Generalized Hibi rings and Hibi ideals

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Let $\mathcal{I}(P)$ be the set of the poset ideals of P. Then $\mathcal{I}(P)$ is a sublattice of the power set of P, and hence it is a distributive lattice.

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Let K be a field. Then the Hibi ring over K attached to P is the toric ring $K[\mathcal{I}(P)] \subset K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ generated by the set of monomials

$$\{u_I : I \in \mathcal{I}(P)\}$$

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One fundamental result concerning Hibi rings is that the toric ideal $L_P = \operatorname{Ker} \varphi$ has a reduced Gröbner basis consisting of the so-called Hibi relations:

$$t_I t_J - t_{I \cap J} t_{I \cup J} \quad \text{with} \quad I \not\subseteq J \quad \text{and} \quad J \not\subseteq I.$$

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More generally, for any finite lattice $\mathcal{L}$, not necessarily distributive, one may consider the K algebra $K[\mathcal{L}]$ with generators y_α, $\alpha \in \mathcal{L}$, and relations $y_\alpha y_\beta = y_{\alpha\wedge\beta} y_{\alpha\vee\beta}$ where $\wedge$ and $\vee$ denote meet and join in $\mathcal{L}$. Hibi showed that $K[\mathcal{L}]$ is a domain if and only if $\mathcal{L}$ is distributive, in other words, if $\mathcal{L}$ is an ideal lattice of a poset.

Let K be a field and $X = (x_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$ a matrix of indeterminates.

We denote by $K[X]$ the polynomial ring over K with the indeterminates x_{ij}, and by A the K-subalgebra of $K[X]$ generated by all maximal minors of X.

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Let $<$ be the lexicographic order on $K[X]$ induced by

$$x_{11} > x_{12} > \cdots > x_{1n} > x_{21} > x_{22} > \cdots > x_{m1} > x_{m2} > \cdots > x_{mn}.$$

We denote by $\delta = [a_1, a_2, \dots, a_m]$ the maximal minor of X with columns $a_1 < a_2 < \cdots < a_m$. Then

$$\mathrm{in}_<(\delta) = x_{1,a_1} x_{2,a_2} \cdots x_{m,a_m}$$

is the 'diagonal' of δ.

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In general $\mathrm{in}_<(A)$ is not finitely generated. A subset $\mathcal{S} \subset A$ is called a Sagbi bases of A with respect to $<$, if the elements $f \in \mathcal{S}$ generate A over K. This concept has been introduced by Robbiano and Sweedler and independently by Kapur and Madlener.

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What is the use of this theorem?

We define a partial on the set $\mathcal{L}$ of maximal minors of X:

$$[a_1, a_2, \ldots, a_m] \leq [b_1, b_2, \ldots, b_m] \quad \Leftrightarrow \quad a_i \leq b_i \quad \text{for all } i$$

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Indeed, let T be the polynomial ring over K in the variables t_δ with $\delta \in \mathcal{L}$, and let $\psi: T \to \text{in}_<(A)$ be the K-algebra homomorphism with $\psi(t_\delta) = \text{in}_<(\delta)$. One shows that the Hibi relations

$$t_{\delta_1} t_{\delta_2} - t_{\delta_1 \vee \delta_2} t_{\delta_1 \wedge \delta_2}, \quad \delta_1, \delta_1 \in \mathcal{L}$$

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Corollary The coordinate ring A of the Grassmannian of m-dimensional K-subspaces of K^n is a Gorenstein ring of dimension $m(n-m)+1$.

Hibi ideals

Let P be a finite poset. The ideal $H_P \subset K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ which his generated by the monomials

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(b) $H_P = \bigcap_{p \leq q} (x_p, y_q)$.

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Application: Let G be a finite simple graph on the vertex set $[n]$. One defines the edge ideal I_G of G as the monomial ideal in $K[x_1, \ldots, x_n]$ with set of generators $\{x_i x_j : \{i, j\} \in E(G)\}$.

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For which graphs is I_G Cohen–Macaulay?

Theorem (H-Hibi) Let G be a bipartite graph with vertex partition $V \cup V'$. Then the following conditions are equivalent:

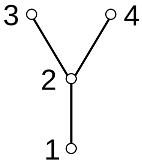
(a) G is a Cohen–Macaulay graph;

(b) $|V| = |V'|$ and the vertices $V = \{x_1, \ldots, x_n\}$ and $V' = \{y_1, \ldots, y_n\}$ can be labelled such that:

  (i) $\{x_i, y_i\}$ are edges for $i = 1, \ldots, n$;

  (ii) if $\{x_i, y_j\}$ is an edge, then $i \leq j$;

  (iii) if $\{x_i, y_j\}$ and $\{x_j, y_k\}$are edges, then $\{x_i, y_k\}$ is an edge.



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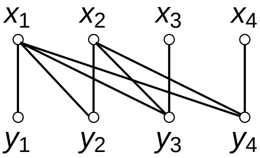

$G(P)$

The Alexander dual: let I be a squarefree monomial ideal. Then

$$I = \bigcap_{j=1}^{r} P_{F_j},$$

where for a subset $F \subset [n]$ we set $P_F = (\{x_i : i \in F\})$.

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Example:
$I = (x_1x_4, x_1x_5, x_2x_5, x_3x_5) = (x_1, x_2, x_3) \cap (x_1, x_5) \cap (x_4, x_5)$.

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Therefore $I^\vee = (x_1x_2x_3, x_1x_5, x_4x_5)$.

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But $H_P^\vee = (\{x_p y_q : p \leq q\})$ is the edge ideal of a bipartite graph satisfying the conditions (i), (ii) and (ii). This proves one direction of the classification theorem of Cohen-Macaulay bipartite graphs.

Generalized Hibi ideals and Hibi rings

Let P be a finite poset and $\mathcal{I}(P)$ the set of poset ideals of P. An r-multichain of $\mathcal{I}(P)$ is a chain of poset ideals of length r,

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We define a partial order on the set $\mathcal{I}_r(P)$ of all r-multichains of $\mathcal{I}(P)$ by setting $\mathcal{I} \leq \mathcal{I}'$ if $I_k \subseteq I'_k$ for $k = 1, \ldots, r$.

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The partially ordered set $\mathcal{I}_r(P)$ is a distributive lattice, if we define the meet of $\mathcal{I} : \ I_1 \subseteq \cdots \subseteq I_r$ and $\mathcal{I}' : \ I'_1 \subseteq \cdots \subseteq I'_r$ as $\mathcal{I} \cap \mathcal{I}'$ where

$$(\mathcal{I} \cap \mathcal{I}')_k = I_k \cap I'_k$$

for $k = 1, \ldots, r$, and the join as $\mathcal{I} \cup \mathcal{I}'$ where

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for $k = 1, \ldots, r$.

With each r-multichain $\mathcal{I}$ of $\mathcal{I}_r(P)$ we associate a monomial $u_{\mathcal{I}}$ in the polynomial ring $S = K[\{x_{ij} : 1 \leq i \leq r, 1 \leq j \leq n\}]$ in rn indeterminates which is defined as

$$u_{\mathcal{I}} = x_{1J_1} x_{2J_2} \cdots x_{rJ_r},$$

where $x_{kJ_k} = \prod_{p_\ell \in J_k} x_{k\ell}$ and $J_k = I_k \setminus I_{k-1}$ for $k = 1, \ldots, r$.

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We denote by $H_{r,P}$ the monomial ideal in S generated by these monomials and by $R_r(P)$ the K-subalgebra generated by the monomial generators of $H_{r,P}$.

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For $r = 2$ the ideal $H_{r,P}$ is just the classical Hibi ideal, and $R_r(P)$ the Hibi ring of the ideal lattice $\mathcal{I}(P)$ of P.

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Theorem The set

$$\Gamma = \{t_{\mathcal{I}} t_{\mathcal{I}'} - t_{\mathcal{I} \cup \mathcal{I}'} t_{\mathcal{I} \cap \mathcal{I}'} \in T : \mathcal{I}, \mathcal{I}' \in \mathcal{I}_r(P) \text{ incomparable}\}$$

is a reduced Gröbner basis of the ideal $L_r = \operatorname{Ker} \varphi$ with respect to the reverse lexicographic order.

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Finally we consider the generalized Hibi ideal $H_{r,P}$ and its Alexander dual.

Let $C \subset P$ a multichain of length r, i.e., $C = \{p_1, p_2, \ldots, p_r\}$ with $p_1 \leq p_2 \leq \cdots \leq p_r$. Let $\mathcal{C}$ be the set of all multichains of length r of P.

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We define the monomial $u_C = \prod_{i=1}^{r} x_{i,p_i}$ and let

$$I_{r,P} = (\{u_C : C \in \mathcal{C}\})$$

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$$I_{r,P} = (\{u_C : C \in \mathcal{C}\})$$

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The ideals $I_{r,P}$ may be interpreted as facet ideals of a completely balanced simplicial complexes, as introduced by Stanley.

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Corollary The facet ideal of a completely balanced simplicial complex arising from a poset is Cohen–Macaulay.

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