Multiplicity-free case of Fomin-Fulton-Li-Poon conjecture

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Fomin-Fulton-Li-Poon conjecture

[FFLP] Fomin, S.; Fulton, W.; Li, C.-K.; Poon, Y.-T., *Eigenvalues, singular values, and Littlewood-Richardson coefficients.* Amer. J. Math. 127 (2005), no. 1, 101127. arXiv:math.AG/0301307.

- Given a pair of partitions (μ, ν), the $\star$-operation builds a new pair of partitions (λ, ρ) from the sizes of the parts of μ and ν,

$$(\mu, \nu) \longrightarrow (\mu, \nu)^\star = (\lambda, \rho), \quad |\mu| + |\nu| = |\lambda| + |\rho|.$$

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$$c^\theta_{\mu,\nu} \le c^\theta_{\lambda,\rho}, \qquad s_\lambda s_\rho - s_\mu s_\nu \quad \text{is Schur positive.}$$

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- The fixed points of the $\star$-operation are the pairs of partitions $\nu_1 \geq \mu_1 \geq \nu_2 \geq \mu_2 \geq \cdots$. The effect of the $\star$-operation is such that the pair (λ, ρ) is closer to be interlaced than (μ, ν).

The $\star$-operation

The $\star$-operation of Fomin, Fulton, Li and Poon

$$\lambda_k := \mu_k - k + \#\{j : 1 \leq j \leq n, \nu_j - j \geq \mu_k - k\},$$
$$\rho_j := \nu_j - j + 1 + \#\{k : 1 \leq k \leq n, \mu_k - k > \nu_j - j\}.$$

has an equivalent recursive definition obtained by Bergeron, Biagioli, Rosas [BBR](*Inequalities between Littlewood-Richardson coefficients*, J. Comb. Th., 2006)

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Lemma (BBR)

Let ν be a partition. Then

$$\underline{\nu} = \rho(0, \nu) = (\nu_1, \nu_2 - 1, \cdots, \nu_k - (k-1))$$
$$\overline{\nu} = \lambda(0, \nu) = (1, 2, \cdots, k-1, \nu_{k+1}, \cdots, \nu_n)_{\geq}$$

where k is the side of the Durffee square of ν.

$c^{\nu}_{0,\nu} = 1 = c^{\nu}_{\underline{\nu},\overline{\nu}}$.

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Lemma (BBR)

The conjecture holds for the pair (μ,ν) iff it holds for the pair (ν',μ').

Littlewood-Richardson rule

- A semi-standard tableau T with content $\mu = (\mu_1, \ldots, \mu_s)'$ whose word is a shuffle of the s words $12\cdots\mu_1$, $12\cdots\mu_2$, $\ldots$, $12\cdots\mu_s$ is said to be a *Littlewood–Richardson tableau* of content μ.

- Littlewood-Richardson tableaux of shape $\theta/\nu = (6544221)/(6421)$ and contents respectively $(3,3,2,2,1)'$ and $(6,4,1)'$

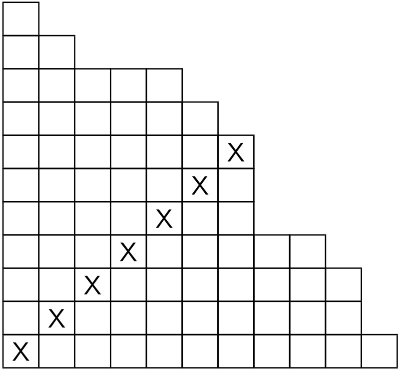
x					
x	x				
x	x				
	x	x	x		
		x	x		
				x	

3					
2	3				
1	2				
	1	2	2		
		1	1		
				1	

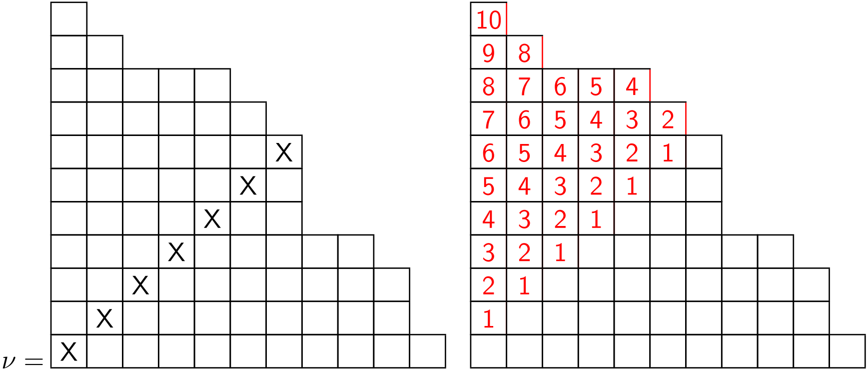
6					
4	5				
3	4				
	1	2	3		
		1	2		
				1	

.

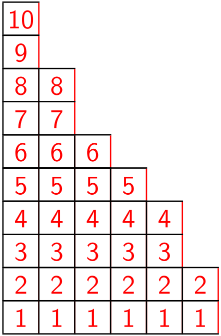
The ⋆-operation

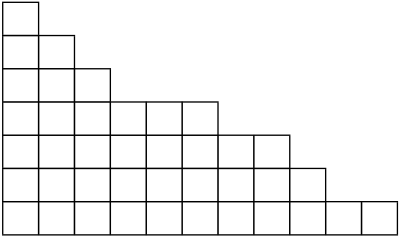
$\nu =$ 

The $\star$-operation



$\star$-operation

$(0, \nu) \rightarrow \overline{\nu} =$ 

, $\underline{\nu} =$ 

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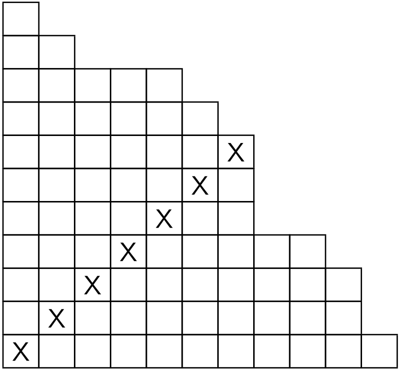
$\nu =$ 

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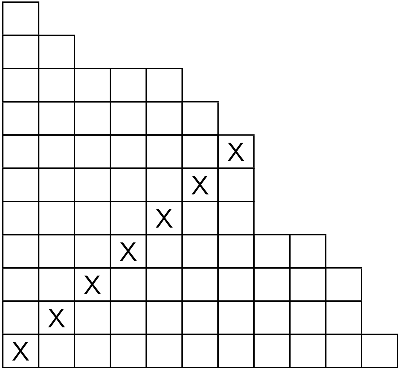
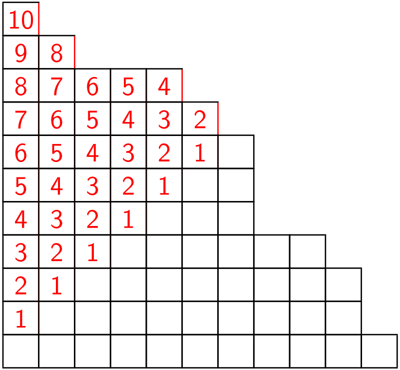
$\nu =$ 

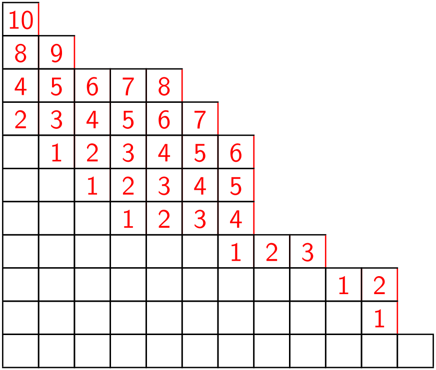


Canonical filling

There is only one LR tableau of shape $\nu/\underline{\nu}$ *and content* $\overline{\nu}$, the LR tableau of shape $\nu/\underline{\nu}$ with maximal filling with respect to the dominance order. This filling is said to be the *canonical filling* of ν.

$$c^{\nu}_{0,\nu} = 1 = c^{\nu}_{\underline{\nu},\overline{\nu}}.$$

$(\nu/0, \nu) \longrightarrow (\nu/\underline{\nu}, \overline{\nu}) =$



Augmented canonical filling

The *augmented canonical filling* $C(\nu)$ is defined to be the (infinite) tableau obtained by drawing canonical filling of $\nu/\underline{\nu}$, augmented in a way that each row starts as in the canonical filling and then increases by one from left to right. For each row $k > \ell(\nu)$, we start with entry k.

$C(\nu) =$

13	14	...													
12	13	14	...												
10	11	12	13	...											
8	9	10	11	12	...										
4	5	6	7	8	9	10	...								
2	3	4	5	6	7	8	9	...							
	1	2	3	4	5	6	7	8	...						
		1	2	3	4	5	6	7	8	...					
			1	2	3	4	5	6	7	...					
						1	2	3	4	5	6	...			
									1	2	3	4	5	...	
										1	2	3	4	...	
											1	2	3	4	...

Enriched $\star$- operation

One computes the image of $(\mu, \nu)^\star$ for an arbitrary partition μ, building from the computation of the canonical filling of ν.

Lemma

The image of (μ, ν) under the enriched star operation is the pair of semistandard Young tableaux (λ, ρ) where,

1. To compute $\lambda(\mu, \nu)$, we place in the kth row of $\bar{\nu}$ a sequence consisting of j times letter k, where j is the number of columns of $C(\nu)$ that do not contain entry k and that are $\leq \mu_k$.
2. To compute $\rho(\mu, \nu)$, we place each entry k of $C(\nu)$ to the right of $\underline{\nu}$ and in its original position, as long as k belongs to a column of $C(\nu)$ that is $\leq \mu_k$.

Multiplicity–free pairs of partitions

- A pair of partitions (μ, ν) is multiplicity–free if the Littlewood–Richardson coefficients $c^{\theta}_{\mu,\nu}$ are always either 0 or 1, for all partitions θ.

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- J. Stembridge, Ann. Comb. 2001, characterized all multiplicity–free pairs of partitions.

Theorem

The product $s_\mu s_\nu$ is multiplicity-free if and only if

(a) μ or ν is a one-line rectangle (Pieri rules), or
(b) μ and ν are both rectangles, or
(c) μ is rectangle and ν is a near rectangle or vice-versa, or
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The validity of FFLP conjecture for Stembridge's shapes

- *Shape-by-shape* proof.

  We describe a family of moves on fillings of tableaux that allow us to explicitly construct a Littlewood-Richardson filling of type (ρ, λ, θ) from a Littlewood-Richardson filling of type (ν, μ, θ).

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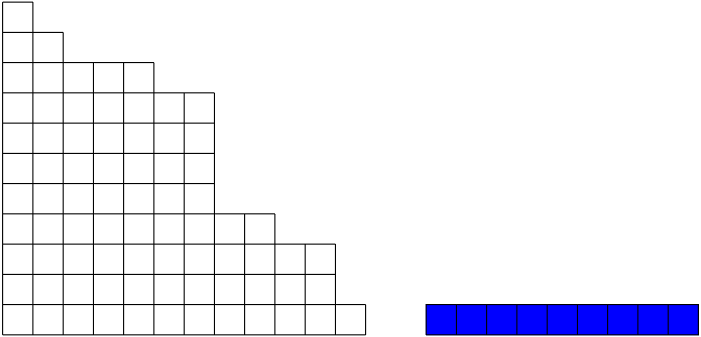
- $c^{\theta}_{\nu,\mu} = 1 \leq c^{\theta}_{\rho,\lambda}$.

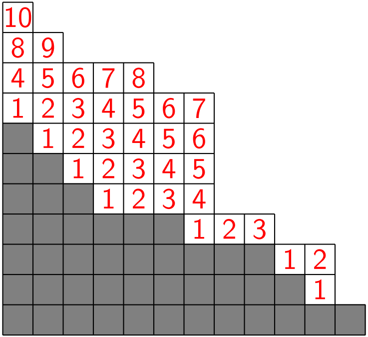
Row rectangle

- $(\nu, \mu = (9))$



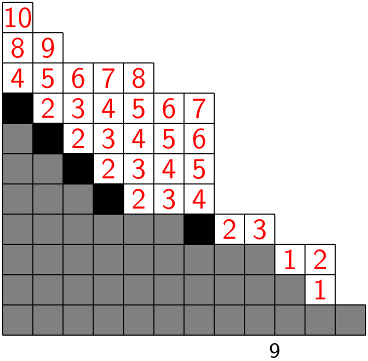
Row rectangle

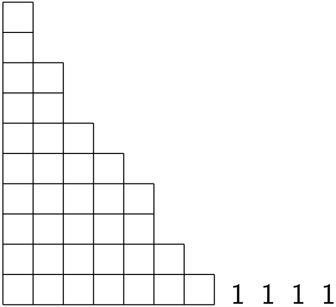
- $(\nu/\underline{\nu}, \overline{\nu})$



Row rectangle

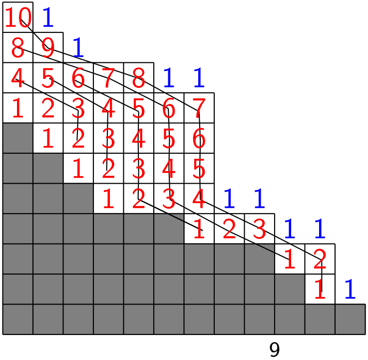
- $(\nu, \mu = (9)) \longrightarrow (\rho, \lambda)$.





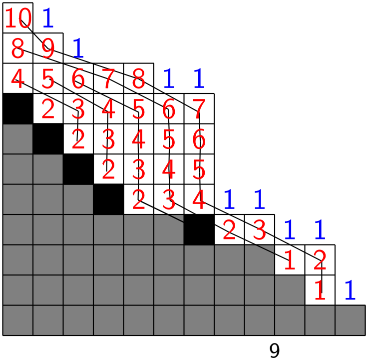
Row rectangle

- $(\nu, \mu = (9), \theta)$, $c^{\theta}_{\nu,\mu} = 1$



Row rectangle

- (ρ, λ, θ)



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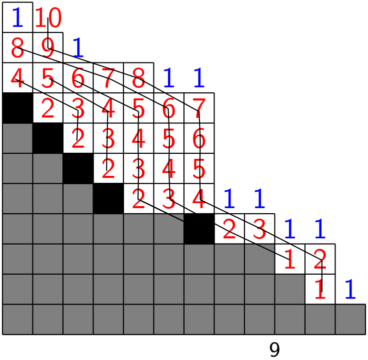

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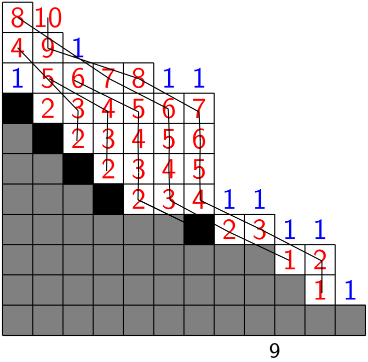

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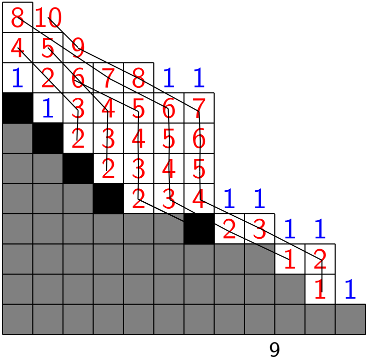

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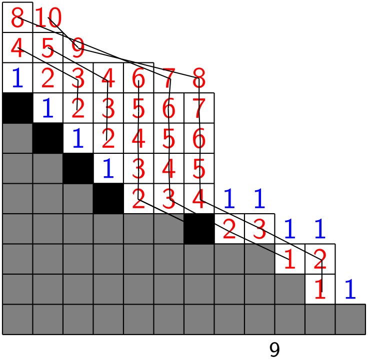

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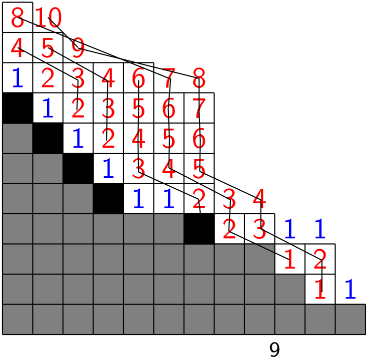
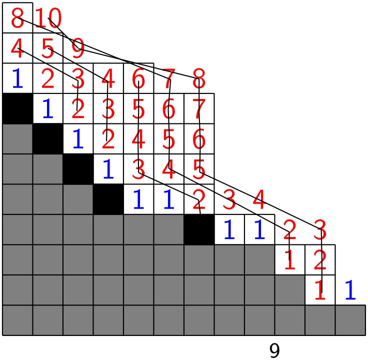


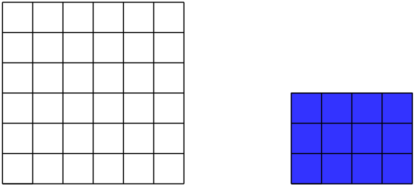
Row rectangle

- (ρ, λ, θ), $\quad c^{\theta}_{\rho,\lambda} \geq 1.$



Rectangle, Rectangle

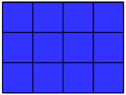
- Case 1: $\mu = (4^3) \subseteq \nu = (6^6)$



Rectangle, Rectangle

- Case 1: $\mu = (4^3) \subseteq \nu = (6^6)$

	1	2	3	4	5
		1	2	3	4
			1	2	3
				1	2
					1



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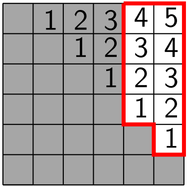
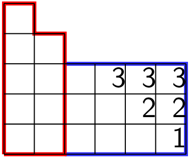
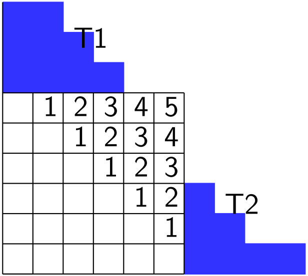


,



Rectangle, Rectangle

- Case 1: $\mu = (4^3) \subseteq \nu$, $c^{\theta}_{\nu,\mu} = 1$



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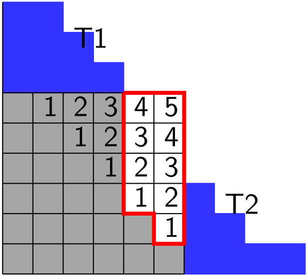
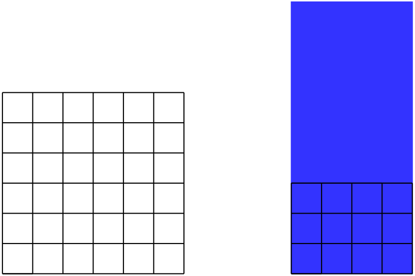


Tall Rectangle, Rectangle

- Case 2: $\ell(\mu) > \ell(\nu)$, $\mu = (4^9)$ $\nu = (6^6)$



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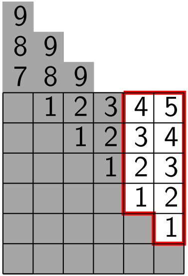

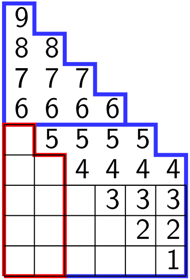

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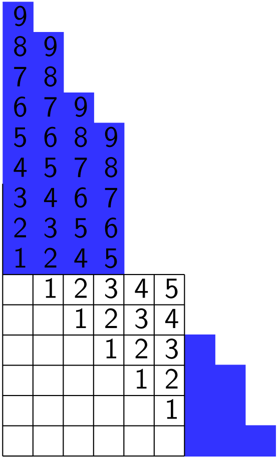
- Case 2: $c^{\theta}_{\nu,\mu} = 1$



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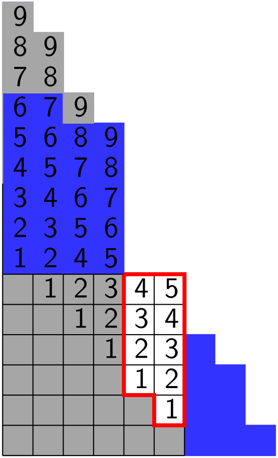

Tall Rectangle, Rectangle

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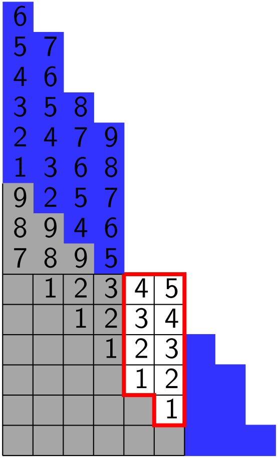
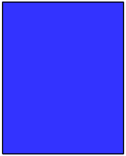


Leg Rectangle, Rectangle

- Case 1: $\nu = (6^6, 1^4)$, $\mu \subseteq 6^{6-1}$, $\mu = 4^5$,

8					
7					
6					
	1	2	3	4	5
		1	2	3	4
			1	2	3
				1	2
					1



Leg Rectangle, Rectangle

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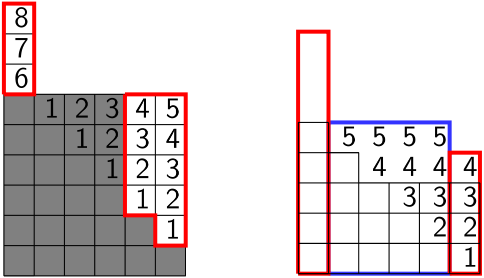

Leg Rectangle, Rectangle

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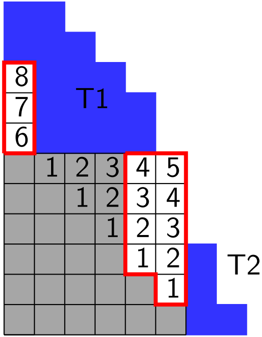

Leg Rectangle, Rectangle

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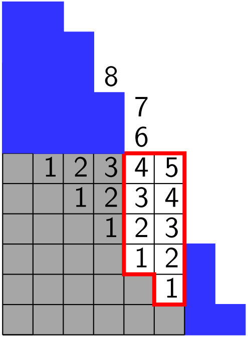

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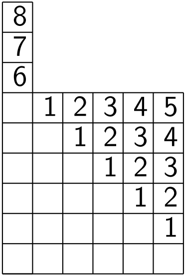
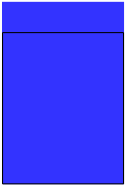
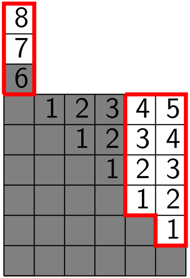
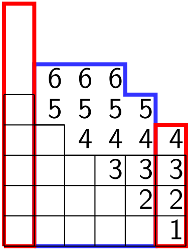




Leg Rectangle, Rectangle

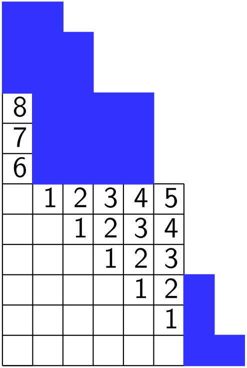
- Case 2: (ρ, λ)





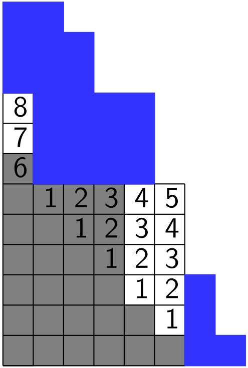
Leg Rectangle, Rectangle

- Case 2: $c^{\theta}_{\nu,\mu} = 1$.



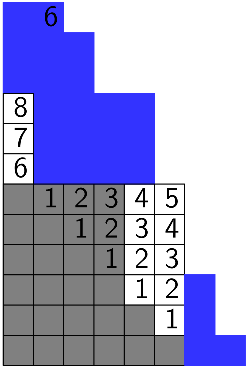
Leg Rectangle, Rectangle

- Case 2: (ν, μ, θ).



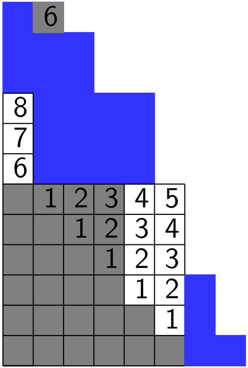
Leg Rectangle, Rectangle

- Case 2: (ρ, λ, θ).



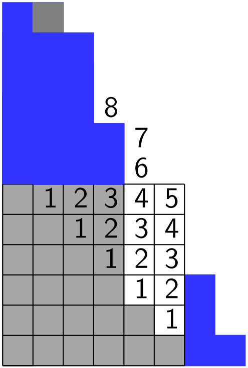
Leg Rectangle, Rectangle

- Case 2: (ρ, λ, θ).



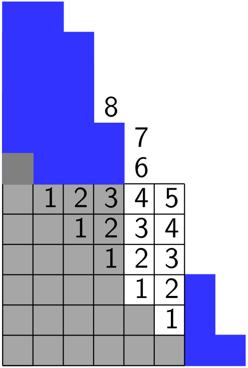
Leg Rectangle, Rectangle

- Case 2: (ρ, λ, θ)



Leg Rectangle, Rectangle

- Case 2: $c^{\theta}_{\rho,\lambda} \geq c^{\theta}_{\nu,\mu} = 1$



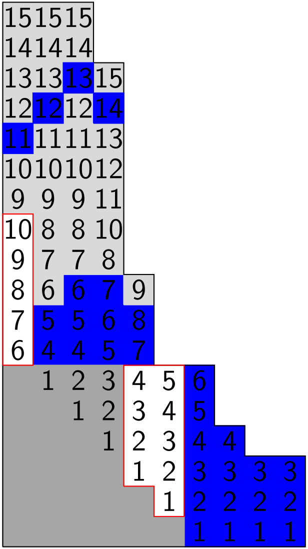
Leg Rectangle, Rectangle

- Case 3: $\mu = (4^{15})$

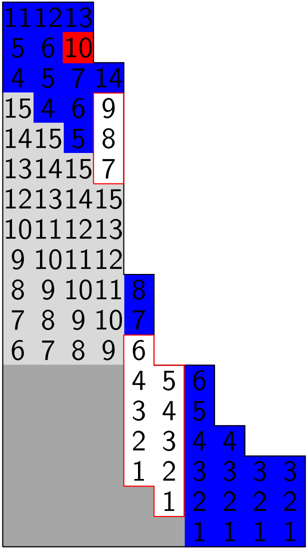
10					
9					
8					
7					
6					
	1	2	3	4	5
		1	2	3	4
			1	2	3
				1	2
					1

15	15	15	15
14	14	14	14
13	13	13	13
12	12	12	12
11	11	11	11
10	10	10	10
9	9	9	9
8	8	8	8
7	7	7	7
6	6	6	6
5	5	5	5
4	4	4	4
3	3	3	3
2	2	2	2
1	1	1	1

- Case 3: $(\nu, \mu = (4^{15}); \theta)$



- Case 3: $(\nu, \mu = (4^{15}); \theta)$



- Case 3: $(\rho, \lambda; \theta)$