $\zeta \in T$ to mean that ζ appears in a vertex of the connected component $H_T^{q,s}$.

Definition 5.6. In the same way, we define $\mathrm{std}_0 T$ of T to be the reverse standard tableau of shape λ obtained by the following process:

(1) Denote by $|T|_i$ the number of occurrences of i in T.
(2) Read the tableau T from *the left to the right and the bottom to the top* and replace occurrences of i in the order of their appearance by the numbers $N - |T|_0 - \cdots - |T|_{i-1}$, $N - |T|_0 - \cdots - |T|_{i-1} - 1$, $\ldots N - |T|_0 - \cdots - |T|_i$.

Let T be a filling of shape λ. Then $\mathrm{std}_1 T$ of T is defined to be the reverse standard tableau of shape λ obtained by the following process:

(1) Denote by $|T|_i$ the number of occurrences of i in T.
(2) Read the tableau T from *the bottom to the top and the left to the right* and replace occurrences of i in the order of their appearance by the numbers $N - |T|_0 - \cdots - |T|_{i-1}$, $N - |T|_0 - \cdots - |T|_{i-1} - 1$, $\ldots N - |T|_0 - \cdots - |T|_i$.

Example 5.7. To construct $\mathrm{std}_0 \begin{pmatrix} 0 & 1 & \\ 0 & 0 & 2 \end{pmatrix}$ we first write:

$$\begin{array}{cc|cc|c} 0 & 0 & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & . & . \\ . & . & . & 1 & . \\ . & . & . & . & 2 \end{array}$$

and we renumber entries in increasing order from the bottom to the top and the right to the left:

$$\begin{array}{cc|cc|c} 0 & 0 & 0 & 1 & 2 \\ \hline 5 & 4 & 3 & . & . \\ . & . & . & 2 & . \\ . & . & . & . & 1 \end{array}$$

We obtain $\mathrm{std}_0\left(\begin{array}{ccc}0&1&\\0&0&2\end{array}\right) = \begin{array}{ccc}4&2&\\5&3&1\end{array}$.

Pictorially, we construct $\mathrm{std}_1\left(\begin{array}{ccc}0&1&\\0&0&2\end{array}\right)$ writing:

$$\begin{array}{ccc|cc}0&0&2&0&1\\\hline 0&0&.&0&.\\ .&.&.&.&1\\ .&.&2&.&.\end{array}$$

and renumbering entries in increasing order from the bottom to the top and the right to the left:

$$\begin{array}{ccc|cc}0&0&2&0&1\\\hline 5&4&.&3&.\\ .&.&.&.&2\\ .&.&1&.&.\end{array}$$

This gives $\mathrm{std}_1\left(\begin{array}{ccc}0&1&\\0&0&2\end{array}\right) = \begin{array}{ccc}3&2&\\5&4&1\end{array}$.

Alternatively, one has

$$\begin{aligned}\mathrm{std}_0(T)[i,j] := \#\{(k,l) : T[k,l] > T[i,j]\} + \#\{(k,l) : k > i, T[k,l] = T[i,j]\}\\ + \#\{(i,l) : l \geq j, T[i,l] = T[i,j]\}\end{aligned}$$

and

$$\begin{aligned}\mathrm{std}_1(T)[i,j] := \#\{(k,l) : T[k,l] > T[i,j]\} + \#\{(k,l) : l > j, T[k,l] = T[i,j]\}\\ + \#\{(k,j) : k \geq i, T[k,j] = T[i,j]\}.\end{aligned}$$

We can characterize the root and the sink of a connected component.

Lemma 5.8. *One has:*

(1) $\mathbb{T}_{\mathrm{root}(T)} = \mathrm{std}_0 T$ *and* $\mathbb{T}_{\mathrm{sink}(T)} = \mathrm{std}_1 T$*,*
(2) $v_{\mathrm{root}(T)} = v^R$ *and* $v_{\mathrm{sink}(T)} = v^+$*.*

Proof. First observe that $T(\mathrm{std}_0(T), v) = T(\mathrm{std}_1(T), v) = T$ by construction. So we have $(v^R, \mathrm{std}_0(T)), (v^+, \mathrm{std}_1(T)) \in H_T^{q,s}$.

Since, v^R is an increasing partition, each arrow

$$\boxed{(\mathbb{T}, u)} \xrightarrow{\boxed{s_i}} \boxed{(\mathrm{std}_0(T), v^R)}$$

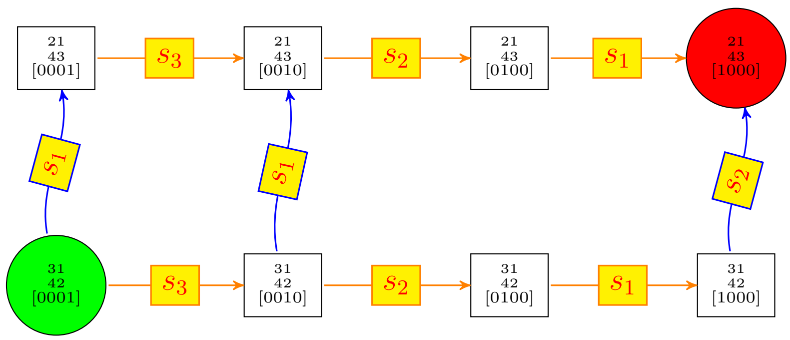
is a jump (i.e., $u = v^R$). Let $[i,j]$ be a cell of $\mathrm{std}_0(T)$ and $k = \mathrm{std}_0(T)[i,j]$. Let $[i',j']$ be the cell such that $k+1 = \mathrm{std}_0(T)[i',j']$. From the definition of $\mathrm{std}_0(T)$, we have either $T[i,j] \neq T[i',j']$ or $j = j'$ or $i < i'$ and $j > j'$ (that is, $\mathrm{CT}_{\mathrm{std}_0(T)}[k] < \mathrm{CT}_{\mathrm{std}_0(T)}[k+1]-1$). Hence, such a row does not exist and $(\mathrm{std}_0(T), v^R)$ has no antecedent in $H_T^{q,s}$. This is equivalent to $\mathrm{std}_0(T) = \mathbb{T}_{\mathrm{root}(T)}$.

In an equivalent way, we find that there is no arrow in $H_T^{q,s}$ of the form

$$(\mathrm{std}_1(T), v^+) \xrightarrow{s_i} (\mathbb{T}, u)$$

Consequently, we have $\mathrm{std}_1(T) = \mathbb{T}_{\mathrm{sink}(T)}$. $\square$

Example 5.9. We write Example 5.5 in terms of tableaux:



We observe that $\mathrm{std}_0\begin{pmatrix}01\\00\end{pmatrix} = \begin{smallmatrix}31\\42\end{smallmatrix} = \mathbb{T}_{\mathrm{root}\left(\begin{smallmatrix}01\\00\end{smallmatrix}\right)}$ and $\mathrm{std}_1\begin{pmatrix}01\\00\end{pmatrix} = \begin{smallmatrix}21\\43\end{smallmatrix} = \mathbb{T}_{\mathrm{sink}\left(\begin{smallmatrix}01\\00\end{smallmatrix}\right)}$.

Remark 5.10. As a consequence, we obtain that, if m_i denotes the number of occurrences of i among the entries of T, then

$$r_{\mathrm{root}(T)} = [\dots, m_0 + \cdots + m_i + 1, \dots, m_0 + \cdots + m_{i+1} + 1, \dots,\\ m_0 + 1, \dots, m_0 + m_1, 1, \dots, m_0],$$

and $r_{\mathrm{sink}(T)} = [1, \dots, N]$.

The notion of (± 1)-compatibility is easily detectable from the root and the sink.

Lemma 5.11. *If $H_T^{q,s}$ is 1-compatible then, for each i, i and $i+1$ are not in the same column of $\mathbb{T}_{\mathrm{root}(T)}$. If $H_T^{q,s}$ is (-1)-compatible then, for each i, i and $i+1$ are not in the same row of $\mathbb{T}_{\mathrm{sink}(T)}$.*

Proof. From Lemma 5.8, we have $\mathbb{T}_{\mathrm{root}(T)} = \mathrm{std}_0(T)$ and $\mathbb{T}_{\mathrm{sink}(T)} = \mathrm{std}_1(T)$. But if k and $k+1$ are in the same column of $\mathrm{std}_0(T)$, where we suppose that $\mathrm{std}_0(T)[i,j] = k$, then $\mathrm{std}_0(T)[i,j+1] = k+1$, and the only possibility is that $T[i,j] = T[i,j+1]$, which contradicts the fact that T is a column-strict tableau. Similarly, if k and $k+1$ are in the same row of $\mathrm{std}_1(T)$, then $T[i,j] = T[i+1,j]$ for some (i,j), which contradicts the fact that T is a row-strict tableau. $\square$

Now we have all the material for an interpretation of the (± 1)-compatibility in terms of spectral vectors.

Proposition 5.12. *If $H_T^{q,s}$ is 1-compatible then, for each i, $\zeta_{\mathrm{root}(T)}[i] \not\sim \zeta_{\mathrm{root}(T)}[i+1]$ implies $\zeta_{\mathrm{root}(T)}[i] = s\zeta_{\mathrm{root}(T)}[i+1]$. If $H_T^{q,s}$ is (-1)-compatible then, for each i, $\zeta_{\mathrm{sink}(T)}[i] \not\sim \zeta_{\mathrm{sink}(T)}[i+1]$ implies $\zeta_{\mathrm{sink}(T)}[i] = s^{-1}\zeta_{\mathrm{sink}(T)}[i+1]$.*

Proof. This is just the translation of Lemma 5.11 in terms of spectral vectors. $\square$

5.2. **Invariant subspaces.** The Yang–Baxter graph and the previous section imply that we can characterize the irreducible subspaces U of polynomials invariant under $\mathcal{H}_N(s)$ and $\{\boldsymbol{\xi}_i : 1 \le i \le N\}$, that is, $U\mathbf{T}_i, U\boldsymbol{\xi}_i \subset U$.

Definition 5.13. Let T be a tableau with increasing row and column entries. We denote by $\mathcal{M}_T$ the space generated by the polynomials P_ζ with $\zeta \in T$.

Example 5.14. For instance, $\mathcal{M}_{\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}}$ is spanned by

$$\{P_{[s,s^{-1},1,q]},\ P_{[s,s^{-1},q,1]},\ P_{[s,q,s^{-1},1]},\ P_{[q,s,s^{-1},1]}P_{[s^{-1},s,1,q]},\ P_{[s^{-1},s,q,1]},\ P_{[s^{-1},q,s,1]},\ P_{[q,s^{-1},s,1]}\}.$$

The spaces $\mathcal{M}_T$ are the irreducible invariant subspaces.

Proposition 5.15. *We have $\mathcal{M}_T\mathbf{T}_i, \mathcal{M}_T\boldsymbol{\xi}_i \subset \mathcal{M}_T$. Furthermore, if U is a proper subspace of $\mathcal{M}_T$, then $U\mathbf{T}_i \not\subset U$ or $U\boldsymbol{\xi}_i \not\subset U$.*

Proof. Let U be a subspace of $\mathcal{M}_T\mathbf{T}_i$ such that $U\mathbf{T}_i, U\boldsymbol{\xi}_i \subset U$. The operators $\boldsymbol{\xi}_i$ being simultaneously diagonalizable, U is spanned by a set of polynomials $\{P_{\zeta_1}, \dots, P_{\zeta_k}\}$ with $k \in \mathbb{N}$ and $\zeta_i \in \mathbb{T}$. But from the Yang–Baxter construction, if there exists $\zeta \in T$ such that $P_\zeta \in U$, then, for each $\zeta \in T$, $P_\zeta \in U$. So U is not a proper subspace. $\square$

In the rest of the section, we investigate the dimension of the spaces $\mathcal{M}_T$. The dimension of such a space equals the number of permutations of the vector of the entries of T multiplied by the number of tableaux $\mathbb{T}$ appearing in $\mathcal{H}_T$. The first number is easy to obtain, but for the second we need some additional considerations.

Suppose that μ, λ are partitions with $\mu \subset \lambda$ ($\mu[i] \le \lambda[i]$ for all i), $|\mu| = k, |\lambda| = n$, then the set $\{(i,j) : 1 \le i \le \ell(\lambda), \mu[i] < j \le \lambda[i]\}$ is the skew diagram $\lambda\backslash\mu$. The basic step in determining the dimension of a connected component is to find the number (denoted $\dim(\lambda\backslash\mu)$) of RST's of shape $\lambda\backslash\mu$, that is, the number of ways the numbers $(n-k), (n-k-1), \dots, 1$ can be entered in $\lambda\backslash\mu$ so that the entries decrease in each row and in each column. Equivalently, we ask for the number of standard Young tableaux of shape $\lambda\backslash\mu$ (see [12] or [15] for the definition). There is a classical formula for this number (see [15, Cor. 7.16.3]). If $\det(a_{ij})$ denotes the determinant of the matrix $(a_{ij})_{i,j=1}^p$, where $p \ge \ell(\lambda)$ (the formula is independent of p), the formula says that

$$\dim(\lambda\backslash\mu) = (n-k)! \det\left[\frac{1}{(\lambda[i] - \mu[j] - i + j)!}\right],$$

Here, $\frac{1}{p!} = 0$ for $p = -1, -2, \dots$ (as in the zeros of $\frac{1}{\Gamma(p+1)}$).

Now consider a tableau T. Let M denote the maximum entry (also of any v in this component) and let

$$\mu_m = \{(i,j) \in T : T(i,j) \le m\}, \quad 0 \le m \le M.$$

Then each μ_m is the Ferrers diagram of a partition, $\mu_m \subset \mu_{m+1}$ (possibly $\mu_m = \mu_{m+1}$ for some m and $\dim(\mu_{m+1}\backslash\mu_m) = 1$ trivially), and $v^+[j] = m$ if j is an entry in $\mu_m\backslash\mu_{m-1}$. The number of RST's in the connected component of T is

$$\dim(\mu_0\backslash[0]) \prod_{m=1}^{M} \dim(\mu_m\backslash\mu_{m-1}),$$

and the number of permutations of v^+ is $N!/\left(|\mu_0|!\prod_{m=1}^{M}(|\mu_m|-|\mu_{m-1}|)!\right)$; the dimension of the component is

$$N!\det\left[\frac{1}{(\mu_0[i]-i+j)!}\right]\prod_{m=1}^{M}\det\left[\frac{1}{(\mu_m[i]-\mu_{m-1}[j]-i+j)!}\right].$$

This product can be restricted to the values of m for which $\mu_{m-1}\neq\mu_m$, that is, the set of entries of v^+.

Example 5.16. (1) Consider again the tableau $T=\begin{matrix}0&1\\0&0\end{matrix}$. Then $\mu_0=[2,1]$ and $\mu_1=[2,2]$. Hence,

$$\dim(\mu_0\backslash[0])=3!\det\begin{bmatrix}\frac{1}{2}&\frac{1}{6}\\1&1\end{bmatrix}=2,$$
$$\dim(\mu_1\backslash\mu_0)=1!\det\begin{bmatrix}1&\frac{1}{2}\\0&1\end{bmatrix}=1.$$

Consequently, the number of tableaux $\mathbb{T}$ in T equals 2. The tableaux are $\begin{matrix}2&1\\4&3\end{matrix}$ and $\begin{matrix}3&1\\4&2\end{matrix}$. So the dimension of $\mathcal{M}_T$ is $\frac{4!}{3!1!}\times 2=8$.

(2) Consider the bigger example given by the tableau $T=\begin{matrix}1&2&\\0&0&1\end{matrix}$ (see Figure 4). Here $\mu_0=[2]$, $\mu_1=[3,1]$, and $\mu_2=[3,2]$. We compute

$$\dim(\mu_0\backslash[0])=2!\det\begin{bmatrix}\frac{1}{2}&\frac{1}{6}\\0&1\end{bmatrix}=1,$$
$$\dim(\mu_1\backslash\mu_0)=2!\det\begin{bmatrix}1&\frac{1}{24}\\0&1\end{bmatrix}=2,$$
$$\dim(\mu_2\backslash\mu_1)=1!\det\begin{bmatrix}1&\frac{1}{6}\\0&1\end{bmatrix}=1.$$

There are two tableaux; the graph decomposes into two parts when we remove the jump edges. The dimension of $\mathcal{M}_T$ is $\frac{5!}{2!2!1!}\times 2=60$.

(3) Consider $T=\begin{matrix}0&1&\\0&1&2\end{matrix}$ (see Figure 4). One has $\mu_0=[1,1]$, $\mu_1=[2,2]$, and $\mu_2=[3,2]$. Hence, we have only one tableau in the connected component. Graphically, there is no jump (blue arrow) in the connected component $H_T^{q,s}$. The dimension of $\mathcal{M}_T$ is 30.

5.3. **Symmetrizer/Antisymmetrizer.** We define the operator

$$\mathbf{S}_N:=\sum_{\sigma\in\mathfrak{S}_N}\widetilde{\mathbf{T}}_\sigma,$$

where $\widetilde{\mathbf{T}}_\sigma=\mathbf{T}_{i_1}\cdots\mathbf{T}_{i_k}$ if there is a shortest expression $\sigma=s_{i_1}\cdots s_{i_k}$. The operator $\mathbf{S}_N$ is an s-deformation of the classical symmetrizer in the following sense.

Proposition 5.17. *For each i, one has*

$$\mathbf{S}_N\mathbf{T}_i = s\mathbf{S}_N.$$

Proof. It suffices to split the sum as

$$\mathbf{S}_N\mathbf{T}_i = \sum_{\substack{\sigma\in\mathfrak{S}_N\\ \ell(\sigma s_i)>\ell(\sigma)}} \widetilde{\mathbf{T}}_\sigma\mathbf{T}_i + \sum_{\substack{\sigma\in\mathfrak{S}_N\\ \ell(\sigma s_i)<\ell(\sigma)}} \widetilde{\mathbf{T}}_\sigma\mathbf{T}_i. \tag{5.1}$$

We use the quadratic relation to write the second sum as

$$\sum_{\substack{\sigma\in\mathfrak{S}_N\\ \ell(\sigma s_i)<\ell(\sigma)}} \widetilde{\mathbf{T}}_\sigma\mathbf{T}_i = (s-1)\sum_{\substack{\sigma\in\mathfrak{S}_N\\ \ell(\sigma s_i)<\ell(\sigma)}} \widetilde{\mathbf{T}}_{\sigma s_i}\mathbf{T}_i + s\sum_{\substack{\sigma\in\mathfrak{S}_N\\ \ell(\sigma s_i)<\ell(\sigma)}} \widetilde{\mathbf{T}}_{\sigma s_i}.$$

But

$$\sum_{\substack{\sigma\in\mathfrak{S}_N\\ \ell(\sigma s_i)<\ell(\sigma)}} \widetilde{\mathbf{T}}_{\sigma s_i}\mathbf{T}_i = \sum_{\substack{\sigma\in\mathfrak{S}_N\\ \ell(\sigma s_i)>\ell(\sigma)}} \widetilde{\mathbf{T}}_\sigma$$

Hence,

$$\sum_{\substack{\sigma\in\mathfrak{S}_N\\ \ell(\sigma s_i)<\ell(\sigma)}} \widetilde{\mathbf{T}}_\sigma\mathbf{T}_i = (s-1)\sum_{\substack{\sigma\in\mathfrak{S}_N\\ \ell(\sigma s_i)>\ell(\sigma)}} \widetilde{\mathbf{T}}_\sigma + s\sum_{\substack{\sigma\in\mathfrak{S}_N\\ \ell(\sigma s_i)<\ell(\sigma)}} \widetilde{\mathbf{T}}_{\sigma s_i}.$$

Substituting this in (5.1), we obtain the result. □

As a consequence, we obtain the following result.

Corollary 5.18. *The operator $\mathbf{S}_N$ satisfies*

$$\mathbf{S}_N^2 = \phi_N(s)\mathbf{S}_N,$$

where $\phi_N(s) := \prod_{j=2}^{N}\frac{1-s^j}{1-s}$ is the Poincaré polynomial of $\mathfrak{S}_N$.

Proof. From Proposition 5.17, one obtains

$$\mathbf{S}_N^2 = \mathbf{S}_N\sum_{\sigma\in\mathfrak{S}_N}\widetilde{\mathbf{T}}_u = \sum_{\sigma\in\mathfrak{S}_n} s^{\ell(\sigma)}\mathbf{S}_N = \phi_N(s)\mathbf{S}_N.$$

□

Alternatively, we define

$$\mathbf{S}'_N = \sum_{\substack{\sigma\in\mathfrak{S}_N\\ \ell(\sigma)=k,\sigma=s_{i_1}\cdots s_{i_k}}} \mathbf{T}_{i_1}^{-1}\cdots\mathbf{T}_{i_k}^{-1}.$$

This operator satisfies

$$\mathbf{S}'_N\mathbf{T}_i = s\mathbf{S}'_N \tag{5.2}$$

and

$$\mathbf{S}'^2_N = \phi_N\left(\frac{1}{s}\right)\mathbf{S}'_N. \tag{5.3}$$

The action of the symmetrizer on leading terms has some nice properties.

Lemma 5.19. *Let v and $\mathbb{T}$ be such that $\mathrm{COL}_{\mathbb{T}}[r_v[i]] = \mathrm{COL}_{\mathbb{T}}[r_v[i]+1]$ and $v[i] = v[i+1]$ for some i. Then*
$$x^{v,\mathbb{T}}\mathbf{S}_N = 0.$$

Proof. We have
$$x^{v,\mathbb{T}}\mathbf{T}_i = x^v\delta_i^x\mathbb{T} + x^v s_i\mathbb{T}R_vT_i.$$
But $v[i] = v[i+1]$ implies $x^v\delta_i^x = 0$ and, since $\mathrm{COL}_{\mathbb{T}}[r_v[i]] = \mathrm{COL}_{\mathbb{T}}[r_v[i]+1]$, we have $\mathbb{T}T_{r_v[i]} = -\mathbb{T}$. Hence
$$x^{v,\mathbb{T}}\mathbf{T}_i = x^v\mathbb{T}T_{r_v[i]}R_v = -x^{v,\mathbb{T}}. \tag{5.4}$$
Now we split the sum $x^{v,\mathbb{T}}\mathbf{S}_N$ into two sums,
$$\begin{aligned} x^{v,\mathbb{T}}\mathbf{S}_N &= x^{v,\mathbb{T}}\sum_{\ell(s_i\sigma)<\ell(\sigma)}\widetilde{\mathbf{T}}_\sigma + x^{v,\mathbb{T}}\sum_{\ell(s_i\sigma)>\ell(\sigma)}\widetilde{\mathbf{T}}_\sigma \\ &= x^{v,\mathbb{T}}\sum_{\ell(s_i\sigma)<\ell(\sigma)}T_i\widetilde{\mathbf{T}}_{s_i\sigma} + x^{v,\mathbb{T}}\sum_{\ell(s_i\sigma)>\ell(\sigma)}\widetilde{\mathbf{T}}_\sigma. \end{aligned}$$
From eq. (5.4), one obtains
$$\begin{aligned} x^{v,\mathbb{T}}\mathbf{S}_N &= -x^{v,\mathbb{T}}\sum_{\ell(s_i\sigma)<\ell(\sigma)}\widetilde{\mathbf{T}}_{s_i\sigma} + x^{v,\mathbb{T}}\sum_{\ell(s_i\sigma)>\ell(\sigma)}\widetilde{\mathbf{T}}_\sigma \\ &= -x^{v,\mathbb{T}}\sum_{\ell(s_i\sigma)>\ell(\sigma)}\widetilde{\mathbf{T}}_\sigma + x^{v,\mathbb{T}}\sum_{\ell(s_i\sigma)>\ell(\sigma)}\widetilde{\mathbf{T}}_\sigma \\ &= 0. \end{aligned}$$
□

In the same way, we define
$$\mathbf{A}_N = \sum_{\sigma\in\mathfrak{S}_N}(-s)^{\ell(\sigma)}\overline{\mathbf{T}}_\sigma,$$
where $\overline{\mathbf{T}}_\sigma = \mathbf{T}_{i_1}^{-1}\cdots\mathbf{T}_{i_k}^{-1}$ given there is a shortest expression $\sigma = s_{i_1}\cdots s_{i_k}$. The operator $\mathbf{A}_N$ satisfies the following relation.

Proposition 5.20. *For each i, we have*
$$\mathbf{A}_N\mathbf{T}_i = -\mathbf{A}_N.$$

Proof. The proof is very close to the proof of Proposition 5.17 and is left to the reader. □

The operator $\mathbf{A}_N$ satisfies the following analog of Corollary 5.18.

Corollary 5.21. *We have*
$$\mathbf{A}_N^2 = \phi_N(s)\mathbf{A}_N.$$

Lemma 5.22. *Let v and $\mathbb{T}$ be such that $\mathrm{ROW}_{\mathbb{T}}[r_v[i]] = \mathrm{ROW}_{\mathbb{T}}[r_v[i]+1]$ and $v[i] = v[i+1]$ for some i. Then*
$$x^{v,\mathbb{T}}\mathbf{A}_N = 0.$$

Lemma 5.23. *Let $v = [v[1] < \cdots < v[N]]$ and $\mathbb{T}$ be such that, for each i, $v[i] = v[i+1]$ implies $\mathrm{COL}_{\mathbb{T}}[r_v[i]] = \mathrm{COL}_{\mathbb{T}}[r_v[i]+1]$. Then the coefficient of $x^{v,\mathbb{T}}$ in $x^{v,\mathbb{T}}\mathbf{A}_N$ equals $\prod_i s^{m_i}\phi_{m_i}(s)$, where m_i denotes the number of parts i in v.*

5.4. **Symmetric/Antisymmetric polynomials.** For $\zeta = \zeta_{v,\mathbb{T}}$ and $\zeta s_i = \zeta_{v',\mathbb{T}'}$, we set $\mathfrak{s}^i_\zeta := P_{\zeta s_i} + \frac{s-\frac{\zeta[i+1]}{\zeta[i]}}{1-\frac{\zeta[i+1]}{\zeta[i]}} P_\zeta$ and $\mathfrak{a}^i_\zeta := P_{\zeta s_i} - \frac{1-s\frac{\zeta[i+1]}{\zeta[i]}}{1-\frac{\zeta[i+1]}{\zeta[i]}} P_\zeta$.

Lemma 5.24. *If $\zeta[i+1] \succ \zeta[i]$, then we have*

$$\mathfrak{s}^i_\zeta \mathbf{T}_i = s\mathfrak{s}^i_\zeta \quad \text{and} \quad \mathfrak{a}^i_\zeta \mathbf{T}_i = -\mathfrak{a}^i_\zeta.$$

Proof. We prove only the result for $\mathfrak{s}^i_\zeta$ since the proof is very similar for $\mathfrak{a}^i_\zeta$. Recall that Proposition 4.19 yields

$$P_\zeta \mathbf{T}_i = P_{\zeta s_i} - (1-s)\frac{\zeta[i]}{\zeta[i]-\zeta[i+1]} P_\zeta$$

and

$$P_{\zeta s_i}\mathbf{T}_i = \frac{(\zeta[i+1]-s\zeta[i])(s\zeta[i+1]-\zeta[i])}{(\zeta[i+1]-\zeta[i])^2} P_\zeta - (1-s)\frac{\zeta[i+1]}{\zeta[i+1]-\zeta[i]} P_{\zeta s_i}.$$

Hence,

$$\begin{aligned}\mathfrak{s}^i_\zeta \mathbf{T}_i &= \left(\frac{(\zeta[i+1]-s\zeta[i])(s\zeta[i+1]-\zeta[i])}{(\zeta[i+1]-\zeta[i])^2} - \frac{(1-s)\zeta[i](s-\frac{\zeta[i+1]}{\zeta[i]})}{(\zeta[i]-\zeta[i+1])(1-\frac{\zeta[i+1]}{\zeta[i]})}\right) P_\zeta \\ &\qquad + \left(\frac{s-\frac{\zeta[i+1]}{\zeta[i]}}{1-\frac{\zeta[i+1]}{\zeta[i]}} - (1-s)\frac{\zeta[i+1]}{\zeta[i+1]-\zeta[i]}\right) P_{\zeta s_i} \\ &= sP_{\zeta s_i} + s\left(\frac{s-\frac{\zeta[i+1]}{\zeta[i]}}{1-\frac{\zeta[i+1]}{\zeta[i]}}\right) \\ &= s\mathfrak{s}^i_\zeta.\end{aligned}$$

□

Let $\mathfrak{f} = \sum_{\zeta\in T} b_\zeta P_\zeta \in \mathcal{M}_T$ be a symmetric polynomial, i.e., $\mathfrak{f}\mathbf{T}_i = s\mathfrak{f}$ for each i.

Lemma 5.25. *If $\zeta[i+1] \succ \zeta[i]$, then $\frac{b_\zeta}{b_{\zeta s_i}} = \frac{s\zeta[i]-\zeta[i+1]}{\zeta[i]-\zeta[i+1]}$.*

Proof. Since $\mathfrak{f}T_i = s\mathfrak{f}$, we have

$$\left(b_\zeta P_\zeta + b_{\zeta s_i} P_{\zeta s_i}\right)\mathbf{T}_i = s\left(b_\zeta P_\zeta + b_{\zeta s_i} P_{\zeta s_i}\right).$$

Then $b_\zeta P_\zeta + b_{\zeta s_i} P_{\zeta s_i}$ is proportional to $\mathfrak{s}^i_\zeta$. This finishes the proof of the lemma. □

Since each vertex of T is connected to $\mathrm{sink}(T)$ by a series of edges

$$\boxed{\zeta} \longrightarrow \boxed{s_i} \longrightarrow \boxed{\zeta s_i}$$

the polynomial $\mathfrak{f}$ is unique up to a global multiplicative coefficient, and $b_\zeta \neq 0$ for all ζ if $\mathfrak{f} \neq 0$.

If $T[i,j] = T[i,j+1]$ for some (i,j) then $\zeta_{\mathrm{root}(T)}[k] = q^n s^m \not\succ \zeta_{\mathrm{root}(T)}[k+1] = q^n s^{m+1}$ for some k. Indeed, $T[i,j] = T[i,j+1]$ implies $v_{\mathrm{root}(T)}[k] = v_{\mathrm{root}(T)}[k+1]$, hence $r_{v_{\mathrm{root}(T)}}[k]+1 = r_{v_{\mathrm{root}(T)}}[k+1]$. It follows that $m = \mathrm{CT}_{\mathbb{T}_{\mathrm{root}(T)}}[\ell]$ and $m+1 = \mathrm{CT}_{\mathbb{T}_{\mathrm{root}(T)}}[\ell+1]$ for some ℓ.

Example 5.26. If $T = \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}$, we have

$$\mathrm{root}(T) = \left(\begin{matrix} 3 & 1 \\ 4 & 2 \end{matrix}, [s, s^{-1}, 1, q], [0,0,0,1], [2,3,4,1] \right).$$

We have $T[1,1] = T[1,2] = 0$. The corresponding cells in the tableau $\mathbb{T}_{\mathrm{root}(T)}$ are $\mathbb{T}_{\mathrm{root}(T)}[1,1] = 4$ and $\mathbb{T}_{\mathrm{root}(T)}[1,2] = 3$. So $\ell = 3$, $k = 2$, and $m = -1 = \mathrm{CT}_{\mathbb{T}_{\mathrm{root(T)}}}[3] = \mathrm{CT}_{\mathbb{T}_{\mathrm{root(T)}}}[4] - 1$.

From $\mathfrak{f}T_k = s\mathfrak{f}$, one deduces $b_{\zeta_{\mathrm{root}(T)}} = s(s-1)^{-1}\zeta_{\mathrm{root}(T)}[k] - \frac{\zeta_{\mathrm{root}(T)}[k+1]}{\zeta_{\mathrm{root}(T)}[k]} b_{\zeta_{\mathrm{root}(T)}}$. Finally, $\frac{\zeta_{\mathrm{root}(T)}[k]}{\zeta_{\mathrm{root}(T)}[k] - \zeta_{\mathrm{root}(T)}[k+1]} = \frac{1}{1-s}$ implies $b_{\zeta_{\mathrm{root}(T)}} = 0$ and $\mathfrak{f} = 0$.

In the other cases, the coefficients b_ζ are not zero, and they can be computed via the recurrence given in Lemma 5.25. More precisely, setting $b_{\zeta_{\mathrm{root}(T)}} = 1$, and $b_{\zeta s_i} = \frac{\zeta[i]-\zeta[i+1]}{s\zeta[i]-\zeta[i+1]} b_\zeta$ if $\zeta[i+1] \succ \zeta[i]$, we define the polynomial

$$\mathfrak{M}_T = \sum_{\zeta \in T} b_\zeta P_\zeta,$$

which is the unique generator of the subspace of symmetric polynomials of $\mathcal{M}_T$.
So, one arrives at the following result.

Theorem 5.27. *The subspace of $\mathcal{M}_T$ of symmetric polynomials is*

1. *a 1-dimension space generated by $\mathfrak{M}_T$ if T is a strict-column tableau;*
2. *a 0-dimension space in the other cases.*

Example 5.28. Consider the graph $H^{q,s}_{\substack{11\\00}}$ (see Figure 5). The polynomial

$$\mathfrak{M}_{\substack{11\\00}} = P_{[s,1,q,qs^{-1}]} + \frac{1-q}{s-q} P_{[s,q,1,qs^{-1}]} + \frac{(1-q)}{(s^2-q)} P_{[q,s,1,qs^{-1}]} + \frac{(1-q)}{(s^2-q)} P_{[s,q,qs^{-1},1]} + \frac{(1-q)(s-q)}{(s^2-q)^2} P_{[q,s,qs^{-1}]} + \frac{(1-q)(s-q)}{(s^2-q)(s^3-q)} P_{[q,qs^{-1},s,1]}$$

is symmetric.

In the same way, define $b^a_{\zeta_{\mathrm{root}(T)}} = 1$, and $b^a_{\zeta s_i} = -\frac{\zeta[i]-\zeta[i+1]}{\zeta[i]-s\zeta[i+1]} b^a_\zeta$ if $\zeta[i+1] \succ \zeta[i]$, and the polynomial

$$\mathfrak{M}^a_T = \sum_{\zeta \in T} b^a_\zeta P_\zeta.$$

We then have the following result.

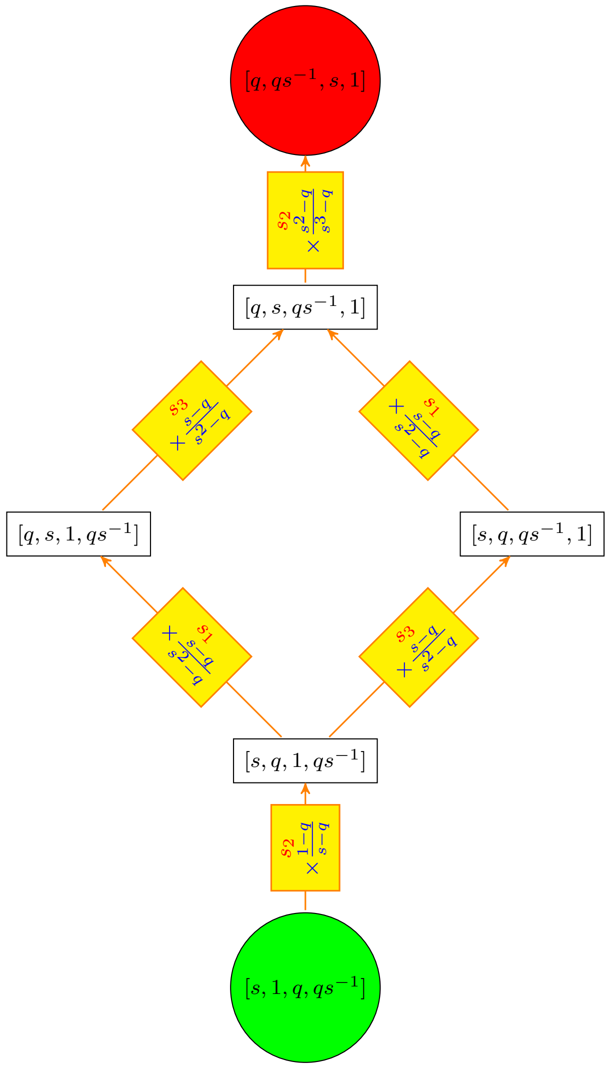


FIGURE 5. The graph $H^{q,s}_{\substack{11\\00}}$

Theorem 5.29. *The subspace of $\mathcal{M}^a_T$ of antisymmetric polynomials is*

(1) *a 1-dimension space generated by $\mathfrak{M}^a_T$ if T is a strict-row tableau;*
(2) *a 0-dimension space in the other cases.*

5.5. **The group of permutations leaving T invariant.** Let T be a filling of shape λ with increasing rows and strictly increasing columns.

To each i we associate the pair $\mathrm{COORD}_T[i] = \left(\mathrm{COL}_{\mathrm{std}_1(T)}[i], \mathrm{ROW}_{\mathrm{std}_1(T)}[i]\right)$. An elementary transposition s_i acts on T by permuting the cells $\mathrm{COORD}_T[i]$ and $\mathrm{COORD}_T[i+1]$.

For a tableau $\mathbb{T}$, we denote by $\mathfrak{S}_{\mathbb{T}}$ the maximal subgroup of $\mathfrak{S}_N$ leaving invariant the sets of entries of each line.

Example 5.30. For instance, consider the tableau $\mathbb{T} = \begin{smallmatrix} 3 & 2 & \\ 5 & 4 & 1 \end{smallmatrix}$. We have $\mathfrak{S}_{\mathbb{T}} = \mathfrak{S}_{\{1,4,5\}} \times \mathfrak{S}_{\{2,3\}}$.

We denote also by $\mathfrak{S}_T$ the maximal subgroup of $\mathfrak{S}_{\mathrm{std}_1(T)}$ leaving T invariant.

Example 5.31. Let $T = \begin{smallmatrix} 1 & 1 & \\ 0 & 0 & 1 \end{smallmatrix}$. Then we have $\mathrm{std}_1(T) = \begin{smallmatrix} 3 & 2 & \\ 5 & 4 & 1 \end{smallmatrix}$ and

$$\mathfrak{S}_T = \mathfrak{S}_{\{2,3\}} \times \mathfrak{S}_{\{4,5\}} \times \mathfrak{S}_{\{1\}} \subset \mathfrak{S}_{\mathrm{std}_1(T)} = \mathfrak{S}_{\{1,4,5\}} \times \mathfrak{S}_{\{2,3\}}.$$

Let $\mathfrak{S}_r(T)$ be the subgroup of $\mathfrak{S}_N$ leaving the partition $v_{\mathrm{sink}(T)}$ invariant.

Example 5.32. Again, with $T = \begin{smallmatrix} 1 & 1 & \\ 0 & 0 & 1 \end{smallmatrix}$, we have $v_{\mathrm{sink}(T)} = [1,1,1,0,0]$ and

$$\mathfrak{S}_r(T) = \mathfrak{S}_{\{1,2,3\}} \times \mathfrak{S}_{\{4,5\}}.$$

Observe that $\mathfrak{S}_T = \mathfrak{S}_{\mathrm{std}_1(T)} \cap \mathfrak{S}_r(T)$. This implies that for each $\sigma \in \mathfrak{S}_T$ one has $(v_{\mathrm{sink}(T)}, \mathrm{std}_1(T))\sigma = (v_{\mathrm{sink}(T)}, \mathrm{std}_1(T))$.

Remark 5.33. In terms of spectral vectors, we have $\zeta_{\mathrm{sink}(T)}\sigma = \zeta_{\mathrm{sink}(T)}$ (here we use the action defined in eq. (4.4)). The property of T to have only strictly increasing columns can also be interpreted in terms of spectral vectors. Indeed, for each i, we have

$$\zeta_{\mathrm{sink}(T)}[i] \succ \zeta_{\mathrm{sink}(T)}[i+1] \quad \text{or} \quad \zeta_{\mathrm{sink}(T)}[i] = q^n s^{m+1} \not\succ \zeta_{\mathrm{sink}(T)}[i+1] = q^n s^m. \tag{5.5}$$

Example 5.34. Consider the tableau $T = \begin{smallmatrix} 1 & 1 & \\ 0 & 0 & 1 \end{smallmatrix}$. We compute $\zeta_{\mathrm{sink}(T)}$ from the vector $v_{\mathrm{sink}(T)} = [1,1,1,0,0]$ and the tableau $\mathrm{std}_1(T) = \begin{smallmatrix} 3 & 2 & \\ 5 & 4 & 1 \end{smallmatrix}$. Here, $r_{\mathrm{sink}(T)} = [1,2,3,4,5]$, hence $\zeta_{\mathrm{sink}(T)} = [s^2q, q, s^{-1}q, s, 1]$. Observe that $\zeta_{\mathrm{sink}(T)}[1] \succ \zeta_{\mathrm{sink}(T)}[2]$, $\zeta_{\mathrm{sink}(T)}[2] \not\succ \zeta_{\mathrm{sink}(T)}[3]$, with $\frac{\zeta_{\mathrm{sink}(T)}[2]}{\zeta_{\mathrm{sink}(T)}[3]} = s$, $\zeta_{\mathrm{sink}(T)}[3] \succ \zeta_{\mathrm{sink}(T)}[4]$ and $\zeta_{\mathrm{sink}(T)}[4] \not\succ \zeta_{\mathrm{sink}(T)}[5]$, with $\frac{\zeta_{\mathrm{sink}(T)}[4]}{\zeta_{\mathrm{sink}(T)}[5]} = s$.

Let σ_T be the minimal permutation such that $\zeta_{\mathrm{root}T}\sigma_T = \zeta_{\mathrm{sink}(T)}$. It can be used to characterize the group $\mathfrak{S}_T$.

Lemma 5.35. *The group $\mathfrak{S}_T$ is the subgroup of $\mathfrak{S}_N$ consisting of the permutations σ such that $\ell(\sigma_T\sigma) = \ell(\sigma_T) + \ell(\sigma)$.*

Furthermore, we will use the following result.

Lemma 5.36. *For each permutation σ such that $\zeta_{root(T)} \preceq \zeta_{root(T)}\sigma$ one has:*

$$P_{\zeta_{\mathrm{root}(T)}} \widetilde{\mathbf{T}}_\sigma = P_{\zeta_{\mathrm{root}(T)}\sigma} + \sum_{\zeta' \prec \zeta_{\mathrm{root}(T)}\sigma} (*) P_{\zeta'}.$$

Proof. We prove the result by induction on the length of σ. If $\sigma = id$, then the result is obvious. Now suppose that $\sigma = \sigma' s_j$, with $\ell(\sigma) = \ell(\sigma')+1$, and so $\zeta_{\mathrm{root}(T)}\sigma \prec \zeta_{\mathrm{root}(T)}\sigma s_j$. Then $\widetilde{\mathbf{T}}_\sigma = \widetilde{\mathbf{T}}_{\sigma'}\mathbf{T}_j$. Furthermore, using the induction hypothesis, we obtain

$$P_{\zeta_{\mathrm{root}(T)}} \widetilde{\mathbf{T}}_\sigma = P_{\zeta_{\mathrm{root}(T)}\sigma'}\mathbf{T}_j + \sum_{\zeta' \prec \zeta_{\mathrm{root}(T)}\sigma} (*) P_{\zeta'}\mathbf{T}_j. \tag{5.6}$$

But $P_{\zeta_{\mathrm{root}(T)}\sigma'}\mathbf{T}_j = P_{\zeta_{\mathrm{root}(T)}\sigma s_j} + (*)P_{\zeta_{\mathrm{root}(T)}\sigma}$. Furthermore, since $\zeta' \prec \zeta_{\mathrm{root}(T)}\sigma'$, we have $\zeta' s_j \prec \zeta_{\mathrm{root}(T)}\sigma$. But

$$P_{\zeta'}\mathbf{T}_j = (*)P_{\zeta' s_j} + (*)P_{\zeta'}.$$

Hence, by substituting this in (5.6), we arrive at the claimed result. □

We are now in the position to deduce the following auxiliary result.

Lemma 5.37. *Denote by β_T^σ the coefficient of $P_{\zeta_{\mathrm{sink}(T)}}$ in $P_{\zeta_{\mathrm{root}(T)}}\widetilde{\mathbf{T}}_\sigma$. Then we have:*

(1) *If $\sigma_T^{-1}\sigma \notin \mathfrak{S}_T$, then $\beta_T^\sigma = 0$.*
(2) *If $\sigma_T^{-1}\sigma \in \mathfrak{S}_T$, then $\beta_T^\sigma = s^{\ell(\sigma)-\ell(\sigma_T)}$.*

Proof. Part (1) is a direct consequence of Lemma 5.36. To show Part (2), we first use Lemma 5.36 and write $P_{\zeta_{\mathrm{root}(T)}}\widetilde{\mathbf{T}}_{\sigma_T} = P_{\zeta_{\mathrm{sink}(T)}} + \sum_{\zeta\prec\zeta_{\mathrm{sink}(T)}}(*)P_\zeta$. Now set $\tau := \sigma_T^{-1}\sigma \in \mathfrak{S}_T$ and observe that, for each element $\tau' \in \mathfrak{S}_T$, $\zeta\tau' = \zeta_{\mathrm{sink}(T)}$ implies $\zeta = \zeta_{\mathrm{sink}(T)}$. Hence, the coefficient of $\zeta_{\mathrm{sink}(T)}$ in $\sum_{\zeta\prec\zeta_{\mathrm{sink}(T)}}(*)P_\zeta\widetilde{\mathbf{T}}_\tau$ is 0. It follows that β_T^σ equals the coefficient of $\zeta_{\mathrm{sink}(T)}$ in $P_{\zeta_{\mathrm{sink}(T)}}\widetilde{\mathbf{T}}_\tau$. But $\mathfrak{S}_T$ is generated by transpositions s_i such that $\zeta_{\mathrm{sink}(T)}[i] = qs^{m+1} \not\sim \zeta_{\mathrm{sink}(T)}[i+1] = q^n s^m$ (see eq. (5.5)). This implies $P_{\zeta_{\mathrm{sink}(T)}}s_i = sP_{\zeta_{\mathrm{sink}(T)}}$. Hence, $P_{\zeta_{\mathrm{sink}(T)}}\tau = s^{\ell(\tau)}P_{\zeta_{\mathrm{sink}(T)}}$. Since, from Lemma 5.35, $\ell(\tau) = \ell(\sigma) - \ell(\sigma_T)$, we obtain the desired result. □

Proposition 5.38. *The coefficient of $P_{\zeta_{\mathrm{sink}(T)}}$ in $P_{\zeta_{\mathrm{root}(T)}}\mathbf{S}_N$ equals the Poincaré polynomial $\phi_T(s)$ of $\mathfrak{S}_T$.*

Proof. We write

$$P_{\zeta_{\mathrm{root}(T)}}\mathbf{S}_N = P_{\zeta_{\mathrm{root}(T)}}\sum_{\sigma\in\mathfrak{S}_T}\widetilde{\mathbf{T}}_{\sigma_T}\widetilde{\mathbf{T}}_\sigma + P_{\zeta_{\mathrm{root}(T)}}\sum_{\ell(\sigma_T\sigma)<\ell(\sigma_T)+\ell(\sigma)}\widetilde{\mathbf{T}}_\sigma.$$

By Lemma 5.37, the coefficient of $P_{\zeta_{\mathrm{sink}(T)}}$ in

$$P_{\zeta_{\mathrm{root}(T)}}\sum_{\ell(\sigma_T\sigma)<\ell(\sigma_T)+\ell(\sigma)}\widetilde{\mathbf{T}}_\sigma = P_{\zeta_{\mathrm{root}(T)}}\sum_{\sigma_T^{-1}\sigma\notin\mathfrak{S}_T}\widetilde{\mathbf{T}}_\sigma$$

is 0. Furthermore, Lemma 5.37 implies

$$P_{\zeta_{\mathrm{root}(T)}}\sum_{\sigma\in\mathfrak{S}_T}\widetilde{\mathbf{T}}_{\sigma_T}\widetilde{\mathbf{T}}_\sigma = P_{\zeta_{\mathrm{sink}(T)}}\sum_{\sigma\in\mathfrak{S}_T}\widetilde{\mathbf{T}}_\sigma + \sum_{\zeta\prec\zeta_{\mathrm{sink}(T)}}(*)P_\zeta\widetilde{\mathbf{T}}_\sigma.$$

But, since $\zeta \neq \zeta_{\mathrm{sink}(T)}$, the coefficient of $P_{\zeta_{\mathrm{sink}(T)}}$ in $P_\zeta\widetilde{\mathbf{T}}_\sigma$ is 0. Hence, the coefficient of $P_{\zeta_{\mathrm{sink}(T)}}$ in $P_{\zeta_{\mathrm{root}(T)}}\mathbf{S}_N$ equals the coefficient of $P_{\zeta_{\mathrm{sink}(T)}}$ in $P_{\zeta_{\mathrm{sink}(T)}}\sum_{\sigma\in\mathfrak{S}_T}\widetilde{\mathbf{T}}_\sigma$. The result now follows from Lemma 5.37. □

The polynomial $\mathfrak{M}_T$ is proportional to any $P_\zeta\mathbf{S}_N$ for $\zeta \in T$. In fact, we can compute the proportion factor.

Theorem 5.39. *We have*

$$\mathfrak{M}_T = \frac{b_{\zeta_{\mathrm{sink}(T)}}}{\phi_T(s)}P_{\zeta_{\mathrm{root}(T)}}\mathbf{S}_N$$

.

Proof. It suffices to compare the coefficient of $P_{\zeta_{\mathrm{sink}(T)}}$ in $\mathfrak{M}_T$ (given by Theorem 5.27) and in $P_\zeta \mathbf{S}_N$ (given by Proposition 5.38). $\square$

Example 5.40. Consider the tableau $T = \begin{matrix} 1 & & \\ 0 & 0 & 1 \end{matrix}$. Here, $\zeta_{\mathrm{root}(T)} = [s, 1, qs^2, qs^{-1}]$ and $\zeta_{\mathrm{sink}(T)} = [qs^{-1}, qs^2, s, 1]$. The images of $\zeta_{\mathrm{root}(T)}$ by

$$\begin{array}{rcllllll} \mathfrak{S}_4 & = & \{[1,2,3,4], & [1,2,4,3], & [1,3,2,4], & [1,3,4,2], & [1,4,2,3], & [1,4,3,2], \\ & & [2,1,3,4], & [2,1,4,3], & [2,3,1,4], & [2,3,4,1], & [2,4,1,3], & [2,4,3,1], \\ & & [3,1,2,4], & [3,1,4,2], & [3,2,1,4], & [3,2,4,1], & [3,4,2,3], & [3,4,2,1] \\ & & [4,1,2,3], & [4,1,3,2], & [4,2,1,3], & [4,2,3,1], & [4,3,1,2], & [4,3,2,1]\} \end{array}$$

are

$$\begin{array}{llll} [s,1,qs^2,qs^{-1}], & [s,1,qs^{-1},qs^2], & [s,qs^{-1},1,qs^2], & [s,qs^{-1},qs^2,1], \\ [s,qs^2,1,qs^{-1}], & [s,qs^2,qs^{-1},1], & [s,1,qs^2,qs^{-1}], & [s,1,qs^{-1},qs^2], \\ [s,qs^{-1},1,qs^2], & [s,qs^{-1},qs^2,1], & [s,qs^2,1,qs^{-1}], & [s,qs^2,qs^{-1},1], \\ [qs^2,s,1,qs^{-1}], & [qs^2,s,qs^{-1},1], & [qs^2,s,1,qs^{-1}], & [qs^2,s,qs^{-1},1], \\ [qs^2,qs^{-1},s,1], & [qs^2,qs^{-1},s,1], & [qs^{-1},s,1,qs^2], & [qs^{-1},s,qs^2,1], \\ [qs^{-1},s,1,qs^2], & [qs^{-1},s,qs^2,1], & [qs^{-1},qs^2,s,1], & [qs^{-1},qs^2,s,1], \end{array}$$

respectively. Only two permutations give $\zeta_{\mathrm{sink}(T)}$: $[4,3,1,2]$ and $[4,3,2,1]$. Indeed, one computes σ_T by choosing a maximal path in the Yang–Baxter graph: $\sigma_T = s_2 s_3 s_1 s_2 s_1 = [4,3,1,2]$. The group $\mathfrak{S}_T$ is the order-two group $\mathfrak{S}_T = \mathfrak{S}_{\{3,4\}}$. We see that acting by $\mathbf{T}_3$ on $P_{[qs^{-1},qs^2,s,1]}$ gives $sP_{[qs^{-1},qs^2,s,1]}$. Hence,

$$P_{[qs^{-1},qs^2,s,1]}(1 + \mathbf{T}_3) = (1+s)P_{[qs^{-1},qs^2,s,1]} = \phi_T(s)P_{[qs^{-1},qs^2,s,1]}.$$

Note that $\phi_T(s)$ is the product of the $\phi_\lambda(s)$ for each row $\lambda = [a_1^{m_1}, \dots, a_k^{m_k}]$ of T, where $\phi_\lambda(s) = \prod_i \phi_{m_1}(s)$.

In the same way, we prove a similar formula for antisymmetric polynomials.

Theorem 5.41. *We have*

$$\mathfrak{M}_T^a = \frac{b^a_{\zeta_{\mathrm{sink}(T)}}}{\phi_{\overline{T}}(s)} P_{\zeta_{\mathrm{root}(T)}} \mathbf{A}_N,$$

where $\overline{T}$ denotes the conjugate of T (that is, the tableau obtained by exchanging rows and columns).

Proof. Similarly to Lemma 5.37, we denote by $\overline{\beta}_T^\sigma$ the coefficient of $P_{\zeta_{\mathrm{sink}(T)}}$ in $P_{\zeta_{\mathrm{root}(T)}} \overline{\mathbf{T}}_\sigma$. We then have the following:

(1) If $\sigma_{\overline{T}}^{-1}\sigma \notin \mathfrak{S}_{\overline{T}}$, then $\overline{\beta}_T^\sigma = 0$.
(2) If $\sigma_{\overline{T}}^{-1}\sigma \in \mathfrak{S}_{\overline{T}}$, then $\overline{\beta}_T^\sigma = (-1)^{\ell(\sigma) - \ell(\sigma_{\tilde{T}})}$.

Using these properties, we prove as in Proposition 5.38 that the coefficient of $P_{\zeta_{\mathrm{sink}(T)}}$ in $P_{\zeta_{\mathrm{root}(T)}} \mathbf{A}_N$ equals the Poincaré polynomial $\phi_{\overline{T}}(s)$. The result follows. $\square$

Example 5.42. Consider the tableau $T = \begin{array}{ll} 1 & \\ 0 & \\ 0 & 1 \end{array}$. Here, $\zeta_{\mathrm{root}(T)} = [s^{-1}, 1, qs, qs^{-2}]$ and $\zeta_{\mathrm{sink}(T)} = [qs^{-2}, qs, s^{-1}, 1]$. The images of $\zeta_{\mathrm{root}(T)}$ by $\mathfrak{S}_4$ are:

$$\begin{array}{llll}
[s^{-1}, 1, sq, qs^{-2}], & [s^{-1}, 1, qs^{-2}, sq], & [s^{-1}, sq, 1, qs^{-2}], & [s^{-1}, sq, qs^{-2}, 1],\\
[s^{-1}, qs^{-2}, 1, sq], & [s^{-1}, qs^{-2}, sq, 1], & [s^{-1}, 1, sq, qs^{-2}], & [s^{-1}, 1, qs^{-2}, sq],\\
[s^{-1}, sq, 1, qs^{-2}], & [s^{-1}, sq, qs^{-2}, 1], & [s^{-1}, qs^{-2}, 1, sq], & [s^{-1}, qs^{-2}, sq, 1],\\
[sq, s^{-1}, 1, qs^{-2}], & [sq, s^{-1}, qs^{-2}, 1], & [sq, s^{-1}, 1, qs^{-2}], & [sq, s^{-1}, qs^{-2}, 1],\\
[sq, qs^{-2}, s^{-1}, 1], & [sq, qs^{-2}, s^{-1}, 1], & [qs^{-2}, s^{-1}, 1, sq], & [qs^{-2}, s^{-1}, sq, 1],\\
[qs^{-2}, s^{-1}, 1, sq], & [qs^{-2}, s^{-1}, sq, 1], & [qs^{-2}, sq, s^{-1}, 1], & [qs^{-2}, sq, s^{-1}, 1].
\end{array}$$

Only two permutations give $\zeta_{\mathrm{sink}(T)}$: $[4, 3, 1, 2]$ and $[4, 3, 2, 1]$. These permutations generate $\mathfrak{S}_{\overline{T}}$ with $\overline{T} = \begin{array}{lll} 1 & & \\ 0 & 0 & 1 \end{array}$.

5.6. **Minimal symmetric/antisymmetric polynomials.** We have seen that for a given isotype λ the symmetric polynomials are indexed by column-strict tableaux T of shape λ. There exists only one tableau filling of λ such that the sum of its entries is minimal. This tableau is obtained by filling the first row with 0's, the second with 1's, etc. Let

$$T_\lambda := \begin{array}{lllll} m-1 & \dots & m-1 & & \\ \vdots & & \vdots & & \\ 1 & \dots & \dots & 1 & \\ 0 & \dots & \dots & \dots & 0 \end{array}$$

if $\lambda = [\lambda_1, \dots, \lambda_m]$, with $\lambda_1 \geq \cdots \geq \lambda_m$, and the number of i's among the entries of T_λ equals λ_i.

Example 5.43. Let $\lambda = [5, 3, 2, 2, 1]$, then

$$T_\lambda = \begin{array}{lllll} 4 & & & & \\ 3 & 3 & & & \\ 2 & 2 & & & \\ 1 & 1 & 1 & & \\ 0 & 0 & 0 & 0 & 0 \end{array}.$$

Corollary 5.44. *The space of the minimal symmetric polynomials for isotype λ is spanned by $\mathfrak{M}_{T_\lambda}$, and similarly the space of minimal antisymmetric polynomials is spanned by $\mathfrak{M}^a_{T_{\overline{\lambda}}}$, where $\overline{\lambda}$ denotes the conjugate partition of λ.*

Example 5.45. Consider the isotype $\lambda = [5, 3, 2, 2, 1]$. Then $\overline{\lambda} = [5, 4, 2, 1, 1]$ and

$$T_{\overline{\lambda}} = \begin{array}{lllll} 4 & & & & \\ 3 & & & & \\ 2 & 2 & & & \\ 1 & 1 & 1 & 1 & \\ 0 & 0 & 0 & 0 & 0 \end{array}.$$

Hence, the space of minimal antisymmetric polynomials for isotype λ is spanned by

$$\mathfrak{M}^a_{\begin{array}{lllll} 0 & & & & \\ 0 & 1 & & & \\ 0 & 1 & & & \\ 0 & 1 & 2 & & \\ 0 & 1 & 2 & 3 & 4 \end{array}}.$$

6. Bilinear form

6.1. **Bilinear form on the space V_λ.** To define a pairing for V_λ, introduce the dual Hecke algebra $\mathcal{H}_N\left(q^{-1}, s^{-1}\right)$; we use $*$ to indicate objects associated with $\mathcal{H}_N\left(q^{-1,} s^{-1}\right)$, e.g., $\mathbf{T}_i^*$, $\left(c_0 + c_1 s\right)^* = c_0 + \frac{c_1}{s}$. Recall that, when acting on V_λ, $\mathbf{T}_i = T_i$. There is a bilinear form on $V_\lambda^* \times V_\lambda$, $(u^*, v) \mapsto \langle u^*, v\rangle \in \mathbb{Q}(s, q)$, such that $\langle u^* T_i^*, v T_i\rangle = \langle u^*, v\rangle$ for $1 \leq i < N$, and such that $\mathbb{T}_1, \mathbb{T}_2 \in \mathrm{Tab}_\lambda, \mathbb{T}_1 \neq \mathbb{T}_2$ implies $\langle \mathbb{T}_1^*, \mathbb{T}_2\rangle = 0$; the latter property follows from the eigenvalues of L_i, since $\langle u^* \phi_i^*, v\phi_i\rangle = \langle u^*, v\rangle$. We establish a formula for $\langle \mathbb{T}^*, \mathbb{T}\rangle$ after the following recurrence relation.

Lemma 6.1. *If* $\mathbb{T} \in \mathrm{Tab}_\lambda$ *and* $m := \mathrm{CT}_{\mathbb{T}}[i] - \mathrm{CT}_{\mathbb{T}}[i+1]) \geq 2$ *then* $\mathbb{T}^{(i,i+1)} \in \mathrm{Tab}_\lambda$ *and*

$$\left\langle (\mathbb{T}^{(i,i+1)})^*, \mathbb{T}^{(i,i+1)}\right\rangle = \frac{\left(1 - s^{m-1}\right)\left(1 - s^{m+1}\right)}{\left(1 - s^m\right)^2} \left\langle \mathbb{T}^*, \mathbb{T}\right\rangle.$$

Proof. The equation $\mathbb{T} T_i = \mathbb{T}^{(i,i+1)} - \frac{1-s}{1-s^{-m}} \mathbb{T}$ implies

$$\left\langle \mathbb{T}^*, \mathbb{T}\right\rangle = \left\langle \mathbb{T}^{(i,i+1)*}, \mathbb{T}^{(i,i+1)}\right\rangle + \frac{\left(1 - s^{-1}\right)\left(1 - s\right)}{\left(1 - s^m\right)\left(1 - s^{-m}\right)} \left\langle \mathbb{T}^*, \mathbb{T}\right\rangle,$$

thus

$$\left\langle \mathbb{T}^{(i,i+1)*}, \mathbb{T}^{(i,i+1)}\right\rangle = \left(1 - \frac{s^{m-1}\left(1 - s\right)^2}{\left(1 - s^m\right)^2}\right) \left\langle \mathbb{T}^*, \mathbb{T}\right\rangle.$$

∎

Definition 6.2. For $\mathbb{T} \in \mathrm{Tab}_\lambda$ let

$$\nu\left(\mathbb{T}\right) := \prod_{\substack{1 \leq i < j \leq N \\ \mathrm{CT}_{\mathbb{T}}[i] - \mathrm{CT}_{\mathbb{T}}[j] \leq -2}} \frac{\left(1 - s^{\mathrm{CT}_{\mathbb{T}}[j] - \mathrm{CT}_{\mathbb{T}}[i] - 1}\right)\left(1 - s^{\mathrm{CT}_{\mathbb{T}}[j] - \mathrm{CT}_{\mathbb{T}}[i] + 1}\right)}{\left(1 - s^{\mathrm{CT}_{\mathbb{T}}[j] - \mathrm{CT}_{\mathbb{T}}[i]}\right)^2}.$$

Proposition 6.3. *The bilinear form defined by* $\langle \mathbb{T}_1^*, \mathbb{T}_2\rangle = 0$ *for* $\mathbb{T}_1 \neq \mathbb{T}_2$ *and* $\langle \mathbb{T}^*, \mathbb{T}\rangle = \nu\left(\mathbb{T}\right)$ *(for* $\mathbb{T}, \mathbb{T}_1, \mathbb{T}_2$*) and extended by linearity satisfies* $\langle P^* T_i^*, Q T_i\rangle = \langle P^*, Q\rangle$ *for all* P^*, Q, i.

Proof. It suffices to show that $\langle \mathbb{T}^* T_i^*, \mathbb{T} T_i\rangle = \langle \mathbb{T}^*, \mathbb{T}\rangle$ for all $\mathbb{T}$. If $\mathbb{T} T_i = s\mathbb{T}$ then $\mathbb{T}^* T_i^* = s^{-1}\mathbb{T}^*$ and $\langle \mathbb{T}^* T_i^*, \mathbb{T} T_i\rangle = s^{-1} s \langle \mathbb{T}^*, \mathbb{T}\rangle$. The case $\mathbb{T} T_i = -\mathbb{T}$ is treated similarly. Otherwise, consider the pair $\left(\mathbb{T}, \mathbb{T}^{(i,i+1)}\right)$ with $\mathrm{CT}_{\mathbb{T}}[i] - \mathrm{CT}_{\mathbb{T}}[i+1] \geq 2$. There is only one factor in $\nu\left(\mathbb{T}^{(i,i+1)}\right)$ different from $\nu\left(\mathbb{T}\right)$, the one corresponding to $j = i+1$. The proof then follows from Lemma 6.1 and $\mathrm{CT}_{\mathbb{T}^{(i,i+1)}}[i] = \mathrm{CT}_{\mathbb{T}}[i+1]$, $\mathrm{CT}_{\mathbb{T}^{(i,i+1)}}[i+1] = \mathrm{CT}_{\mathbb{T}}[i]$. ∎

Any other bilinear form satisfying $\langle P^* T_i^*, Q T_i\rangle = \langle P^*, Q\rangle$ is a constant multiple of the above form.

6.2. **Bilinear form on the space $\mathcal{M}_\lambda$.** Consider the bilinear form $\langle\, , \,\rangle$ defined by

$$\langle \mathbb{T}_1^*, \mathbb{T}_2\rangle = \delta_{\mathbb{T}_2,\mathbb{T}_2}\nu(\mathbb{T}_1) \tag{6.1}$$

and

$$\langle Px_i, Q\rangle = \langle P, Q\mathbf{D}_i\rangle. \tag{6.2}$$

This form has the following property.

Proposition 6.4. *We have*

$$\langle P(\mathbf{T}_i^*)^{\pm 1}, Q\rangle = \langle P, Q\mathbf{T}_i^{\mp 1}\rangle.$$

Proof. We proceed by induction on the degree of the polynomials. The start of the induction is given by the inner product on the tableaux. Using the induction hypothesis, we have from eq. (3.8) and Proposition 3.6

$$\langle Px_i\mathbf{T}_i^*, Q\rangle = \langle Px_i, Q\mathbf{T}_i^{-1}\rangle, \tag{6.3}$$

$$\langle Px_{i+1}(\mathbf{T}_i^*)^{-1}, Q\rangle = \langle Px_{i+1}, Q\mathbf{T}_i\rangle, \tag{6.4}$$

and

(6.5)
$$\langle Px_j(\mathbf{T}_i^*)^{\pm 1}, Q\rangle = \langle Px_j, Q\mathbf{T}_i^{\mp 1}\rangle = \langle P, Q\mathbf{T}_i^{\mp 1}\mathbf{D}_j\rangle = \langle P, Q\mathbf{D}_j\mathbf{T}_i^{\mp 1}\rangle \text{ when } |i-j|>1.$$

Indeed, one has

$$\langle Px_{i+1}(\mathbf{T}_i^*)^{-1}, Q\rangle = \frac{1}{s}\langle P\mathbf{T}_i^{*-1}x_{i+1}, Q\rangle = \frac{1}{s}\langle P\mathbf{T}_i^{*-1}, Q\mathbf{D}_{i+1}\rangle = \frac{1}{s}\langle P, Q\mathbf{D}_{i+1}\mathbf{T}_i\rangle,$$

using the induction hypothesis. Hence,

$$\langle Px_{i+1}(\mathbf{T}_i^*)^{-1}, Q\rangle = \langle P, Q\mathbf{D}_{i+1}\mathbf{T}_i\rangle = \langle P, Q\mathbf{T}_i^{-1}\mathbf{D}_i\rangle = \langle Px_i, Q\mathbf{T}_i^{-1}\rangle$$

which gives (6.3). The proofs of (6.4) and (6.5) are similar.

Now, by Proposition 3.6, one has

$$\langle Px_{i+1}\mathbf{T}_i^*, Q\rangle = \langle P\mathbf{T}_i^*x_i - (1-\frac{1}{s})Px_{i+1}, Q\rangle = \langle P, Q(\mathbf{D}_i\mathbf{T}_i^{-1} - (1-\frac{1}{s})\mathbf{D}_{i+1})\rangle$$

by induction hypothesis. Hence, by (3.8), one obtains

$$\langle Px_{i+1}\mathbf{T}_i^*, Q\rangle = \langle P, Q\mathbf{T}_i^{-1}\mathbf{D}_{i+1}\rangle = \langle Px_{i+1}, Q\mathbf{T}_i^{-1}\rangle. \tag{6.6}$$

Similarly, one has

$$\langle Px_i\mathbf{T}_i^{*-1}, Q\rangle = \langle Px_i, Q\mathbf{T}_i\rangle. \tag{6.7}$$

Eqs. (6.3), (6.4),(6.5), (6.6) and (6.7) give the result. □

Now one has also the following equalities involving the operator $\mathbf{w}$:

$$\mathbf{D}_{i+1}\mathbf{w}^{-1} = \mathbf{w}^{-1}\mathbf{D}_i,\ x_{i+1}\mathbf{w} = \mathbf{w}x_{i+1}, \quad i \neq 1,$$

and

$$\mathbf{D}_N\mathbf{w}^{-1} = q\mathbf{w}^{-1}\mathbf{D}_1,\ x_1\mathbf{w} = q\mathbf{w}x_N.$$

This entails the following identity.

Proposition 6.5. *We have*

$$\langle P\mathbf{w}^{*\pm 1}, Q\rangle = \langle P, Q\mathbf{w}^{\mp 1}\rangle.$$

From Proposition 6.5 and 6.5 one deduces the following theorem.

Theorem 6.6. *We have*

(1) $\langle P\boldsymbol{\xi}_i^*, Q\rangle = \langle P, Q\boldsymbol{\xi}_i^{-1}\rangle$,
(2) $\langle P_\zeta^*, P_{\zeta'}\rangle = (*)\delta_{\zeta,\zeta'}$,

where $(*)$ *denotes a certain coefficient which remains to be computed.*

6.3. **Computation of $\langle P_\zeta^*, P_\zeta\rangle$.** First we establish some recurrences.

Proposition 6.7. *Let* $\zeta = \zeta_{v,\mathbb{T}}$ *for some* $\mathbb{T} \in \mathrm{Tab}_\lambda$ *and* $v \in \mathbb{N}^N$. *Suppose that* $\zeta[i+1] \succ \zeta[i]$ *for some* i. *Then*

$$\left\langle P_{\zeta s_i}^*, P_{\zeta s_i}\right\rangle = \frac{\left(1 - s\frac{\zeta[i+1]}{\zeta[i]}\right)\left(s - \frac{\zeta[i+1]}{\zeta[i]}\right)}{s\left(1 - \frac{\zeta[i+1]}{\zeta[i]}\right)^2}\left\langle P_\zeta^*, P_\zeta\right\rangle.$$

Proof. From eq. (4.2), we infer $P_\zeta \mathbf{T}_i = -\frac{1-s}{1-\frac{\zeta[i+1]}{\zeta[i]}}P_\zeta + P_{\zeta s_i}$. Thus

$$\begin{aligned}\left\langle P_\zeta^*, P_\zeta\right\rangle &= \left\langle P_\zeta^*\mathbf{T}_i^*, P_\zeta\mathbf{T}_i\right\rangle \\ &= \left(\frac{1-s}{1-\frac{\zeta[i+1]}{\zeta[i]}}\right)\left(\frac{1-s}{1-\frac{\zeta[i+1]}{\zeta[i]}}\right)^*\left\langle P_\zeta^*, P_\zeta\right\rangle + \left\langle P_{\zeta s_i}^*, P_{\zeta s_i}\right\rangle.\end{aligned}$$

Hence

$$\begin{aligned}\left\langle P_{\zeta s_i}^*, P_{\zeta s_i}\right\rangle &= \left(1 - \frac{(1-s)\left(1-s^{-1}\right)}{\left(1 - \frac{\zeta[i+1]}{\zeta[i]}\right)\left(1 - \frac{\zeta[i]}{\zeta[i+1]}\right)}\right)\left\langle P_\zeta^*, P_\zeta\right\rangle \\ &= \frac{\left(1 - s\frac{\zeta[i+1]}{\zeta[i]}\right)\left(s - \frac{\zeta[i+1]}{\zeta[i]}\right)}{s\left(1 - \frac{\zeta[i+1]}{\zeta[i]}\right)^2}\left\langle P_\zeta^*, P_\zeta\right\rangle.\end{aligned}$$

□

Definition 6.8. We define

$$\mathcal{E}_a(\zeta) = \prod_{(i,j)\in\mathrm{inv}(\zeta)} \frac{1 - s^a\frac{\zeta[j]}{\zeta[i]}}{1 - \frac{\zeta[j]}{\zeta[i]}}$$

and

$$\mathcal{E}(\zeta) = \mathcal{E}_1(\zeta)\mathcal{E}_{-1}(\zeta).$$

Proposition 6.9. *Let* $\zeta = \zeta_{v,\mathbb{T}}$ *for some* $v \in \mathbb{N}^N$ *and* $\mathbb{T} \in \mathrm{Tab}_\lambda$. *One has*

$$\left\langle P_\zeta^*, P_\zeta\right\rangle = \mathcal{E}(\zeta)^{-1}\left\langle P_{\zeta^+}^*, P_{\zeta^+}\right\rangle.$$

Proof. we argue by induction on $\#\mathrm{inv}\,(\zeta)$. The statement is trivially true for $\#\mathrm{inv}\,(\zeta) = 0$, that is, if $\zeta = \zeta^+$. Suppose the statement is true for all $\zeta' = \zeta_{v',\mathbb{T}'}$ with $\#\mathrm{inv}\,(\zeta') \leq n$ and $\#\mathrm{inv}\,(\zeta) = n+1$. Thus $\zeta[i] < \zeta[i+1]$ for some $i < N$. By Proposition 6.7, we have

$$\left\langle P_\zeta^*, P_\zeta \right\rangle = \frac{\left(1 - \frac{\zeta[i+1]}{\zeta[i]}\right)^2}{\left(1 - s\frac{\zeta[i+1]}{\zeta[i]}\right)\left(1 - s^{-1}\frac{\zeta[i+1]}{\zeta[i]}\right)} \left\langle P_{\zeta s_i}^*, P_{\zeta s_i} \right\rangle,$$

thus

$$\frac{\left\langle P_\zeta^*, P_\zeta \right\rangle}{\left\langle P_{\zeta s_i}^*, P_{\zeta \cdot s_i} \right\rangle} = \frac{\mathcal{E}\left(\zeta \cdot s_i\right)}{\mathcal{E}\left(\zeta\right)}.$$

This completes the induction since $\#\mathrm{inv}\,(\zeta s_i) = \#\mathrm{inv}\,(\zeta) - 1$. $\square$

Alternatively, the computation of $\left\langle P_\zeta^*, P_\zeta \right\rangle$ can be related to the root or the sink of the connected component of ζ.

Proposition 6.10. *Let $\zeta = \zeta_{v,\mathbb{T}}$ for some v and $\mathbb{T}$. Let $H_T^{q,s}$ be the connected component of ζ. We define the values*

$$\mathcal{S}(\zeta) = \prod_{(i,j)\in\mathrm{inv}_\prec(\zeta)} \frac{(1 - s\frac{\zeta[j]}{\zeta[i]})(1 - s^{-1}\frac{\zeta[j]}{\zeta[i]})}{(1 - \frac{\zeta[j]}{\zeta[i]})^2}$$

and

$$\mathcal{R}(\zeta) = \prod_{(i,j)\in\mathrm{inv}_\succ(\zeta)} \frac{(1 - s\frac{\zeta[j]}{\zeta[i]})(1 - s^{-1}\frac{\zeta[j]}{\zeta[i]})}{(1 - \frac{\zeta[j]}{\zeta[i]})^2}.$$

Then one has

(1) $\left\langle P_\zeta^*, P_\zeta \right\rangle = \mathcal{S}(\zeta)^{-1} \left\langle P^*_{\zeta_{\mathrm{sink}(T)}}, P_{\zeta_{\mathrm{sink}(T)}} \right\rangle$,
(2) $\left\langle P_\zeta^*, P_\zeta \right\rangle = \mathcal{R}(\zeta) \left\langle P^*_{\zeta_{\mathrm{root}(T)}}, P_{\zeta_{\mathrm{root}(T)}} \right\rangle$.

Proof. The proof works as in Proposition 6.9, using induction on $\sharp\mathrm{inv}_\triangleleft(\zeta)$ ($\triangleleft \in \{\prec, \succ\}$), since there is a unique maximal (respectively minimal) element in the connected component: the sink (respectively the root). These elements are connected to ζ by a sequence of steps or jumps. $\square$

There holds as well the following identity.

Proposition 6.11. *Let $\zeta = \zeta_{v,\mathbb{T}}$ for some $v \in \mathbb{N}^N$ and $\mathbb{T} \in \mathrm{Tab}_\lambda$. Then one has*

$$\langle P^*_{\zeta\Psi^q}, P_{\zeta\Psi^q} \rangle = (1 - q\zeta[1])\langle P_\zeta^*, P_\zeta \rangle.$$

Proof. From Proposition 4.19 one has

$$\langle P^*_{\zeta\Psi^q}, P_{\zeta\Psi^q} \rangle = \langle P_\zeta^* \mathbf{\Phi}^*, P_\zeta \mathbf{\Phi} \rangle = \langle P_\zeta^* \left(\mathbf{T}_1^{-1} \cdots \mathbf{T}_{N-1}^{-1}\right)^* x_N, P_\zeta \mathbf{T}_1^{-1} \cdots \mathbf{T}_{N-1}^{-1} x_N \rangle.$$

But Proposition 6.4 implies

$$\begin{aligned}\langle P_\zeta^* &\left(\mathbf{T}_1^{-1} \cdots \mathbf{T}_{N-1}^{-1}\right)^* x_N, P_\zeta \mathbf{T}_1^{-1} \cdots \mathbf{T}_{N-1}^{-1} x_N \rangle \\ &= \langle P_\zeta^*, P_\zeta \mathbf{T}_1^{-1} \cdots \mathbf{T}_{N-1}^{-1} x_N \mathbf{D}_N \mathbf{T}_{N-1} \cdots \mathbf{T}_1 \rangle,\end{aligned}$$

and, by $\mathbf{D}_N = (1-\boldsymbol{\xi}_N)x_N^{-1}$, we obtain

$$\begin{aligned}\langle P^*_{\zeta\Psi^q}, P_{\zeta\Psi^q}\rangle &= \langle P^*_\zeta, P_\zeta\rangle - \langle P^*_\zeta, P_{\zeta\Psi^q}\boldsymbol{\xi}_N x_N^{-1}\mathbf{T}_{N-1}\cdots\mathbf{T}_1\rangle \\ &= \langle P^*_\zeta, P_\zeta\rangle - (\zeta\Psi^q)[N]\langle P^*_\zeta, P_{\zeta\Psi^q}\boldsymbol{\Phi}^{-1}\rangle.\end{aligned}$$

Using again Proposition 4.19, we find

$$\langle P^*_{\zeta\Psi^q}, P_{\zeta\Psi^q}\rangle = (1-(\zeta\Psi^q)[N])\langle P^*_\zeta, P_\zeta\rangle.$$

Since $(\zeta\Psi^q)[N] = q\zeta[1]$, we arrive at the desired result. $\square$

Definition 6.12. We let $\chi(i,j)=1$ if $j<i$ and $\chi(i,j)=0$ if $j\geq i$.
Let $\rho(a,b) = \frac{(a-s^{-1}b)(a-sb)}{(a-b)^2}$ and

$$\triangle(\zeta) := \prod_{j=1}^N \prod_{\substack{\zeta[i]\succ\zeta[j]q^k \\ k\geq\chi(i,j)}} \rho(\zeta[j]q^k, \zeta[i]).$$

Let $\square(q^n s^m) = (q; qs^m)_n$, where $(a;q)_n = (1-a)(1-qa)\cdots(1-q^{n-1}a)$, and $\square(\zeta) = \prod_{i=1}^N \square(\zeta[i])$.

Example 6.13. Let $\zeta := [q^2s^{-1}, qs^2, qs, q]$ be the spectral vector associated with $v = [2,1,1,1]$ and $\mathbb{T} = \begin{matrix}1 & & \\ 4 & 3 & 2\end{matrix}$. Then we have

$$\square(\zeta) = \square(q^2s^{-1})\square(qs^2)\square(qs)\square(q) = (1-qs^{-1})(1-q^2s^{-1})(1-qs^2)(1-qs)(1-q).$$

With the aim of computing $\triangle(\zeta)$, we list the triples (i,j,k) such that $\zeta[i]\succ\zeta[j]q^k$. Here we find 6 triples:

$$(1,2,0),\ (1,2,1),\ (1,3,0),\ (1,3,1),\ (1,4,0),\ (4,2,0).$$

Note that $(1,4,1)$ does not occur in the list since $q^2s^{-1}\not\succ q^2$. Furthermore, there is no factor corresponding to $(4,2,0)$ in $\triangle(\zeta)$ because $\chi(4,2)=1$. Hence, $\triangle(\zeta)$ is a product of 5 factors:

$$\begin{aligned}\triangle(\zeta) &= \rho(qs^2, q^2s^{-1})\rho(q^2s^2, q^2s^{-1})\rho(qs, q^2s^{-1})\rho(q^2s, q^2s^{-1})\rho(q, q^2s^{-1}) \\ &= \frac{(q-s^4)(s^2+1)(-1+q)}{(-s^3+q)(s^2+1+s)(q-s)}\end{aligned}$$

With these notations, one has the following auxiliary result.

Lemma 6.14. (1) *If $\zeta = \zeta_{0^N,\mathbb{T}}$ then $\triangle(\zeta) = \nu(\mathbb{T})$ and $\square(\zeta)=1$.*
(2) *If $\zeta = \zeta_{v,\mathbb{T}}$ with $\zeta[\ell+1]\succ\zeta[\ell]$ then $\triangle(\zeta s_\ell) = \rho(\zeta[\ell],\zeta[\ell+1])\triangle(\zeta)$ and $\square(\zeta s_\ell) = \square(\zeta)$.*
(3) *If $\zeta = \zeta_{v,\mathbb{T}}$ then $\triangle(\zeta\Psi^q) = \triangle(\zeta)$ and $\square(\zeta\Psi^q) = (1-q\zeta[1])\square(\zeta)$.*

Proof. (1) First note that, if $\zeta = \zeta_{0^N,\mathbb{T}}$, then there is no occurrence of q in ζ, so we have $\square(\zeta) = 1$. Moreover, we have

$$\begin{aligned}\nu(\mathbb{T}) &= \prod_{\substack{1\leq i<j\leq N\\ \mathrm{CT}_{\mathbb{T}}[i]-\mathrm{CT}_{\mathbb{T}}[j]\leq -2}} \rho(\zeta[i],\zeta[j])\\ &= \prod_{j=1}^{N}\prod_{\zeta[i]\succ\zeta[j]} \rho(\zeta[j],\zeta[i]).\end{aligned}$$

(2) Obviously we have $\square(\zeta s_\ell) = \square(\zeta)$. Furthermore,

$$\begin{aligned}\frac{\triangle(\zeta s_\ell)}{\triangle(\zeta)} &= \frac{\displaystyle\prod_{\substack{\zeta s_\ell[\ell]\succ\zeta s_\ell[\ell+1]q^k\\ k\geq\chi(\ell,\ell+1)}} \rho(\zeta s_\ell[\ell+1]q^k,\zeta s_\ell[\ell])}{\displaystyle\prod_{\substack{\zeta[\ell+1]\succ\zeta[\ell]q^k\\ k\geq\chi(\ell+1,\ell)}} \rho(\zeta[\ell]q^k,\zeta[\ell+1])}\\ &= \frac{\displaystyle\prod_{\substack{\zeta[\ell+1]\succ\zeta[\ell]q^k\\ k\geq 0}} \rho(\zeta[\ell]q^k,\zeta[\ell+1])}{\displaystyle\prod_{\substack{\zeta[\ell+1]\succ\zeta[\ell]q^k\\ k\geq 1}} \rho(\zeta[\ell]q^k,\zeta[\ell+1])}\\ &= \rho(\zeta[\ell],\zeta[\ell+1]).\end{aligned}$$

This proves the result.

(3) One has $\square(\zeta\Psi^q) = (1-(\zeta\Psi^q)[N])\square(\zeta) = (1-q\zeta[1])\square(\zeta)$. Furthermore,

$$\frac{\triangle(\zeta\Psi^q)}{\triangle(\zeta)} = \prod_{i=1}^{N-1}\left[\frac{\displaystyle\prod_{\substack{(\zeta\Psi^q)[i]\succ\zeta\Psi^q[N]q^k\\ k\geq 0}} \rho((\zeta\Psi^q)[N]q^k,(\zeta\Psi^q)[i])}{\displaystyle\prod_{\substack{\zeta[i+1]\succ\zeta[1]q^k\\ k\geq 1}} \rho(\zeta[1]q^k,\zeta[i+1])}\right.$$
$$\left.\times\frac{\displaystyle\prod_{\substack{(\zeta\Psi^q)[N]\succ\zeta\Psi^q[i]q^k\\ k\geq 1}} \rho((\zeta\Psi^q)[i]q^k,(\zeta\Psi^q)[N])}{\displaystyle\prod_{\substack{\zeta[1]\succ\zeta[i+1]q^k\\ k\geq 0}} \rho(\zeta[i+1]q^k,\zeta[i])}\right].$$

But $(\zeta\Psi^q)[N] = q\zeta[1]$ and $(\zeta\Psi^q)[i] = \zeta[i+1]$. Hence, $(\zeta\Psi^q)[i]\succ(\zeta\Psi^q)[N]q^k$ for $k\geq 0$ implies $\zeta[i+1]\geq\zeta[1]q^{k+1}$. In the same way $(\zeta\Psi^q)[N]\succ\zeta[i]q^k$ for $k\geq 1$

implies $\zeta[1] \succ \zeta[i+1]q^{k-1}$. Hence, the quotient simplifies to

$$\frac{\triangle(\zeta\Psi^q)}{\triangle(\zeta)} = 1,$$

as expected.

$\square$

We deduce the following result.

Theorem 6.15. *Let $\zeta = \zeta_{v,\mathbb{T}})$. Then the value of the pairing $\langle P_\zeta^*, P_\zeta \rangle$ is*

$$\langle P_\zeta^*, P_\zeta \rangle = \square(\zeta)\triangle(\zeta).$$

Proof. Comparing the statement of Lemma 6.14 to Propositions 6.7, 6.11, and 6.3, we see that $\langle P_\zeta^*, P_\zeta \rangle$ and $\square(\zeta)\triangle(\zeta)$ satisfy the same recurrence rules and have the same values when $\zeta = \zeta_{0^N,\mathbb{T}}$. $\square$

6.4. **Computation of $\langle \mathfrak{M}_T^*, \mathfrak{M}_T \rangle$.** First observe that

$$\langle P, Q\mathbf{S}_N \rangle = \langle P{\mathbf{S}'_N}^*, Q \rangle. \tag{6.8}$$

We use Theorem 5.39 to write

$$\langle \mathfrak{M}_T^*, \mathfrak{M}_T \rangle = \frac{b_{\zeta_{\mathrm{sink}(T)}}}{\phi_T(s)} \langle \mathfrak{M}_T^*, P_{\zeta_{\mathrm{root}(T)}} \mathbf{S}_N \rangle.$$

Hence, by eq. (6.8), we have

$$\langle \mathfrak{M}_T^*, \mathfrak{M}_T \rangle = \frac{b_{\zeta_{\mathrm{sink}(T)}}}{\phi_T(s)} \langle \mathfrak{M}_T^* \mathbf{S'}_N^*, P_{\zeta_{\mathrm{root}(T)}} \rangle.$$

Since $\mathfrak{M}_T$ is symmetric, eq. (5.3) gives

$$\langle \mathfrak{M}_T^*, \mathfrak{M}_T \rangle = b_{\zeta_{\mathrm{sink}(T)}} \frac{\phi_N(s)}{\phi_T(s)} \langle \mathfrak{M}_T^*, P_{\zeta_{\mathrm{root}(T)}} \rangle.$$

Hence,

$$\langle \mathfrak{M}_T^*, \mathfrak{M}_T \rangle = \frac{\phi_N(s)}{\phi_T(s)} b_{\zeta_{\mathrm{sink}(T)}} b^*_{\zeta_{\mathrm{root}(T)}} \langle P^*_{\zeta_{\mathrm{root}(T)}}, P_{\zeta_{\mathrm{root}(T)}} \rangle.$$

Using the normalization described in Section 5.4, we have $b_{\zeta_{\mathrm{root}(T)}} = 1$.

Theorem 6.16. *We have*

$$\langle \mathfrak{M}_T^*, \mathfrak{M}_T \rangle = \frac{\phi_N(s)}{\phi_T(s)} b_{\zeta_{\mathrm{sink}(T)}} \langle P^*_{\zeta_{\mathrm{root}(T)}}, P_{\zeta_{\mathrm{root}(T)}} \rangle.$$

In the same way, for antisymmetric polynomials we obtain the following result.

Theorem 6.17. *We have*

$$\langle {\mathfrak{M}_T^a}^*, \mathfrak{M}_T^a \rangle = \frac{\phi_N(s)}{\phi_{\overline{T}}(s)} b^a_{\zeta_{\mathrm{sink}(T)}} \langle P^*_{\zeta_{\mathrm{root}(T)}}, P_{\zeta_{\mathrm{root}(T)}} \rangle.$$

Proof. The proof works as in the symmetric case, but it uses the operator $\mathbf{A}'_N$ with the property that

$$\langle P, Q\mathbf{A}_N\rangle = \langle P{\mathbf{A}'_N}^*, Q\rangle.$$

This operator is the antisymmetrizer

$$\mathbf{A}'_N = \sum_{\sigma\in\mathfrak{S}_N}(-s)^{\ell(T)}\widetilde{\mathbf{T}}_\sigma$$

satisfying

$${\mathbf{A}'_N}^2 = \phi_N(\frac{1}{s})\mathbf{A}'_N.$$

Hence, by a similar reasoning we arrive at the claimed result. □

6.5. **Hook-length type formula for minimal polynomials.** The topic of this section is simpler formulae for $\langle\mathfrak{M}^*_{T_\lambda}, \mathfrak{M}_{T_\lambda}\rangle$ for a decreasing partition λ in the situation where the entries of T are constant in each row. The formulae are then specialized to the minimal symmetric/antisymmetric polynomials. In this case they are expressions in terms of hook-lengths.

First consider a partition μ where $\mu = [\mu_1^{\lambda[m]}, \dots, \mu_m^{\lambda[1]}]$ with $\mu_1 > \cdots > \mu_m$. Let

$$\mathbb{T} = \begin{matrix} \lambda[m] & \dots & 1 & & \\ \lambda[m-1]+\lambda[m] & \dots & \dots & \lambda[m]+1 & \\ \vdots & & & \vdots & \\ \lambda[1]+\cdots+\lambda[m] & \dots & \dots & \dots & \lambda[2]+\cdots+\lambda[m]+1 \end{matrix}$$

be the RST obtained by filling the shape λ with $1,\dots,N(=\lambda[1]+\cdots+\lambda[N])$ row by row and

$$T = \begin{matrix} \mu_1 & \dots & \mu_1 & & \\ \vdots & & \vdots & & \\ \mu_{m-1} & \dots & \dots & \mu_{m-1} & \\ \mu_m & \dots & \dots & \dots & \mu_m \end{matrix}$$

be the column strict tableau obtained by filling the shape λ with the entries of μ row by row. Then $\mu = v_{\mathrm{sink}(T)}$ and $\mathbb{T} = \mathbb{T}_{\mathrm{sink}(T)}$. Hence,

$$\zeta_{\mathrm{sink}(T)} = [q^{\mu_1}s^{\lambda[m]-m}, \dots, q^{\mu_1}s^{1-m}, q^{\mu_2}s^{1-m+\lambda[m-1]}, \dots, q^{\mu_2}s^{2-m}, \dots, q^{\mu_m}s^{-1+\lambda[1]}, \dots, q^{\mu_m}]. \tag{6.9}$$

Example 6.18. Let $\lambda = [3,3,2]$ and $\mu = [3,3,2,2,2,1,1,1]$. We construct

$$\mathbb{T} = \begin{matrix} 2 & 1 & \\ 5 & 4 & 3 \\ 8 & 7 & 6 \end{matrix}$$

and

$$T = \begin{matrix} 3 & 3 & \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{matrix}.$$

Here, $\zeta_{\mathrm{sink}(T)} = [q^3s^{-1}, q^3s^{-2}, q^2s, q^2, q^2s^{-1}, qs^2, qs, q]$.

We have

$$\begin{aligned}\langle P^*_{\zeta_{\mathrm{root}(T)}}, P^*_{\zeta_{\mathrm{root}(T)}}\rangle &= \mathcal{S}(\zeta_{\mathrm{root}(T)})^{-1}\langle P^*_{\zeta_{\mathrm{sink}(T)}}, P_{\zeta_{\mathrm{sink}(T)}}\rangle \\ &= \mathcal{S}(\zeta_{\mathrm{root}(T)})^{-1}\triangle(\zeta_{\mathrm{sink}(T)})\square(\zeta_{\mathrm{sink}(T)}),\end{aligned}$$

where

$$\begin{aligned}\zeta_{\mathrm{root}(T)} = [q^{\mu_m}s^{-1+\lambda[1]}, \dots, q^{\mu_m}, \dots, q^{\mu_2}s^{1-m+\lambda[m-1]}, \dots, q^{\mu_2}s^{2-m},& \\ q^{\mu_1}s^{\lambda[m]-m}, \dots, q^{\mu_1}s^{1-m}].&\end{aligned} \tag{6.10}$$

By telescoping, we find

$$\mathcal{S}(\zeta_{\mathrm{root}(T)}) = \prod_{1\le i<j\le m}\frac{(1-q^{\mu_j-\mu_i}s^{j-i-\lambda[m-i+1]})(1-q^{\mu_j-\mu_i}s^{j-i+\lambda[m-j+1]})}{(1-q^{\mu_j-\mu_i}s^{j-i})(1-q^{\mu_j-\mu_i}s^{j-i+\lambda[m-j+1]-\lambda[m-i+1]})}. \tag{6.11}$$

First we compute $\triangle(\zeta_{\mathrm{sink}(T)})$, and following eq. (6.9) we write

$$\triangle(\zeta_{\mathrm{sink}(T)}) = \langle \mathbb{T}^*, \mathbb{T}\rangle\Diamond,$$

with

$$\Diamond := \prod_{1\le i<j\le m}\prod_{k=0}^{\mu_i-\mu_j-1}\prod_{a=1}^{\lambda[m-i+1]}\prod_{b=1}^{\lambda[m-j+1]}\frac{(1-q^{\mu_j-\mu_i+k}s^{j-i+b-a-1})(1-q^{\mu_j-\mu_i+k}s^{j-i+b-a+1})}{(1-q^{\mu_j-\mu_i+k}s^{j-i+b-a})^2}.$$

Indeed, $\langle P^*_{\zeta_{\mathrm{sink}(T)}}, P_{\zeta_{\mathrm{sink}(T)}}\rangle$ splits into two factors: the first factor $\langle \mathbb{T}^*, \mathbb{T}\rangle$ does not depend on q, all the factors of the second factor $\Diamond\square(\zeta_{\mathrm{sink}(T)})$ involve q. By telescoping, we have

$$\begin{aligned}&\prod_{b=1}^{\lambda[m-j+1]}\frac{(1-q^{\mu_j-\mu_i+k}s^{j-i+b-a-1})(1-q^{\mu_j-\mu_i+k}s^{j-i+b-a+1})}{(1-q^{\mu_j-\mu_i+k}s^{j-i+b-a})^2} \\ &\qquad = \frac{(1-q^{\mu_j-\mu_i+k}s^{j-i-a})(1-q^{\mu_j-\mu_i+k}s^{j-i+\lambda[m-j+1]-a+1})}{(1-q^{\mu_j-\mu_i+k}s^{j-i-a+1})(1-q^{\mu_j-\mu_i+k}s^{j-i+\lambda[m-j+1]-a})},\end{aligned} \tag{6.12}$$

$$\prod_{a=1}^{\lambda[m-i+1]}\frac{1-q^{\mu_j-\mu_i+k}s^{j-i-a}}{1-q^{\mu_j-\mu_i+k}s^{j-i-a+1}} = \frac{1-q^{\mu_j-\mu_i}s^{j-i-\lambda[m-i+1]}}{1-q^{\mu_j-\mu_i}s^{j-i}}, \tag{6.13}$$

and

$$\prod_{a=1}^{\lambda[m-i+1]}\frac{1-q^{\mu_j-\mu_i+k}s^{j-i+\lambda[m-j+1]-a+1}}{1-q^{\mu_j-\mu_i+k}s^{j-i+\lambda[m-j+1]-a}} = \frac{1-q^{\mu_j-\mu_i+k}s^{j-i+\lambda[m-j+1]}}{1-q^{\mu_j-\mu_i+k}s^{j-i+\lambda[m-j+1]-\lambda[m-i+1]}}. \tag{6.14}$$

So, equations (6.12), (6.13), and (6.14) give

$$\Diamond = \prod_{1\le i<j\le m}\prod_{k=0}^{\mu_i-\mu_j-1}\frac{(1-q^{\mu_j-\mu_i+k}s^{j-i-\lambda[m-i+1]})(1-q^{\mu_j-\mu_i+k}s^{j-i+\lambda[m-j+1]})}{(1-q^{\mu_j-\mu_i+k}s^{j-i})(1-q^{\mu_j-\mu_i+k}s^{j-i+\lambda[m-j+1]-\lambda[m-i+1]})}.$$

Note that, by eq. (6.11), we have

$$\frac{\Diamond}{\mathcal{S}(\zeta_{\mathrm{root}(T)})} = \prod_{1\le i<j\le m}\prod_{k=1}^{\mu_i-\mu_j-1}\frac{(1-q^{\mu_j-\mu_i+k}s^{j-i-\lambda[m-i+1]})(1-q^{\mu_j-\mu_i+k}s^{j-i+\lambda[m-j+1]})}{(1-q^{\mu_j-\mu_i+k}s^{j-i})(1-q^{\mu_j-\mu_i+k}s^{j-i+\lambda[m-j+1]-\lambda[m-i+1]})}$$
$$= \prod_{1\le i<j\le m}\frac{(qs^{i-j+\lambda[m-i+1]};q)_{\mu_i-\mu_j-1}(qs^{i-j-\lambda[m-j+1]};q)_{\mu_i-\mu_j-1}}{(qs^{i-j+\lambda[m-i+1]-\lambda[m-j+1]},q)_{\mu_i-\mu_j-1}(qs^{i-j},q)_{\mu_i-\mu_j-1}}.$$

Furthermore,

$$\Box(\zeta_{\mathrm{sink}(T)}) = \prod_{i=1}^{m}\prod_{j=1}^{\lambda[m-i+1]}(qs^{j-m+i-1};q)_{\mu_i}.$$

Hence,

$$\langle P^*_{\zeta_{\mathrm{root}(T)}},P_{\zeta_{\mathrm{root}(T)}}\rangle = \langle\mathbb{T}^*,\mathbb{T}\rangle\prod_{i=1}^{m}\prod_{j=1}^{\lambda[m-i+1]}(qs^{j-m+i-1};q)_{\mu_i}$$
$$\times\prod_{1\le i<j\le m}\frac{(qs^{i-j+\lambda[m-i+1]};q)_{\mu_i-\mu_j-1}(qs^{i-j-\lambda[m-j+1]};q)_{\mu_i-\mu_j-1}}{(qs^{i-j+\lambda[m-i+1]-\lambda[m-j+1]},q)_{\mu_i-\mu_j-1}(qs^{i-j},q)_{\mu_i-\mu_j-1}}. \tag{6.15}$$

We find also

$$b_{\zeta_{\mathrm{sink}(T)}} = \prod_{1\le i<j\le m}\prod_{a=1}^{\lambda[m-i+1]}\frac{1-q^{\mu_j-\mu_i}s^{j-i+1-a}}{1-q^{\mu_j-\mu_i}s^{\lambda[m-j+1]-i+j+1-a}}.$$

Now we specialize $\mu = m-i$. The tableau T then becomes

$$T = \begin{array}{lllll} m-1 & \dots & m-1 & & \\ \vdots & & \vdots & & \\ 1 & \dots & \dots & 1 & \\ 0 & \dots & \dots & \dots & 0 \end{array}.$$

For convenience, consider the normalization

$$\widetilde{\mathfrak{M}}_T := b^{-1}_{\zeta_{\mathrm{sink}(T)}}\mathfrak{M}_T.$$

Furthermore, we set $\nabla_\lambda := \frac{\phi_T(s)}{\phi_N(s)}\frac{\langle\widetilde{\mathfrak{M}}^*_T,\widetilde{\mathfrak{M}}_T\rangle}{\langle\mathbb{T}^*,\mathbb{T}\rangle}$. So, we have

$$\nabla_\lambda = \left(b^{-1}_{\mathrm{sink}(T)}\right)^*\frac{\langle P^*_{\zeta_{\mathrm{root}(T)}},P_{\zeta_{\mathrm{root}(T)}}}{\langle\mathbb{T}^*,\mathbb{T}\rangle}\rangle.$$

From equality (6.15), we obtain

$$\nabla_\lambda = \prod_{i=1}^{m} \prod_{j=1}^{\lambda[m-i+1]} (qs^{j-m+i-1};q)_{i-1} \times \prod_{1\le i<j\le m} \frac{(qs^{i-j+\lambda[m-i+1]};q)_{j-i-1}(qs^{i-j-\lambda[m-j+1]};q)_{j-i-1}}{(qs^{i-j+\lambda[m-i+1]-\lambda[m-j+1]},q)_{j-i-1}(qs^{i-j},q)_{j-i-1}} \times \prod_{a=1}^{\lambda[m-i+1]} \frac{1-q^{j-i}s^{i-j+a-\lambda[m-j+1]-1}}{1-q^{j-i}s^{i-j+a-1}}. \tag{6.16}$$

Note that this formula remains valid when $\lambda[m]=0$:

$$\nabla_{[\lambda[1],\dots,\lambda[m-1],0]} := \nabla_{[\lambda[1],\dots,\lambda[m-1]]}.$$

Let $\lambda' = [\lambda[1],\lambda[2],\dots,\lambda[m-1],\lambda[m]-1]$ be the partition obtained from λ by subtracting 1 from its last part. We denote by T' and $\mathbb{T}'$ the associated tableaux.

Example 6.19. For instance, if $\lambda=[6,3,2]$ then

$$T = \begin{matrix} 2 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \quad \text{and} \quad \mathbb{T} = \begin{matrix} 2 & 1 \\ 5 & 4 & 3 \\ 11 & 10 & 9 & 8 & 7 & 6 \end{matrix}.$$

In this case $\lambda' = [6,3,1]$ and

$$T = \begin{matrix} 2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \quad \text{and} \quad \mathbb{T} = \begin{matrix} 1 \\ 4 & 3 & 2 \\ 10 & 9 & 8 & 7 & 6 & 5 \end{matrix}$$

One has

$$\begin{aligned} \frac{\nabla_\lambda}{\nabla_{\lambda'}} &= (qs^{\lambda[m]-m};q)_{m-1} \\ &\quad \times \prod_{j=2}^{m} \left[\frac{(qs^{1-j+\lambda[m]};q)_{j-2}(qs^{\lambda[m]-\lambda[m-j+1]-j};q)_{j-2}}{(qs^{\lambda[m]-j};q)_{j-2}(qs^{1-j+\lambda[m]-\lambda[m-j+1]};q)_{j-2}} \right. \\ &\quad \left. \times \frac{(1-q^{j-1}s^{\lambda[m]-\lambda[m-j+1]-j})}{(1-q^{j-1}s^{\lambda[m]-j})} \right] \\ &= (qs^{\lambda[m]-m};q)_{m-1} \\ &\quad \times \prod_{j=2}^{m} \left[\frac{(qs^{\lambda[m]-j+1};q)_{j-2}(qs^{\lambda[m]-\lambda[m-j+1]-j};q)_{j-1}}{(qs^{\lambda[m]-j};q)_{j-1}(qs^{1-j+\lambda[m]-\lambda[m-j+1]};q)_{j-2}} \right]. \end{aligned} \tag{6.17}$$

Observing that

$$\prod_{j=2}^{m} \frac{(qs^{\lambda[m]-j+1};q)_{j-2}}{(qs^{\lambda[m]-j};q)_{j-1}} = \frac{1}{(qs^{\lambda[m]-m};q)_{m-1}},$$

we see that eq. (6.17) gives

$$\frac{\nabla_\lambda}{\nabla_{\lambda'}} = \prod_{j=2}^{m} \frac{(qs^{\lambda[m]-\lambda[m-j+1]-j};q)_{j-1}}{(qs^{1-j+\lambda[m]-\lambda[m-j+1]};q)_{j-2}} = \prod_{i=1}^{m-1} \frac{(qs^{\lambda[m]-\lambda[i]+i-m-1};q)_{m-i}}{(qs^{\lambda[m]-\lambda[i]+i-m};q)_{m-i-1}}. \tag{6.18}$$

As usual, we define the arm, leg, and hook lengths of a node $(x,y)\in\lambda$ by

$$\mathrm{arm}_\lambda[x,y] = \lambda[y]-x,\ \mathrm{leg}_\lambda[x,y] = \overline{\lambda}[x]-y \text{ and } \mathrm{hook}_\lambda[x,y] = \mathrm{arm}_\lambda[x,y] + \mathrm{leg}_\lambda[x,y]+1,$$

where $\overline{\lambda}$ is the conjugate of λ.

Remark 6.20. Note that we use French notation for Ferrers diagrams. For instance, the Ferrers diagram $\lambda = [4,2,1]$ is

$$\begin{array}{ccccc} 3 & \square & & & \\ 2 & \square & \square & & \\ 1 & \square & \square & \square & \square \\ y/x & 1 & 2 & 3 & 4 \end{array}$$

The coordinates of the node $\times$ in the diagram

$$\begin{array}{cccc} \square & & & \\ \square & \square & & \\ \square & \times & \square & \square \end{array}$$

are $[2,1]$. We have

$$\mathrm{arm}_\lambda[2,1] = \lambda[2]-2 = 2,\ \mathrm{leg}_\lambda[2,1] = \overline{\lambda}[1]-1 = 1 \text{ and } \mathrm{hook}_\lambda[2,1] = 4.$$

Graphically, the values of $\mathrm{arm}_\lambda[2,1]$ (respectively of $\mathrm{leg}_\lambda[2,1]$, or of $\mathrm{hook}_\lambda[2,1]$) are obtained by counting the numbers of arm (respectively of leg, or of symbols $\{\mathrm{arm}, \mathrm{leg}, \times\}$) in the following diagram:

$$\begin{array}{cccc} \square & & & \\ \square & \mathrm{leg} & & \\ \square & \times & \mathrm{arm} & \mathrm{arm} \end{array}$$

Let

$$H_\lambda := \prod_{y=1}^{\ell(\lambda)-1} \prod_{x=1}^{\lambda[i]} (qs^{-\mathrm{hook}_\lambda[x,y]};q)_{\mathrm{leg}_\lambda[x,y]}.$$

The changes from $H_{\lambda'}$ to H_λ come from the node $\{(\lambda[m], y) : 1 \le i \le m-1\}$; each hook-length and each leg-length increases by 1, thus

$$\frac{H_\lambda}{H_{\lambda'}} = \prod_{i=1}^{m-1} \frac{(s^{\lambda[m]-\lambda[i]+i-m-1}; q)_{m-i}}{(qs^{\lambda[m]-\lambda[i]+i-m}; q)_{m-i-1}}. \tag{6.19}$$

Hence,

$$\frac{\nabla_\lambda}{\nabla_{\lambda'}} = \frac{H_\lambda}{H_{\lambda'}}. \tag{6.20}$$

Using eq. (6.20) we obtain

$$H_\lambda = \nabla_\lambda. \tag{6.21}$$

It remains to compute $\langle \mathbb{T}^*, \mathbb{T} \rangle$. We start from

$$\langle \mathbb{T}^*, \mathbb{T} \rangle = \prod_{\substack{1 \le i < j \le N \\ \mathrm{CT}_{\mathbb{T}}[i] - \mathrm{CT}_{\mathbb{T}}[j] \le -2}} \frac{(1 - s^{\mathrm{CT}_{\mathbb{T}}[i] - \mathrm{CT}_{\mathbb{T}}[j] - 1})(1 - s^{\mathrm{CT}_{\mathbb{T}}[i] - \mathrm{CT}_{\mathbb{T}}[j] + 1})}{(1 - s^{\mathrm{CT}_{\mathbb{T}}[i] - \mathrm{CT}_{\mathbb{T}}[j]})^2},$$

and we analyze this product in terms of nodes:

$$\langle \mathbb{T}^*, \mathbb{T} \rangle = \prod_{(x,y)\in\lambda} \prod_{\substack{1 \le t \le \overline{\lambda}[x]-y, 1 \le z \le \lambda[y] \\ (x-y-t)-(z-t) \le -2}} \frac{(1 - s^{(x-y-t)-(z-t)+1})(1 - s^{(x-y-t)-(z-t)+1})}{(1 - s^{(x-y-t)-(z-t)})^2}. \tag{6.22}$$

Indeed, consider the set $\mathcal{I}_\lambda$ of the pairs $[(x,y),(z,t)]$ of nodes verifying $\mathbb{T}[x,y] < \mathbb{T}[z,t]$ and $(x-y) \le z-t-2$. This set splits into N disjoint (possibly empty) sets:

$$\begin{aligned}\mathcal{E}_{(x,y)} := \{[(x, y+t), (z,y)] : 1 \le t \le \overline{\lambda}[x] - y,& \\ 1 \le z \le \lambda[y], (x-y-t) - (z-t) \le -2\}&.\end{aligned}$$

Example 6.21. Consider the partition $\lambda = [3,2]$. In this case,

$$\mathbb{T} = \begin{matrix} 2 & 1 & \\ 5 & 4 & 3 \end{matrix} \quad \text{with contents} \quad \begin{matrix} -1 & 0 & \\ 0 & 1 & 2 \end{matrix}.$$

Consequently,

$$\begin{aligned}\mathcal{I}_\lambda &= \{[(2,2),(3,1)], [(1,2),(3,1)], [(1,2),(2,1)]\}, \\ \mathcal{E}_{(1,1)} &= \{[(1,2),(2,1)], [(1,2),(3,1)]\}, \\ \mathcal{E}_{(2,1)} &= \{[(2,2),(3,1)]\}\end{aligned}$$

and

$$\mathcal{E}_{(3,1)} = \mathcal{E}_{(1,2)} = \mathcal{E}_{(2,2)} = \emptyset.$$

Hence,

$$\langle \mathbb{T}^*, \mathbb{T}\rangle = \prod_{[(x,y),(z,t)]\in\mathcal{I}_\lambda} \frac{(1-s^{x-y-z+t-1})(1-s^{x-y-z+t+1})}{(1-s^{x-y-z+t})^2}$$
$$= \prod_{(x,y)\in\lambda}\ \prod_{[(z_1,t_1),(z_2,t_2)]\in\mathcal{E}_{(x,y)}} \frac{(1-s^{z_1-t_1+t_2-z_2-1})(1-s^{z_1-t_1+t_2-z_2+1})}{(1-s^{z_1-t_1+t_2-z_2})^2},$$

and we arrive at (6.22).

Let us compute the products

$$E_{(x,y)} := \prod_{[(z_1,t_1),(z_2,t_2)]\in\mathcal{E}_{(x,y)}} \frac{(1-s^{z_1-t_1+t_2-z_2-1})(1-s^{z_1-t_1+t_2-z_2+1})}{(1-s^{z_1-t_1+t_2-z_2})^2}.$$

Observe that, if $[(x,y+t),(z,y)]\in\mathcal{E}_{(x,y)}$, then t and z have bounds $1\le z\le \lambda[y]$ and $1\le t\le \overline{\lambda}[x]-y$, $z+t-x-2\ge 0$. Hence,

$$E_{(x,y)} := \prod_{t=1}^{\overline{\lambda}[x]-y}\ \prod_{z=\max\{1,x+2-t\}}^{\lambda[y]} \frac{(1-s^{x-t-z+1})(1-s^{x-t-z-1})}{(1-s^{x-t-z})^2}.$$

By telescoping, we find

$$E_{(x,y)} = \prod_{t=1}^{\overline{\lambda}[x]-y} \frac{(1-s^{\max\{1,x+2-t\}-x+t-1})(1-s^{\lambda[y]-x+t+1})}{(1-s^{\max\{1,x+2-t\}-x+t})(1-s^{\lambda[y]-x+t})}. \tag{6.23}$$

We also find

$$\prod_{t=1}^{\overline{\lambda}[x]-y} \frac{(1-s^{\lambda[y]-x+t+1})}{(1-s^{\lambda[y]-x+t})} = \frac{1-s^{\lambda[y]-x+\overline{\lambda}[x]-y+1}}{1-s^{\lambda[y]-x+1}} = \frac{1-s^{\sqsubset_{\lambda}[x,y]}}{1-s^{\hookleftarrow_{\lambda}[x,y]}}. \tag{6.24}$$

But, if $\overline{\lambda}[x]-y\le x$, then $\max\{1,x+2-t\}=x+2-t$, for $1\le t\le\overline{\lambda}[x]-y$, and

$$\prod_{t=1}^{\overline{\lambda}[x]-y} \frac{(1-s^{\max\{1,x+2-t\}-x+t-1})}{(1-s^{\max\{1,y+2-t\}-x+t})} = \left(\frac{1-s}{1-s^2}\right)^{\overline{\lambda}[x]-y}. \tag{6.25}$$

If $\overline{\lambda}[x]-y>x$, then we use telescoping to show that

$$\prod_{t=1}^{\overline{\lambda}[x]-y} \frac{(1-s^{\max\{1,x+2-t\}-x+t-1})}{(1-s^{\max\{1,y+2-t\}-x+t})} = \left(\frac{1-s}{1-s^2}\right)^{x} \frac{1-s}{1-s^{\overline{\lambda}[x]-a-b+1}}. \tag{6.26}$$

Eqs. (6.25) and (6.26) give

(6.27)
$$\prod_{t=1}^{\overline{\lambda}[x]-y} \frac{(1-s^{\max\{1,x+2-t\}-x+t-1})}{(1-s^{\max\{1,y+2-t\}-x+t})} = \left(\frac{1-s}{1-s^2}\right)^{\min\left\{x,\lrcorner_{\lambda}[x,y]\right\}} \frac{1-s}{1-s^{\max\left\{1,\lrcorner_{\lambda}[x,y]-x+1\right\}}}.$$

Hence, by (6.24) and (6.27), we obtain
(6.28)

$$E_{(x,y)} = \left(\frac{1-s}{1-s^2}\right)^{\min\left\{x, \lrcorner_{\lambda}[x,y]\right\}} \frac{(1-s)\left(1-s^{\sqsubset_{\lambda}[x,y]}\right)}{\left(1-s^{\max\left\{1, \lrcorner_{\lambda}[x,y]-x+1\right\}}\right)\left(1-s^{\multimap_{\lambda}[x,y]}\right)}.$$

Finally, eqs. (6.21), (6.22), and (6.28) lead to the following result.

Theorem 6.22. *We have*

$$\langle \widetilde{\mathfrak{M}}_T^*, \widetilde{\mathfrak{M}}_T\rangle = \prod_{(x,y)\in\lambda}\left[\left(\frac{1-s}{1-s^2}\right)^{\min\left\{x, \lrcorner_{\lambda}[x,y]\right\}} \times \frac{(1-s)(-s)^{\sqsubset_{\lambda}[x,y]}\left(s^{-\sqsubset_{\lambda}[x,y]};q\right)_{\lrcorner_{\lambda}[x,y]+1}}{\left(1-s^{\max\left\{1, \lrcorner_{\lambda}[x,y]-x+1\right\}}\right)\left(1-s^{\multimap_{\lambda}[x,y]}\right)}\right].$$

For a rational expression $f(s)$ let $\iota f(s) = f(s^{-1})$. Here are some immediate consequences:

$$\begin{aligned}\iota\nu(\mathbb{T}) &= \nu(\mathbb{T}),\\ \mathrm{CT}_{\overline{\mathbb{T}}}(i) &= -\mathrm{CT}_{\mathbb{T}}(i), \quad 1\le i\le N,\\ \zeta_{v,\overline{\mathbb{T}}} &= q^{v[i]}s^{\mathrm{CT}_{\overline{\mathbb{T}}}(i)} = q^{v[i]}s^{-\mathrm{CT}_{\mathbb{T}}(i)} = \iota\zeta_{v,\mathbb{T}}.\end{aligned}$$

If $\mathbb{T}_1, \mathbb{T}_2 \in \mathrm{Tab}_\lambda$ then

$$\frac{\nu(\mathbb{T}_1)}{\nu(\mathbb{T}_2)} = \frac{\nu\left(\overline{\mathbb{T}_2}\right)}{\nu\left(\overline{\mathbb{T}_1}\right)}.$$

If $\rho_q(m,n) = \dfrac{(qs^{n-1};q)_m\,(qs^{n+1};q)_m}{(qs^n;q)_m^2}$ then $\iota\rho_q(m,n) = \rho_q(m,-n)$. Using this in the formula for $\left\langle P_{v,\mathbb{T}}^*, P_{v,\mathbb{T}}\right\rangle$, we obtain

$$\iota\left(\frac{\left\langle P_{v,\mathbb{T}}^*, P_{v,\mathbb{T}}\right\rangle}{\nu(\mathbb{T})}\right) = \frac{\left\langle P_{v,\overline{\mathbb{T}}}^*, P_{v,\overline{\mathbb{T}}}\right\rangle}{\nu\left(\overline{\mathbb{T}}\right)}.$$

Now suppose that λ is a partition of N, and $\mathbb{T}, T$ are the tableaux corresponding to the minimal antisymmetric polynomial.

Example 6.23. If $\lambda = (3,2)$, then

$$T = \begin{array}{lll} 0 & 1 & \\ 0 & 1 & 2 \end{array}, \quad \mathbb{T} = \begin{array}{lll} 4 & 2 & \\ 5 & 3 & 1 \end{array}.$$

As for symmetric polynomials, we set

$$\widetilde{\mathfrak{M}}^a_T = \left(b^a_\zeta\right)^{-1} \mathfrak{M}^a_T.$$

Our formulae show that

$$\begin{aligned}
\iota\left(\frac{\left\langle \widetilde{\mathfrak{M}}^{a*}_T, \widetilde{\mathfrak{M}}^a_T \right\rangle}{\nu\left(\mathbb{T}\right)}\right) &= \frac{\left\langle \widetilde{\mathfrak{M}}^{*}_{\overline{T}}, \widetilde{\mathfrak{M}}_{\overline{T}} \right\rangle}{\nu\left(\overline{\mathbb{T}}\right)} \\
&= \frac{\phi_N\left(s\right)}{\prod_{i=1}^{\ell(\overline{\lambda})} \phi_{\overline{\lambda}[i]}\left(s\right)} \prod_{(i,j)\in\overline{\lambda}} \left(qs^{-\sqsubset_{\overline{\lambda}[i,j]}}; q\right)_{\lrcorner_{\overline{\lambda}[i,j]}} \\
&= \frac{\phi_N\left(s\right)}{\prod_{i=1}^{\ell(\overline{\lambda})} \phi_{\overline{\lambda}[i]}\left(s\right)} \prod_{(j,i)\in\lambda} \left(qs^{-\sqsubset_{\lambda[j,i]}}; q\right)_{\llcorner_{\lambda[j,i]}}.
\end{aligned}$$

This leads to the following theorem.

Theorem 6.24. *We have*

$$\left\langle \widetilde{\mathfrak{M}}^{a*}_T, \widetilde{\mathfrak{M}}^a_T \right\rangle = \nu\left(\mathbb{T}\right) \frac{\phi_N\left(s^{-1}\right)}{\prod_{i=1}^{\lambda[1]} \phi_{\lambda'[i]}\left(s^{-1}\right)} \prod_{(i,j)\in\lambda} \left(qs^{\sqsubset_{\lambda[i,j]}}; q\right)_{\llcorner_{\lambda[i,j]}}.$$

Example 6.25. For $\lambda = (3,2)$, we have

$$\begin{aligned}
\left\langle \widetilde{\mathfrak{M}}^{a*}_T, \widetilde{\mathfrak{M}}^a_T \right\rangle &= \frac{\phi_5\left(s^{-1}\right)}{\phi_2\left(s^{-1}\right)^2} \left(qs^4; q\right)_2 \left(qs^3; q\right)_1 \left(qs^2; q\right)_1 \\
&= s^{-8} \frac{\phi_5\left(s\right)}{\phi_2\left(s\right)^2} \left(1 - qs^4\right)\left(1 - q^2s^4\right)\left(1 - qs^3\right)\left(1 - qs^2\right).
\end{aligned}$$

Note that $\nu(\mathbb{T})$ does not always equal 1. For instance, we have

$$\nu\left(\begin{array}{llll} 6 & & & \\ 7 & & & \\ 8 & 4 & 2 & \\ 9 & 5 & 3 & 1 \end{array}\right) = \frac{1+s^2}{(1+s)^2}.$$

7. Conclusion

Throughout this paper, we have constructed and analyzed a Macdonald-type structure for vector valued polynomials, that is, polynomials whose coefficients belong to an irreducible module of the Hecke algebra. The "classical" Macdonald polynomials are recovered for the trivial representation and then correspond to the shapes $\lambda = (n)$, $n \in \mathbb{N}$. Thanks to the Yang–Baxter graph we have found algorithms and some explicit formulae for computing the Macdonald polynomials, their (anti)symmetrizations, their scalar products, etc., and we have given graphical interpretations of these properties.

We remark that almost everything works as for vector valued Jack polynomials [6], and that the Jack polynomials are recovered as a limit case of Macdonald polynomials, as expected (setting $q = s^\alpha$ and sending s to 1).

It remains to consider some constructions that could illuminate this theory. For instance, the shifted Macdonald polynomials could be defined by slightly changing the raising operators. For the trivial representation, shifted Macdonald polynomials are easier to manipulate than the homogeneous ones since they can be defined by vanishing properties [10, 11]. We have seen in [6], that this is no longer the case for shifted vector valued Jack polynomials for a generic irreducible module. But this research is not yet complete, and we speculate that the vanishing properties arise when considering some polynomial representations of the Hecke algebra.

Comparing the results in [5] and [8], we find similarities between the concepts of singular non-symmetric Macdonald polynomials and highest weight symmetric Macdonald polynomials. We hope that this similarity extends to vector valued polynomials. In this context, minimal symmetric polynomials should play a special role and, perhaps, provide applications to the study of the fractional quantum Hall effect. The fractional quantum Hall effect is a state of matter with elusive physical properties whose theoretical study was pioneered by Laughlin based on wave functions describing the many-body state of the interacting electrons. Some of these wave functions (called Read–Rezayi states; see [13]) are multivariate symmetric polynomials with special vanishing properties, and it was shown, combining minimality of the polynomials for the vanishing properties and results of [7], that they are Jack polynomials for a specialization of the parameter α (see e.g. [2]). It would be interesting to know if we can identify other relevant wave functions from vector valued Jack or Macdonald polynomials.

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Appendix A. Some useful formulae for the affine double Hecke algebra

A.1. **Hecke algebra of type A_{N-1}.** The generators of $\mathcal{H}_N(s)$ are $T_1, T_2, \ldots, T_{N-1}$ with $s^n \neq 1$ for $1 \leq n \leq N$. The generators satisfy the relations:

$$\begin{gathered}(T_i - s)(T_i + 1) = 0, \quad T_i^2 = (s-1)T_i + s,\\ T_i^{-1} = \frac{1}{s}(T_i - s + 1),\\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad 1 \leq i < N,\\ T_iT_j = T_jT_i, \quad |i-j| > 1.\end{gathered}$$

Let $S = T_1T_2 \cdots T_{N-1}$. Then $T_iS = ST_{i-1}$ for $1 < i \leq N-1$ and $T_jS^N = S^NT_j$ for $1 \leq j < N$. Indeed,

$$\begin{aligned}T_iS &= T_1 \cdots T_{i-2}T_iT_{i-1}T_iT_{i+1} \cdots T_{N-1}\\ &= T_1 \cdots T_{i-2}T_{i-1}T_iT_{i-1}T_{i+1} \cdots T_{N-1}\\ &= ST_{i-1},\end{aligned}$$

and

$$\begin{aligned}T_jS^N &= S^{j-1}T_1S^{N-j+1}\\ &= S^{j-1}((s-1)T_1 + s)(T_2 \cdots T_{N-1}S)S^{N-j-1}\\ &= (s-1)S^N + sS^{j-1}(ST_1 \cdots T_{N-2})S^{N-j-1},\\ S^NT_j &= S^{j+1}T_{N-1}S^{N-j-1}\\ &= S^jT_1T_2 \cdots T_{N-2}((s-1)T_{N-1} + s)S^{N-j-1}\\ &= (s-1)S^N + sS^jT_1 \cdots T_{N-2}S^{N-j-1}.\end{aligned}$$

A consequence of the above derivation is

$$T_1S^2 = S^2T_{N-1}.$$

The Murphy elements are $\phi_i = s^{i-N}T_iT_{i+1} \cdots T_{N-1}T_{N-1} \cdots T_i$. Let $\phi_i^{'} = s^{N-i}\phi_i$ and $S_i = T_iT_{i+1} \cdots T_{N-1}$ for $1 \leq i < N$. Then $\phi_i^{'}\phi_{i+1}^{'} \cdots \phi_{N-1}^{'} = S_i^{N+1-i}$. Indeed, for $i = N-1$, both sides equal T_{N-1}^2. Note that $S_iT_j = T_{j+1}S_i$ for $i \leq j < N$. Now suppose the statement is true for some $i > 1$. We compute

$$\begin{aligned}S_{i-1}^{N+1-i} &= S_{i-1}^{N-i}T_{i-1}S_i = T_{N-1}S_{i-1}^{N-i}S_i = T_{N-1}S_{i-1}^{N-i-1}T_{i-1}S_i^2\\ &= T_{N-1}T_{N-2}S_{i-1}^{N-i-1}S_i^2 = T_{N-1}T_{N-2}S_{i-1}^{N-i-2}T_{i-1}S_i^3\\ &= \cdots = T_{N-1}T_{N-2} \cdots T_{i-1}S_i^{N-i+1},\end{aligned}$$

multiply both sides on the left by $S_{i-1} = T_{i-1} \cdots T_{N-1}$, and use the inductive hypothesis, to obtain

$$\begin{aligned}S_{i-1}^{N+2-i} &= T_{i-1} \cdots T_{N-1}T_{N-1} \cdots T_{i-1}S_i^{N+1-i}\\ &= \phi_{i-1}^{'}S_i^{N+1-i} = \phi_{i-1}^{'}\phi_i^{'} \cdots \phi_{N-1}^{'}.\end{aligned}$$

Thus, $S^N = s^{N(N-1)/2}\phi_1\phi_2 \cdots \phi_{N-1}$.

Adjoin an invertible operator w with relations

$$\begin{aligned} wT_i &= T_{i+1}w, \quad 1 \leq i < N-1, \\ w^2T_{N-1} &= T_1w^2, \\ w^NT_i &= T_iw^N, \quad 1 \leq i < N. \end{aligned}$$

A.2. **Action on polynomials.** Let $\mathcal{P} = \mathbb{K}\left[x_1, \ldots, x_N\right]$, where $\mathbb{K}$ is an extension field of $\mathbb{Q}\left(s, q\right)$. On $\mathcal{P}$ there is a representation of $\mathcal{H}_N\left(s\right)$,

$$p\left(x\right)T_i = \left(1-s\right)\frac{p\left(x\right) - p\left(xs_i\right)}{x_i - x_{i+1}} + sp\left(xs_i\right), \quad 1 \leq i < N,$$

where $xs_i = \left(x_1, \ldots, x_{i+1}, x_i, \ldots\right)$ (s_i is the transposition $\left(i, i+1\right)$),

$$p\left(x\right)w = p\left(qx_N, x_1, x_2, \ldots, x_{N-1}\right).$$

Denote the multiplication operator $p\left(x\right) \mapsto x_ip\left(x\right)$ by x_i, $1 \leq i \leq N$. Then we have

$$\begin{aligned} x_iT_j &= T_jx_i, \quad j \neq i, i-1, \\ x_iT_i &= sT_i^{-1}x_{i+1}, x_i = sT_i^{-1}x_{i+1}T_i^{-1}, \\ x_{i+1}w &= wx_i, \quad 1 \leq i < N, \\ x_1w &= qwx_N. \end{aligned}$$

A.3. **q-Dunkl operators.** There are pairwise commuting operators $D_1, \ldots, D_N$ (dual to the multiplication operators) with relations

$$\begin{aligned} D_iT_j &= T_jD_i, \quad j \neq i, i-1, \\ sT_i^{-1}D_i &= D_{i+1}T_i, D_i = \frac{1}{s}T_iD_{i+1}T_i, \\ D_{i+1}w &= wD_i, \quad 1 \leq i < N, \\ qD_1w &= wD_N. \end{aligned}$$

They act on polynomials by

$$\begin{aligned} p\left(x\right)D_N &= \left(p\left(x\right) - s^{N-1}p\left(x\right)T_{N-1}^{-1}T_{N-2}^{-1}\cdots T_1^{-1}w\right)x_N^{-1}, \\ D_i &= \frac{1}{s}T_iD_{i+1}T_i = w^{-1}D_{i+1}w, \quad 1 \leq i < N. \end{aligned}$$

The Cherednik operators satisfy

$$\begin{aligned} \xi_N &= s^{1-N}\left(1 - D_Nx_N\right), \\ \xi_i &= \frac{1}{s}T_i\xi_{i+1}T_i, 1 \leq i < N. \end{aligned}$$