Ideals generated by 2-minors with applications to algebraic statistic

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Contingency tables

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In statistics, a contingency table is used to record and analyze the relation between two or more categorical variables. It displays the (multivariate) frequency distribution of the variables in a matrix format.

The following displays an example of a contingency table

HAIR COLORS

EYE COLORS		Blonde	Red	Black	Totals
	Brown	3	4	20	27
	Green	14	18	8	40
	Blue	16	12	5	33
Totals		33	34	33	100

Hair colors

Eye colors		Blonde	Red	Black	Totals
	Brown	3	4	20	27
	Green	14	18	8	40
	Blue	16	12	5	33
Totals		33	34	33	100

The sequence of row and column sums is called the marginal distribution of the contingency table.

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Say our contingency table has cell frequencies a_{ij}, while our statistical model gives the expected cell frequencies e_{ij}. Then the χ^2-statistic of the contingency table is computed by the formula

$$\chi^2 = \sum_{i,j} \frac{(a_{ij} - e_{ij})^2}{e_{ij}}.$$

Under the hypothesis of independence one has

$$e_{ij} = r_i c_j / N$$

where $r_i = \sum_j a_{ij}$ is the ith row sum, $c_j = \sum_i a_{ij}$ is the jth column sum and $N = \sum_i r_i = \sum_j c_j$ is the total number of samples.

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Does the value of χ^2 fit well our hypothesis of independence???

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But how to produce random contingency tables with the same marginal distribution?

Random Walks

We start at the given table A and take random moves that do not change the marginal distribution. Each single move is given as follows: choose a pair of rows and a pair of columns at random, and modify A at the four entries where the selected rows and columns intersect by adding or subtracting 1 according to the following pattern of signs

$$\begin{matrix} + & - \\ - & + \end{matrix} \qquad \text{or} \qquad \begin{matrix} - & + \\ + & - \end{matrix}$$

with probability $1/2$ each. In this way we obtain a random walk on the set of contingency tables with fixed marginal distribution.

If the move produces negative entries, discard it and continue by choosing a new pair of rows and columns.

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If A is a contingency table of shape $m \times n$, then the number of possible moves is $\binom{m}{2}\binom{n}{2}$, which is a rather big number. In practice one obtains a pretty good selection of randomly selected contingency tables with the same marginal distribution as that of A which allows to test the significance of A, if we restrict the set $\mathcal{S}$ of possible moves.

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We say that two contingency tables A and B are connected via $\mathcal{S}$, if B can be obtained from A by a finite number of moves from $\mathcal{S}$.

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The question arises how to decide whether two contingency tables are connected.

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Then the connectedness problem can be rephrased and generalized as follows: let $\mathcal{B}$ be a subset of vectors of $\mathbb{Z}^n$. One defines the graph $G_{\mathcal{B}}$ whose vertex set is the set $\mathbb{N}^n$ of nonnegative integer vectors.

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Two vectors **a** and **c** in $\mathbb{N}^n$ are connected by an edge of $G_{\mathcal{B}}$ if

$$\mathbf{a} - \mathbf{c} \in \pm\mathcal{B}.$$

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We say that **a** and **c** are connected via $\mathcal{B}$, if they belong to the same connected component of $G_{\mathcal{B}}$.

We fix a field K and define the binomial ideal

$$I_{\mathcal{B}} = (\mathbf{x}^{\mathbf{b}^+} - \mathbf{x}^{\mathbf{b}^-} : \mathbf{b} \in \mathcal{B}) \subset K[x_1, \ldots, x_n],$$

where for a vector $\mathbf{a} \in \mathbb{Z}^n$, the vectors $\mathbf{a}^+, \mathbf{a}^- \in \mathbb{N}^n$ are the unique vectors with $\mathbf{a} = \mathbf{a}^+ - \mathbf{a}^-$ and $\text{supp}(a^+) \cap \text{supp}(a^-) = \emptyset$.

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Theorem. The non-negative integer vectors $\mathbf{a}$ and $\mathbf{c}$ are connected via $\mathcal{B}$ if and only if $\mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{c}} \in I_{\mathcal{B}}$.

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How to decide whether a binomial belongs to a binomial ideal?

Primary Decompositions

Given a binomial ideal I and a binomial f, can we find feasible conditions in terms of the exponents appearing in f that guarantee that $f \in I$?

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The following strategy may be successful in some cases. Write the given binomial ideal I as an intersection $I = \bigcap_{k=1}^{r} J_k$ of ideals J_k. Then $f \in I$ if and only if $f \in J_k$ for all k.

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This strategy is useful only if each of the ideals J_k has a simple structure, so that it is possible to describe the conditions that guarantee that f belongs to J_k.

A natural choice for such an intersection is a primary decomposition of I. In the case that I is a radical ideal the natural choice for the ideals J_k are the minimal prime ideals of I.

We apply the above strategy to contingency tables A of shape $2 \times n$.

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Given a set of moves. We want to decide when two contingency tables A and B are connected.

In algebraic terms: given a matrix

$$\begin{pmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \end{pmatrix}.$$

We let S be the set of 2-minors corresponding to the given set of moves.

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This set of minors is indexed by the edges of a graph G:

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We call

$$J_G = (x_i y_j - x_j y_i : \{i,j\} \in E(G))$$

the edge ideal of G.

Theorem Let G be a finite graph on the vertex set $[n]$ and J_G its edge ideal. Then J_G has a squarefree initial ideal with respect to the lexicographic order induced by
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Which are the minimal prime ideals of J_G??

Let G be a simple graph on $[n]$. For each subset $S \subset [n]$ we define a prime ideal $P_S(G) \subset K[x_1, \cdots, x_n, y_1, \cdots, y_n]$.

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Let $T = [n] \setminus S$, and let $G_1, \ldots, G_{c(S)}$ be the connected component of G_T. Here G_T is the induced subgraph of G whose edges are exactly those edges $\{i, j\}$ of G for which $i, j \in T$. For each G_i we denote by $\tilde{G}_i$ the complete graph on the vertex set $V(G_i)$. We set

$$P_S(G) = (\bigcup_{i \in S} \{x_i, y_i\}, J_{\tilde{G}_1}, \ldots, J_{\tilde{G}_{c(S)}}).$$

$P_S(G)$ is a prime ideal containing J_G.

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Theorem $J_G = \bigcap_{S \subset [n]} P_S(G)$

Theorem Let G be a connected simple graph on the vertex set $[n]$, and $S \subset [n]$. Then $P_S(G)$ is a minimal prime ideal of J_G if and only if $S = \emptyset$, or $S \neq \emptyset$ and each $i \in S$ is a cut point of $G_{([n]\setminus S)\cup\{i\}}$, i.e., one has $c(S \setminus \{i\}) < c(S)$.

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Consider for example the path graph G of length 4.

Then the only subsets $S \subset [4]$, besides the empty set, for which each $i \in S$ is a cut-point of the graph $G_{([4]\setminus S)\cup\{i\}}$, are the sets $S = \{2\}$ and $S = \{3\}$. Thus

$$J_G = I_2(X) \cap (x_2, x_2, x_3y_4 - x_4y_3) \cap (x_3, y_3, x_1y_2 - x_2y_1),$$

where

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix}.$$

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two contingency tables of shape 2×4. Let $\mathcal{S}$ be the set of adjacent moves

$$\pm\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix},$$

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Then A and B are connected via $\mathcal{S}$ if and only if the following conditions are satisfied:

(a) $\sum_{j=1}^{4} a_{ij} = \sum_{j=1}^{4} b_{ij}$ for $i = 1, 2$;

(b) $a_{1j} + a_{2j} = b_{1j} + b_{2j}$ for $j = 1, 2, 3, 4$;

(c) either $a_{12} + a_{22} \geq 1$ and $b_{12} + b_{22} \geq 1$, or $a_{ij} = b_{ij}$ for $i, j \leq 2$, and $a_{13} + a_{14} = b_{13} + b_{14}$ and $a_{23} + a_{24} = b_{23} + b_{24}$;

(d) either $a_{13} + a_{23} \geq 1$ and $b_{13} + b_{23} \geq 1$, or $a_{ij} = b_{ij}$ for $i, j \geq 3$, and $a_{11} + a_{12} = b_{11} + b_{12}$ and $a_{21} + a_{22} = b_{21} + b_{22}$.

Ideals generated by 2-minors

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Consider the ideal I generated by the 2-minors

$$ae - bd, bf - ce, dh - eg, ei - fh$$

of the matrix

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Then $cdh - aei \in \sqrt{I} \setminus I$. So I is not a radical ideal.

Let $X = (x_{ij})_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}}$ be a matrix of indeterminates, and let S be the polynomial ring over a field K in the variables x_{ij}.

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The 2-minor $\delta = [a_1, a_2 | b_1, b_2]$ is called adjacent if $a_2 = a_1 + 1$ and $b_2 = b_1 + 1$.

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Let $\mathcal{C}$ be any set of adjacent 2-minors. We call such a set a configuration of adjacent 2-minors. A configuration of adjacent 2-minors may be identified with a polyomino. We denote by $I(\mathcal{C})$ the ideal generated by the elements of $\mathcal{C}$.

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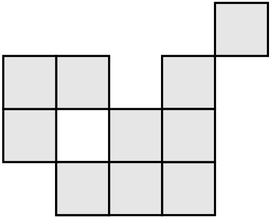
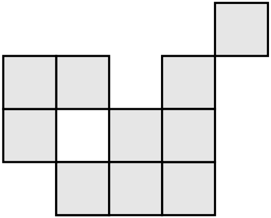
The set of vertices of $\mathcal{C}$, denoted $V(\mathcal{C})$, is the union of the vertices of its adjacent 2-minors. Two distinct minors in $\delta, \gamma \in \mathcal{C}$ are called connected if there exist $\delta_1 \ldots, \delta_r \in \mathcal{C}$ such that $\delta = \delta_1$, $\gamma = \delta_r$, and δ_i and δ_{i+1} have a common edge.



A Configuration of adjacent 2-minors



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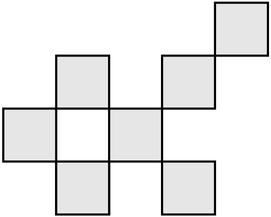

A Chess board configuration

Theorem Let $\mathcal{C}$ be a configuration of adjacent 2-minors. Then the following conditions are equivalent:

(a) $I(\mathcal{C})$ is a prime ideal.

(b) $\mathcal{C}$ is a chessboard configuration with no cycle of length 4.

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Here is another case of primality of ideals of 2-minors discovered by my student Qureshi.

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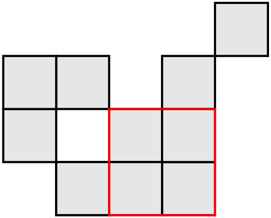
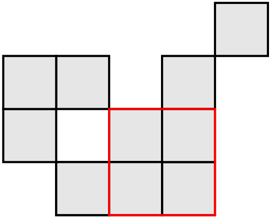
Let $\mathcal{C}$ be a configuration of 2-minors. A minor $[a_1, a_2|b_1, b_2]$ is called an inner minor of $\mathcal{C}$, if all adjacent 2-minors $[a, a+1|b, b+1]$ with $a_1 \leq a < a_2$ and $b_1 \leq b < b_2$ belong to $\mathcal{C}$.



An inner minor



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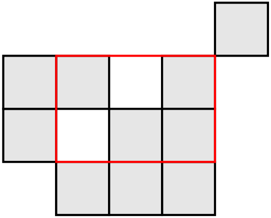

Not an inner minor

A configuration $\mathcal{C}$ is called rectangular, if each minor $[a_1, a_2|b_1, b_2]$ with $(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2) \in V(\mathcal{C})$ is an inner minor of $\mathcal{C}$. In the language of polyminoes, a rectangular configuration is a convex polyomino.

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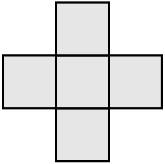

A rectangular configuration

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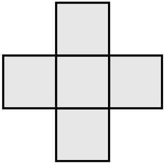

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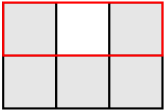

Not rectangular

Theorem (Quereshi) Let $\mathcal{C}$ be a rectangular configuration. Then the ideal generated by all inner 2-minors is a prime ideal.

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