# A survey on Stanley decompositions

Jürgen Herzog
Universität Duisburg-Essen

Ellwangen, March 2011

# Outline

# Outline

# Outline

# Outline

# Outline

# The conjecture

Richard Stanley in his article "Linear Diophantine equations and local cohomology", Invent. Math. **68** (1982) made a striking conjecture concerning the depth of multigraded modules.

# The conjecture

Richard Stanley in his article "Linear Diophantine equations and local cohomology", Invent. Math. **68** (1982) made a striking conjecture concerning the depth of multigraded modules.

Here we concentrate on the case that $M$ is a finitely generated $\mathbb{Z}^n$-graded $S$-module, where $S = K[x_1, \ldots, x_n]$ is the polynomial ring.

# The conjecture

Richard Stanley in his article "Linear Diophantine equations and local cohomology", Invent. Math. **68** (1982) made a striking conjecture concerning the depth of multigraded modules.

Here we concentrate on the case that $M$ is a finitely generated $\mathbb{Z}^n$-graded $S$-module, where $S = K[x_1, \ldots, x_n]$ is the polynomial ring.

An important special case for a $\mathbb{Z}^n$-graded $S$-module is $M = I/J$ where $J \subset I \subset S$ are monomial ideals.

A Stanley decomposition $\mathcal{D}$ of $M$ is direct sum of $\mathbb{Z}^n$-graded $K$-vector spaces

$$\mathcal{D} : M = \bigoplus_{j=1}^{r} m_j K[Z_j],$$

where each $m_j \in M$ is homogeneous, $Z_j \subset X = \{x_1, \ldots, x_n\}$ and each $m_j K[Z_j]$ is a free $K[Z_j]$-module.

A Stanley decomposition $\mathcal{D}$ of $M$ is direct sum of $\mathbb{Z}^n$-graded $K$-vector spaces

$$\mathcal{D} : M = \bigoplus_{j=1}^{r} m_j K[Z_j],$$

where each $m_j \in M$ is homogeneous, $Z_j \subset X = \{x_1, \ldots, x_n\}$ and each $m_j K[Z_j]$ is a free $K[Z_j]$-module.

We set $\text{sdepth}(\mathcal{D}) = \min\{|Z_j| \ j = 1, \ldots, r\}$, and

$$\text{sdepth}\, M = \max\{\text{sdepth}(\mathcal{D}) : \mathcal{D} \text{ is a Stanley decomposition of } M\}.$$

is called the Stanley depth of $M$.

A Stanley decomposition $\mathcal{D}$ of $M$ is direct sum of $\mathbb{Z}^n$-graded $K$-vector spaces

$$\mathcal{D} : M = \bigoplus_{j=1}^{r} m_j K[Z_j],$$

where each $m_j \in M$ is homogeneous, $Z_j \subset X = \{x_1, \ldots, x_n\}$ and each $m_j K[Z_j]$ is a free $K[Z_j]$-module.

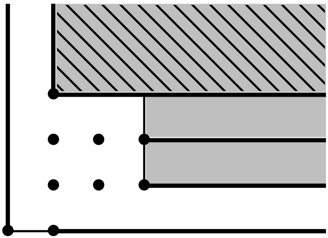
We set $\text{sdepth}(\mathcal{D}) = \min\{|Z_j| \ j = 1, \ldots, r\}$, and

$$\text{sdepth}\, M = \max\{\text{sdepth}(\mathcal{D}) : \mathcal{D} \text{ is a Stanley decomposition of } M\}.$$

is called the Stanley depth of $M$.

**Conjecture** (Stanley) $\text{sdepth}\, M \geq \text{depth}\, M$.

Example: $I = (x_1x_2^3, x_1^3x_2)$



The figure displays Stanley decompositions of

$$I = x_1x_2^3K[x_1, x_2] \oplus x_1^3x_2^2K[x_1] \oplus x_1^3x_2K[x_1],$$

and

$$S/I = K[x_2] \oplus x_1K[x_1] \oplus x_1x_2K \oplus x_1x_2^2K \oplus x_1^2x_2K \oplus x_1^2x_2^2K.$$

# Known cases

The Stanley depth for modules of the form $I/J$ where $J \subset I \subset S_K[x_1, \ldots, x_n]$ are monomial ideals is a pure combinatorial invariant, in particular, it does not depend on the field $K$, while the depth is homological invariant and in case of squarefree monomial ideal, a topological invariant of the attached simplicial complex, and may very well depend on the field $K$.

# Known cases

The Stanley depth for modules of the form $I/J$ where $J \subset I \subset S_K[x_1, \ldots, x_n]$ are monomial ideals is a pure combinatorial invariant, in particular, it does not depend on the field $K$, while the depth is homological invariant and in case of squarefree monomial ideal, a topological invariant of the attached simplicial complex, and may very well depend on the field $K$.

What is know for $I/J$?

- (Jahan, Zheng, H) Stanley's conjecture holds for all algebras $S/I$, $I$ a monomial ideal, if it holds for all such Cohen–Macaulay algebras.

- (Jahan, Zheng, H) Stanley's conjecture holds for all algebras $S/I$, $I$ a monomial ideal, if it holds for all such Cohen–Macaulay algebras.
- (H, Popescu) If $\Delta$ is a simplicial complex. Then the Stanley Reisner ring $K[\Delta]$ of $\Delta$ satisfies Stanley's conjecture, if $\Delta$ is shellable.

- (Jahan, Zheng, H) Stanley's conjecture holds for all algebras $S/I$, $I$ a monomial ideal, if it holds for all such Cohen–Macaulay algebras.
- (H, Popescu) If $\Delta$ is a simplicial complex. Then the Stanley Reisner ring $K[\Delta]$ of $\Delta$ satisfies Stanley's conjecture, if $\Delta$ is shellable.
- (H, Jahan, Yassemi) Stanley's conjecture holds for $S/I$ when $I$ is Cohen-Macaulay ideal of codimension 2 or Gorenstein of codimension 3.

- (Jahan, Zheng, H) Stanley's conjecture holds for all algebras $S/I$, $I$ a monomial ideal, if it holds for all such Cohen–Macaulay algebras.
- (H, Popescu) If $\Delta$ is a simplicial complex. Then the Stanley Reisner ring $K[\Delta]$ of $\Delta$ satisfies Stanley's conjecture, if $\Delta$ is shellable.
- (H, Jahan, Yassemi) Stanley's conjecture holds for $S/I$ when $I$ is Cohen-Macaulay ideal of codimension 2 or Gorenstein of codimension 3.
- (Popescu) If $S$ is a polynomial ring in at most 5 variables, then Stanley's conjecture holds for $S/I$.

- (Jahan, Zheng, H) Stanley's conjecture holds for all algebras $S/I$, $I$ a monomial ideal, if it holds for all such Cohen–Macaulay algebras.
- (H, Popescu) If $\Delta$ is a simplicial complex. Then the Stanley Reisner ring $K[\Delta]$ of $\Delta$ satisfies Stanley's conjecture, if $\Delta$ is shellable.
- (H, Jahan, Yassemi) Stanley's conjecture holds for $S/I$ when $I$ is Cohen-Macaulay ideal of codimension 2 or Gorenstein of codimension 3.
- (Popescu) If $S$ is a polynomial ring in at most 5 variables, then Stanley's conjecture holds for $S/I$.
- (Cimpoeas) If $I \subset S = K[x_1, \ldots, x_n]$ is generated by at most $2n - 1$ monomials, then Stanley's conjecture holds for $I$

- (Jahan, Zheng, H) Stanley's conjecture holds for all algebras $S/I$, $I$ a monomial ideal, if it holds for all such Cohen–Macaulay algebras.
- (H, Popescu) If $\Delta$ is a simplicial complex. Then the Stanley Reisner ring $K[\Delta]$ of $\Delta$ satisfies Stanley's conjecture, if $\Delta$ is shellable.
- (H, Jahan, Yassemi) Stanley's conjecture holds for $S/I$ when $I$ is Cohen-Macaulay ideal of codimension 2 or Gorenstein of codimension 3.
- (Popescu) If $S$ is a polynomial ring in at most 5 variables, then Stanley's conjecture holds for $S/I$.
- (Cimpoeas) If $I \subset S = K[x_1, \dots, x_n]$ is generated by at most $2n - 1$ monomials, then Stanley's conjecture holds for $I$
- (Apel, Okazaki, Yanagawa) If $I$ is a cogeneric monomial ideal, then Stanley's conjecture holds for $S/I$.

# How to compute the Stanley depth

We return to the case $M = I/J$ where $J \subset I \subset S = K[x_1, \ldots, x_n]$ are monomial ideals.

# How to compute the Stanley depth

We return to the case $M = I/J$ where $J \subset I \subset S = K[x_1, \dots, x_n]$ are monomial ideals.

We choose $g \in \mathbb{N}^n$ such that $g \geq a$ for all generators $x^a$ of $I$ and $J$, and consider the finite poset

$$P_{I/J}^{g} = \{a \in \mathbb{N}^n \;\; x^a \in I \setminus J, \; a \leq g\}.$$

We call it the characteristic poset of $I/J$ with respect to $g$.

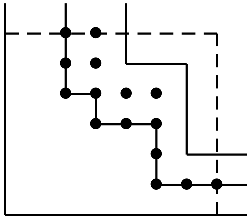
# How to compute the Stanley depth

We return to the case $M = I/J$ where $J \subset I \subset S = K[x_1, \ldots, x_n]$ are monomial ideals.

We choose $g \in \mathbb{N}^n$ such that $g \geq a$ for all generators $x^a$ of $I$ and $J$, and consider the finite poset

$$P_{I/J}^{g} = \{a \in \mathbb{N}^n \ x^a \in I \setminus J, \ a \leq g\}.$$

We call it the characteristic poset of $I/J$ with respect to $g$.



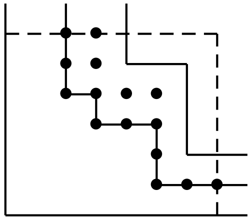
# How to compute the Stanley depth

We return to the case $M = I/J$ where $J \subset I \subset S = K[x_1, \dots, x_n]$ are monomial ideals.

We choose $g \in \mathbb{N}^n$ such that $g \geq a$ for all generators $x^a$ of $I$ and $J$, and consider the finite poset

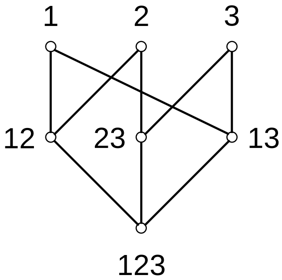
$$P^g_{I/J} = \{a \in \mathbb{N}^n \ \ x^a \in I \setminus J, \ a \leq g\}.$$

We call it the characteristic poset of $I/J$ with respect to $g$.



If $\Delta$ is a simplicial complex and $g = (1, \dots, 1)$, then $P^g_{K[\Delta]}$ can be identified with the face poset of $\Delta$.

The characteristic poset of $\mathfrak{m} = (x_1, x_2, x_3)$ with respect to $g = (1, 1, 1)$ is given by



Given any poset $P$ and $a, b \in P$. Then $[a, b] = \{c \in P : \; a \leq c \leq b\}$ is called an interval.

Given any poset $P$ and $a, b \in P$. Then $[a, b] = \{c \in P : a \leq c \leq b\}$ is called an interval.

Suppose $P$ is a finite poset. A partition of $P$ is a disjoint union

$$\mathcal{P} : P = \bigcup_{i=1}^{r} [a_i, b_i]$$

of intervals.

Given any poset $P$ and $a, b \in P$. Then $[a, b] = \{c \in P : \ a \leq c \leq b\}$ is called an interval.

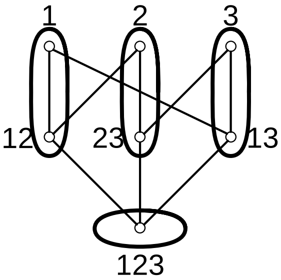
Suppose $P$ is a finite poset. A partition of $P$ is a disjoint union

$$\mathcal{P} : \ P = \bigcup_{i=1}^{r} [a_i, b_i]$$

of intervals.

$$\mathcal{P} : \ P^{g}_{\mathfrak{m}} = [1, 12] \cup [2, 23] \cup [3, 13] \cup [123, 123].$$

is a partition of $P^{g}_{\mathfrak{m}}$.


1
2
3
12
23
13
123

Each partition of $P^g_{I/J}$ gives rise to a Stanley decomposition of $I/J$.

Each partition of $P^g_{I/J}$ gives rise to a Stanley decomposition of $I/J$.

In order to describe the Stanley decomposition of $I/J$ coming from a partition of $P^g_{I/J}$ we shall need the following notation: for each $b \in P^g_{I/J}$, we set

$$Z_b = \{x_j : \ b(j) = g(j)\},$$

Each partition of $P^g_{I/J}$ gives rise to a Stanley decomposition of $I/J$.

In order to describe the Stanley decomposition of $I/J$ coming from a partition of $P^g_{I/J}$ we shall need the following notation: for each $b \in P^g_{I/J}$, we set

$$Z_b = \{x_j : \ b(j) = g(j)\},$$

and define

$$\rho \ P^g_{I/J} \to \mathbb{Z}_{\geq 0}, \quad b \mapsto \rho(b) = |Z_b|.$$

**Theorem** (a) Let $\mathcal{P}: P^g_{I/J} = \bigcup_{i=1}^r [c_i, d_i]$ be a partition of $P^g_{I/J}$. Then

$$\mathcal{D}(\mathcal{P}): I/J = \bigoplus_{i=1}^{r} (\bigoplus_{c} x^c K[Z_{d_i}])$$

is a Stanley decomposition of $I/J$, where the inner direct sum is taken over all $c \in [c_i, d_i]$ for which $c(j) = c_i(j)$ for all $j$ with $x_j \in Z_{d_i}$. Moreover, sdepth $\mathcal{D}(\mathcal{P}) = \min\{\rho(d_i)\ i = 1, \ldots, r\}$.

**Theorem** (a) Let $\mathcal{P}: \; P^g_{I/J} = \bigcup_{i=1}^r [c_i, d_i]$ be a partition of $P^g_{I/J}$. Then

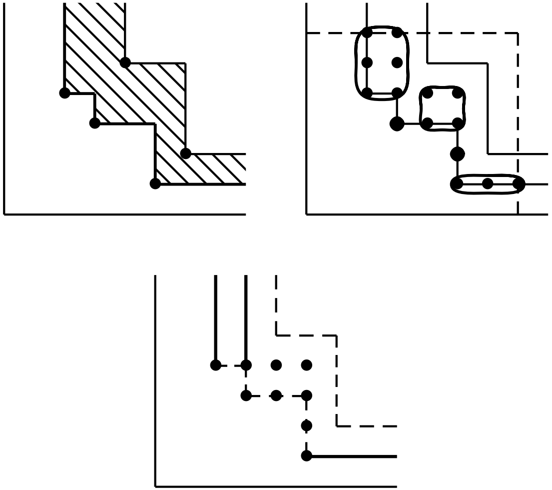
$$\mathcal{D}(\mathcal{P}): \; I/J = \bigoplus_{i=1}^{r} (\bigoplus_{c} x^c K[Z_{d_i}])$$

is a Stanley decomposition of $I/J$, where the inner direct sum is taken over all $c \in [c_i, d_i]$ for which $c(j) = c_i(j)$ for all $j$ with $x_j \in Z_{d_i}$. Moreover, $\operatorname{sdepth} \mathcal{D}(\mathcal{P}) = \min\{\rho(d_i) \; i = 1, \ldots, r\}$.

(b) Let $\mathcal{D}$ be a Stanley decomposition of $I/J$. Then there exists a partition $\mathcal{P}$ of $P^g_{I/J}$ such that

$$\operatorname{sdepth} \mathcal{D}(\mathcal{P}) \geq \operatorname{sdepth} \mathcal{D}.$$

In particular, $\operatorname{sdepth} I/J$ can be computed as the maximum of the numbers $\operatorname{sdepth} \mathcal{D}(\mathcal{P})$, where $\mathcal{P}$ runs over the (finitely many) partitions of $P^g_{I/J}$.



This theorem has been used to compute or to estimate the Stanley depth in several cases:

- (C. Biro, D. Howard, M. Keller, W. Trotter, S. Young) $\operatorname{sdepth}(x_1, \ldots, x_n) = \lceil n/2 \rceil$.

This theorem has been used to compute or to estimate the Stanley depth in several cases:

- (C. Biro, D. Howard, M. Keller, W. Trotter, S. Young) $\mathrm{sdepth}(x_1, \dots, x_n) = \lceil n/2 \rceil$.
- (Shen) Let $I \subset S = K[x_1, \dots, x_n]$ be a complete intersection monomial ideal minimally generated by m elements. Then $\mathrm{sdepth}(I) = n - \lfloor m/2 \rfloor$.

This theorem has been used to compute or to estimate the Stanley depth in several cases:

- (C. Biro, D. Howard, M. Keller, W. Trotter, S. Young) $\operatorname{sdepth}(x_1, \ldots, x_n) = \lceil n/2 \rceil$.
- (Shen) Let $I \subset S = K[x_1, \ldots, x_n]$ be a complete intersection monomial ideal minimally generated by m elements. Then $\operatorname{sdepth}(I) = n - \lfloor m/2 \rfloor$.
- (Floystad, H) Let $s$ be the largest integer such that $n + 1 \geq (2s + 1)(s + 1)$. Then the Stanley depth of any squarefree monomial ideal in $n$ variables is greater or equal to $2s + 1$. Explicitly this lower bound is

$$2\left\lfloor \frac{\sqrt{2n + 2.25} + 0.5}{2} \right\rfloor - 1.$$

# Upper and lower bounds

Let $M$ be a $\mathbb{Z}^n$-graded $S = K[x_1, \ldots, x_n]$-module. Then there exists

$$\mathcal{F}: \; 0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$$

a chain of $\mathbb{Z}^n$-graded submodules of $M$ such that $M_i/M_{i-1} \simeq (S/P_i)(-a_i)$ where $a_i \in \mathbb{Z}^n$ and where each $P_i$ is a monomial prime ideal.

# Upper and lower bounds

Let $M$ be a $\mathbb{Z}^n$-graded $S = K[x_1, \ldots, x_n]$-module. Then there exists

$$\mathcal{F} : \; 0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$$

a chain of $\mathbb{Z}^n$-graded submodules of $M$ such that $M_i/M_{i-1} \simeq (S/P_i)(-a_i)$ where $a_i \in \mathbb{Z}^n$ and where each $P_i$ is a monomial prime ideal.

One has

$$\mathrm{Min}(M) \subset \mathrm{Ass}(M) \subset \{P_1, \ldots, P_r\} \subset \mathrm{Supp}(M),$$

and

# Upper and lower bounds

Let $M$ be a $\mathbb{Z}^n$-graded $S = K[x_1, \dots, x_n]$-module. Then there exists

$$\mathcal{F}: \; 0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$$

a chain of $\mathbb{Z}^n$-graded submodules of $M$ such that $M_i / M_{i-1} \simeq (S/P_i)(-a_i)$ where $a_i \in \mathbb{Z}^n$ and where each $P_i$ is a monomial prime ideal.

One has

$$\mathrm{Min}(M) \subset \mathrm{Ass}(M) \subset \{P_1, \dots, P_r\} \subset \mathrm{Supp}(M),$$

and

$$\begin{aligned} \min\{\dim S/P_1, \dots, S/P_r\} &\leq \operatorname{depth} M, \operatorname{sdepth} M \\ &\leq \min\{\dim S/P : \; P \in \mathrm{Ass}(M)\}. \end{aligned}$$

The upper inequality has been proved by Apel.

Let $M$ be a finitely generated $\mathbb{Z}^n$-graded $S = K[x_1, \ldots, x_n]$-module. It is also $\mathbb{Z}$-graded,i.e.,

$$M = \bigoplus_{i \in \mathbb{Z}} M_i, \quad \text{with} \quad M_i = \bigoplus_{\substack{\mathbf{a} \in \mathbb{Z}^n \\ |\mathbf{a}| = i}} M_{\mathbf{a}}.$$

Let $M$ be a finitely generated $\mathbb{Z}^n$-graded $S = K[x_1, \ldots, x_n]$-module. It is also $\mathbb{Z}$-graded,i.e.,

$$M = \bigoplus_{i \in \mathbb{Z}} M_i, \quad \text{with} \quad M_i = \bigoplus_{\substack{\mathbf{a} \in \mathbb{Z}^n \\ |\mathbf{a}| = i}} M_{\mathbf{a}}.$$

$$H_M(t) = \sum_{i \in \mathbb{Z}} \dim_K M_i t^i$$

Let $M$ be a finitely generated $\mathbb{Z}^n$-graded $S = K[x_1, \ldots, x_n]$-module. It is also $\mathbb{Z}$-graded,i.e.,

$$M = \bigoplus_{i \in \mathbb{Z}} M_i, \quad \text{with} \quad M_i = \bigoplus_{\substack{\mathbf{a} \in \mathbb{Z}^n \\ |\mathbf{a}| = i}} M_{\mathbf{a}}.$$

$$H_M(t) = \sum_{i \in \mathbb{Z}} \dim_K M_i t^i = \frac{Q(t)}{(1-t)^d}, \quad Q(1) \neq 0$$

is the Hilbert series of $M$.

Let $M$ be a finitely generated $\mathbb{Z}^n$-graded $S = K[x_1, \ldots, x_n]$-module. It is also $\mathbb{Z}$-graded,i.e.,

$$M = \bigoplus_{i \in \mathbb{Z}} M_i, \quad \text{with} \quad M_i = \bigoplus_{\substack{\mathbf{a} \in \mathbb{Z}^n \\ |\mathbf{a}| = i}} M_{\mathbf{a}}.$$

$$H_M(t) = \sum_{i \in \mathbb{Z}} \dim_K M_i t^i = \frac{Q(t)}{(1-t)^d}, \quad Q(1) \neq 0$$

is the Hilbert series of $M$.

Example: $H_S(t) = \frac{1}{(1-t)^n}$ for $S = K[x_1, \ldots, x_n]$.

Given a Stanley decomposition

$$\mathcal{D}: \; M = \bigoplus_{j=1}^{r} m_j K[Z_j], \quad \deg m_j = a_j, \; b_j = |Z_j|.$$

Given a Stanley decomposition

$$\mathcal{D}: \; M = \bigoplus_{j=1}^{r} m_j K[Z_j], \quad \deg m_j = a_j, \; b_j = |Z_j|.$$

One obtains

$$\mathcal{S}: \; H_M(t) = \sum_{j=1}^{r} \frac{t^{a_j}}{(1-t)^{b_j}}.$$

Given a Stanley decomposition

$$\mathcal{D}: \; M = \bigoplus_{j=1}^{r} m_j K[Z_j], \quad \deg m_j = a_j, \; b_j = |Z_j|.$$

One obtains

$$\mathcal{S}: \; H_M(t) = \sum_{j=1}^{r} \frac{t^{a_j}}{(1-t)^{b_j}}.$$

For any such sum decomposition $\mathcal{S}$ of $H_M(t)$ we set

$$\operatorname{hdepth} \mathcal{S} = \min\{b_1, \ldots, r_r\}$$

, and calls

$$\operatorname{hdepth} M = \max\{\operatorname{hdepth} \mathcal{S} : \; \mathcal{S} \text{ is a sum decomposition of } H_M(t)\}.$$

the Hilbert depth of $M$.

Given a Stanley decomposition

$$\mathcal{D}:\ M = \bigoplus_{j=1}^{r} m_j K[Z_j], \quad \deg m_j = a_j,\ b_j = |Z_j|.$$

One obtains

$$\mathcal{S}:\ H_M(t) = \sum_{j=1}^{r} \frac{t^{a_j}}{(1-t)^{b_j}}.$$

For any such sum decomposition $\mathcal{S}$ of $H_M(t)$ we set

$$\operatorname{hdepth} \mathcal{S} = \min\{b_1, \ldots, r_r\}$$

, and calls

$$\operatorname{hdepth} M = \max\{\operatorname{hdepth} \mathcal{S} :\ \mathcal{S} \text{ is a sum decomposition of } H_M(t)\}.$$

the Hilbert depth of $M$. Obviously, $\operatorname{hdepth} M \geq \operatorname{sdepth} M$.

The inequality $\operatorname{hdepth} M \geq \operatorname{depth} M$ is unknown in general.

The inequality $\operatorname{hdepth} M \geq \operatorname{depth} M$ is unknown in general.

The Hilbert depth has been computed is some interesting special cases

- (Bruns, Krattenthaler, Uliczka) Let $M(n,k)$ be the $k$-syzygy module of $K = S/(x_1, \dots, x_n)$. Then

$$\operatorname{hdepth} M(n,k) = n - 1 \quad \mathit{for} \lfloor n/2 \rfloor \leq k < n,$$

and

$$\operatorname{hdepth} M(n,k) \geq \operatorname{sdepth} M(n,k) \geq \lfloor (n+k)/2 \rfloor$$

for $k < n/2$.

The inequality $\operatorname{hdepth} M \geq \operatorname{depth} M$ is unknown in general.

The Hilbert depth has been computed is some interesting special cases

- (Bruns, Krattenthaler, Uliczka) Let $M(n,k)$ be the $k$-syzygy module of $K = S/(x_1,\dots,x_n)$. Then

$$\operatorname{hdepth} M(n,k) = n-1 \quad \textit{for} \lfloor n/2 \rfloor \leq k < n,$$

  and

$$\operatorname{hdepth} M(n,k) \geq \operatorname{sdepth} M(n,k) \geq \lfloor (n+k)/2 \rfloor$$

  for $k < n/2$.

- (Bruns, Krattenthaler, Ulizcka) $\operatorname{hdepth}(x_1,\dots,x_n)^k = \lceil n/(k+1) \rceil$

# Stanley depth for syzygies

We have

$$\operatorname{hdepth} M(n,k) = n-1 \geq \operatorname{depth} M(n,k) = k$$

for $\lfloor n/2 \rfloor \leq k < n$.

# Stanley depth for syzygies

We have

$$\operatorname{hdepth} M(n,k) = n-1 \geq \operatorname{depth} M(n,k) = k$$

for $\lfloor n/2 \rfloor \leq k < n$.

In general, if $M$ is a finitely generated $\mathbb{Z}^n$-graded of depth $t$, and $Z_k(M)$ is its $k$th syzygy module, then

$$\operatorname{depth} Z_k(M) = t+k \quad \text{for } k = 1, \ldots, n-t.$$

In particular for $I \subset S$ one has $\operatorname{depth} I \geq \operatorname{depth} S/I + 1$.

# Stanley depth for syzygies

We have

$$\operatorname{hdepth} M(n,k) = n-1 \geq \operatorname{depth} M(n,k) = k$$

for $\lfloor n/2 \rfloor \leq k < n$.

In general, if $M$ is a finitely generated $\mathbb{Z}^n$-graded of depth $t$, and $Z_k(M)$ is its $k$th syzygy module, then

$$\operatorname{depth} Z_k(M) = t+k \quad \text{for } k = 1, \dots, n-t.$$

In particular for $I \subset S$ one has $\operatorname{depth} I \geq \operatorname{depth} S/I + 1$.

It is not known whether $\operatorname{sdepth} I = \operatorname{sdepth} S/I + 1$.

# Stanley depth for syzygies

We have

$$\operatorname{hdepth} M(n,k) = n-1 \geq \operatorname{depth} M(n,k) = k$$

for $\lfloor n/2 \rfloor \leq k < n$.

In general, if $M$ is a finitely generated $\mathbb{Z}^n$-graded of depth $t$, and $Z_k(M)$ is its $k$th syzygy module, then

$$\operatorname{depth} Z_k(M) = t + k \quad \text{for } k = 1, \ldots, n-t.$$

In particular for $I \subset S$ one has $\operatorname{depth} I \geq \operatorname{depth} S/I + 1$.

It is not known whether $\operatorname{sdepth} I = \operatorname{sdepth} S/I + 1$.

**Theorem** (Floystad, H) $\operatorname{sdepth} Z_k(M) \geq k$ for all $k$.

- J. Apel, On a conjecture of R. P. Stanley; Part I - Monomial Ideals, J. of Alg. Comb. 17, (2003), 3956.
- J. Apel, On a conjecture of R. P. Stanley; Part II - Quotients Modulo Monomial Ideals, J. of Alg. Comb. 17, (2003), 5774.
- W.Bruns, J.Herzog, Cohen–Macaualy rings, Cambridge University Press, Revised edition (1998)
- W.Bruns, C.Krattenthaler, J.Ulizka, Stanley depth and Hilbert decompositions in the Koszul complex, arXiv:0909.0686.
- W. Bruns, C.Krattenthaler, J.Uliczka, Hilbert depth of powers of the maximal ideal, arXiv:1002.1400
- M. Cimpoeas, A note on Stanleys conjecture for monomial ideals, arXiv:0906.1303
- M.Keller, Y.-H.Shen, N.Streib, S.Young, On the Stanley depth of squarefree Veronese ideals, arXiv:0910.4645.
- G.Floystad, J. Herzog, Gröbner bases of syzygies and Stanley depth

J.Herzog, M.Vladiou, X.Zheng, How to compute the Stanley depth of a monomial ideal, J. Algebra **322** (2009), 3151–3169.

J. Herzog, A. SoleymanJahan and S. Yassemi, Stanley decompositions and partitionable simplicial complexes. J. Algebr. Comb. 27, 113125 (2008).

J. Herzog, A. Soleyman Jahan, X. Zheng, Skeletons of monomial ideals, Math. Nachr. 283 (2010), 14031408.

R. Okazaki, K. Yanagawa, Alexander duality and Stanley depth of multigraded modules.

D. Popescu, Stanley depth of multigraded modules, J. Algebra 321(2009), 2782-2797.

Y. Shen, Stanley depth of complete intersection monomial ideals and upper-discrete partitions, J. Algebra 321 (2009), 12851292.

R.Stanley, *Linear Diophantine equations and local cohomology*, Inventiones Mathematicae **68**, (1982),