Generalized hook lengths in symbols and partitions

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A new view of the

Hook formula

– inspired by recent work of Malle and Navarro on the characterization of nilpotent p-blocks of p-modular group algebras by the degrees of their ordinary characters.

Main aspect: for a given d, **separation of hooks** of a given partition into two multisets, according to its d-core and a suitable d-quotient.

B.-Gramain-Olsson: Generalized hook lengths in symbols and partitions, arXiv 1101.5067

For the symmetric groups and a prime p:

p-blocks $\leftrightarrow$ p-core partitions

Degree computation for irreducible characters:

hook formula

But: not adequate for the purpose ...

Malle-Navarro: new degree formula, deduced from a formula for character degrees for classical groups.

The complex irreducible characters of the symmetric group S_n are labelled by partitions of n,

$$\mathrm{Irr}(S_n) = \{[\lambda] \mid \lambda \vdash n\}$$

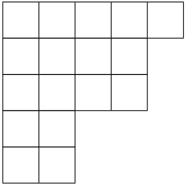
The character degrees $[\lambda](1)$ are given by:

Theorem (Hook formula)

Let $\prod \mathcal{H}(\lambda)$ be the product of all hook lengths in $\lambda \vdash n$. Then

$$[\lambda](1) = \frac{n!}{\prod \mathcal{H}(\lambda)} .$$

Let $\lambda = (5,4,4,2,2) \vdash 17$.



9	8	5	4	1
7	6	3	2	
6	5	2	1	
3	2			
2	1			

$$[\lambda](1) = \frac{17!}{9\cdot 8\cdot 5\cdot 4\cdot 1\cdot 7\cdot 6\cdot 3\cdot 2\cdot 6\cdot 5\cdot 2\cdot 1\cdot 3\cdot 2\cdot 2\cdot 1} = 1361360$$

Fix $d \in \mathbb{N}$.

For a partition λ, denote by

$$\lambda_{(d)} \text{ its } d\text{-core},$$

obtained by removing as many d-hooks as possible.

The removal may be described by the

$$d\text{-quotient } \lambda^{(d)},$$

a d-tuple of partitions.

Useful tool: β-sets and the d-abacus

A β-set is a finite subset of $\mathbb{N}_0$.

For a β-set $X = \{a_1, \ldots, a_s\}_>$, the associated partition $p(X)$ has as its parts the positive numbers among

$$a_i - (s - i),\ i = 1, \ldots, s.$$

For $k \in \mathbb{N}_0$,

$$X^{+k} = \{a + k \mid a \in X\} \cup \{k - 1, \ldots, 1, 0\}$$

is the kth shift of X.

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is the kth shift of X.

- $p(X) = p(X^{+k})$, for all $k \in \mathbb{N}_0$.
- Any β-set Y with $p(Y) = \lambda$ is called a β-set for λ.
- For any λ, the set of first column hook lengths is a β-set for λ.

Hook removal and the d-abacus

Let X be a β-set.
A d-hook of X is a pair $(a, b) \in \mathbb{N}_0^2$ with

$$a \in X,\ b < a,\ b \notin X \text{ and } a - b = d.$$

Removal of this d-hook from X means: replacing a by b.
(This corresponds to the removal of a d-hook from $\lambda = p(X)$.)

Place the elements of X as beads on an abacus with d runners. The removal of a d-hook corresponds to moving a bead to an empty space one level up. Easy computation of d-core!

Example

$X = \{11, 8, 6, 2, 0\}$ is a β-set of $p(X) = \lambda = (7, 5, 4, 1) \vdash 17$. Fix $d = 3$. The 3-abacus representation for X and the corresponding 3-core:

0	1	2
0	1	**2**
3	4	5
6	7	**8**
9	10	**11**

0	1	2
0	1	**2**
3	4	**5**
6	7	**8**
9	10	11

3-core $C_3(X) = \{8, 5, 3, 2, 0\}$

$$c_3(X) = p(C_3(X)) = p(\{8, 5, 3, 2, 0\}) = (4, 2, 1, 1) = \lambda_{(3)}$$

Theorem (Malle-Navarro)

Let p be a prime, $\lambda \vdash n$. Let $\mu \vdash r$ be the p-core of λ, S a symbol associated to the p-quotient $\lambda^{(p)}$, b_i the number of beads on the i^{th} runner of the p-abacus for μ, $c_i = pb_i + i - 1$. Then

$$[\lambda](1) = \frac{n!}{r!} \frac{1}{\prod_{h \text{ hook of } S} |p\ell(h) + c_{i(h)} - c_{j(h)}|} [\mu](1) \,.$$

Proof: by specialization at $q = 1$ of a formula for character degrees of unipotent characters of general linear groups due to Malle (1995).

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Proof: by specialization at $q = 1$ of a formula for character degrees of unipotent characters of general linear groups due to Malle (1995).

Suspicion: This is the hook formula in disguise.

A *d-symbol* is a d-tuple of β-sets $S = (X_0, \ldots, X_{d-1})$.

Let X be a β-set. For $j \in \{0, \ldots, d-1\}$ set

$$X_j^{(d)} = \{k \in \mathbb{N}_0 \mid kd + j \in X\}.$$

This gives a bijection

$$\begin{array}{cccc} s_d : & \{\beta\text{-sets}\} & \rightarrow & \{d\text{-symbols}\} \\ & X & \mapsto & (X_0^{(d)}, \ldots, X_{d-1}^{(d)}) \end{array}$$

A hook of S: $(a, b, i, j) \in \mathbb{N}_0^4$ with $i, j \in \{0, \ldots, d-1\}$, $a \in X_i$, $b \notin X_j$, and either $a > b$, or $a = b$ and $i > j$. $H(S)$ denotes the set of all hooks of S.

Remark. There are canonical bijections between the hooks in X, $\lambda = p(X)$ and $S = s_d(X)$.

Example

β-set $X = \{11, 8, 6, 2, 0\}$ for $p(X) = \lambda = (7, 5, 4, 1) \vdash 17$.
Let $d = 3$; 3-abacus representation for X and $S = s_3(X)$:

$$s_3 : \begin{array}{ccc} 0 & 1 & 2 \\ \hline \mathbf{0} & 1 & \mathbf{2} \\ 3 & 4 & 5 \\ \mathbf{6} & 7 & \mathbf{8} \\ 9 & 10 & \mathbf{11} \end{array} \quad \mapsto \quad \begin{array}{ccc} 0 & 1 & 2 \\ \hline \mathbf{0} & 0 & \mathbf{0} \\ 1 & 1 & 1 \\ \mathbf{2} & 2 & \mathbf{2} \\ 3 & 3 & \mathbf{3} \end{array}$$

$$S = (\{2, 0\}, \emptyset, \{3, 2, 0\})$$

Example: hook $(11, 4)$ in $X \leftrightarrow$ hook $(3, 1, 2, 1)$ in S.

A d-symbol $S = (X_0, \ldots, X_{d-1})$ is called balanced, if

$$|X_0| = \ldots = |X_{d-1}| \text{ and } 0 \notin X_i \text{ for some } i.$$

For any $S = (X_0, \ldots, X_{d-1})$, its balanced quotient is the unique *balanced* d-symbol

$$Q(S) = (X'_0, \ldots, X'_{d-1}) \text{ with } p(X'_i) = p(X_i) \text{ for all } i.$$

The core of S is the d-symbol $C(S)$ with ith component

$$\{|X_i| - 1, \ldots, 1, 0\},\ i = 0, \ldots, d-1.$$

If $X = s_d^{-1}(S)$, we define the balanced d-quotient of X

$$Q_d(X) = s_d^{-1}(Q(S))$$

and the d-quotient partition of $\lambda = p(X)$:

$$q_d(X) = p(Q_d(X))\,.$$

Example

Balanced quotient of $S = s_3(X) = (\{2,0\}, \emptyset, \{3,2,0\})$:

$$Q(S) = (\{2,0\}, \{1,0\}, \{2,1\}).$$

$$Q(S): \begin{matrix} \mathbf{0} & \mathbf{0} & 0 \\ 1 & \mathbf{1} & \mathbf{1} \\ \mathbf{2} & 2 & \mathbf{2} \\ 3 & 3 & 3 \end{matrix} \quad \xmapsto{s_3^{-1}} \quad Q_3(X): \begin{matrix} \mathbf{0} & \mathbf{1} & 2 \\ 3 & \mathbf{4} & \mathbf{5} \\ \mathbf{6} & 7 & \mathbf{8} \\ 9 & 10 & 11 \end{matrix}$$

$$q_3(X) = p(Q_3(X)) = p(\{8,6,5,4,1,0\}) = (3,2,2,2)$$

Note. $|q_3(X)| + |c_3(X)| = 9 + 8 = 17 = |p(X)|.$

Let $S = (X_0, \ldots, X_{d-1})$ be a d-symbol.
We consider only the hooks between the runners i and j:

$$\begin{aligned} H_{ij}(S) &= \{(a,b,i,j) \mid (a,b,i,j) \in H(S)\}, \\ H_{\{ij\}}(S) &= H_{ij}(S) \cup H_{ji}(S)\,. \end{aligned}$$

For $\ell \geq 0$ we define the ℓ-level section

$$H_{ij}^{\ell}(S) = \{(a,b,i,j) \in H_{ij}(S) \mid a - b = \ell\}\,.$$

Theorem (Hook correspondence in symbols)

Let S be a d-symbol with balanced quotient $Q(S) = Q$ and core $C(S) = C$.
For all i, j, we have bijective multiset correspondences

$$H_{\{ij\}}(S) \to H_{\{ij\}}(Q) \cup H_{\{ij\}}(C)\ ,$$

with control on the level sections.
We glue these bijections together to a universal bijection

$$\omega_S : H(S) \to H(Q) \cup H(C)\ .$$

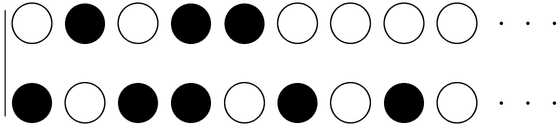
Remark. For $S = (X_0, \ldots, X_{d-1})$, $\Delta = |X_i| - |X_j|$ is crucial for controlling the correspondence of the level sections.

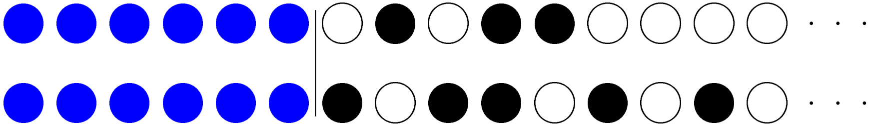
Theorem. Let S, Q, C be as above, $i \neq j$, $\Delta = |X_i| - |X_j| \geq 0$.
When $\Delta > 0$, we have the following equalities:

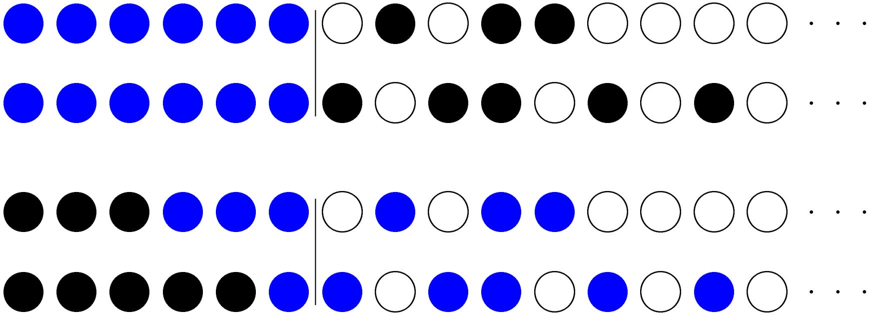
- For all $\ell > \Delta$: $|H_{ij}^{\ell}(S)| = |H_{ij}^{\ell-\Delta}(Q)|$.
- For all $\ell > \Delta$: $|H_{ji}^{\ell-\Delta}(S)| = |H_{ji}^{\ell}(Q)|$.
- For all $0 < \ell < \Delta$: $|H_{ij}^{\ell}(S)| = |H_{ji}^{\Delta-\ell}(Q)| + |H_{ij}^{\ell}(C)|$.
- For $\ell = \Delta$: $|H_{ij}^{\Delta}(S)| = \begin{cases} |H_{ij}^{0}(Q)| = |H_{\{ij\}}^{0}(Q)| & \text{if } i > j \\ |H_{ji}^{0}(Q)| = |H_{\{ij\}}^{0}(Q)| & \text{if } i < j \end{cases}$.
- For $\ell = 0$:
  $$|H_{ji}^{\Delta}(Q)| + |H_{ij}^{0}(C)| = \begin{cases} |H_{ij}^{0}(S)| = |H_{\{ij\}}^{0}(S)| & \text{if } i > j \\ |H_{ji}^{0}(S)| = |H_{\{ij\}}^{0}(S)| & \text{if } i < j \end{cases}.$$
- $|H_{ij}^{\Delta}(S)| + |H_{\{ij\}}^{0}(S)| = |H_{ji}^{\Delta}(Q)| + |H_{\{ij\}}^{0}(Q)| + |H_{ij}^{0}(C)|$.

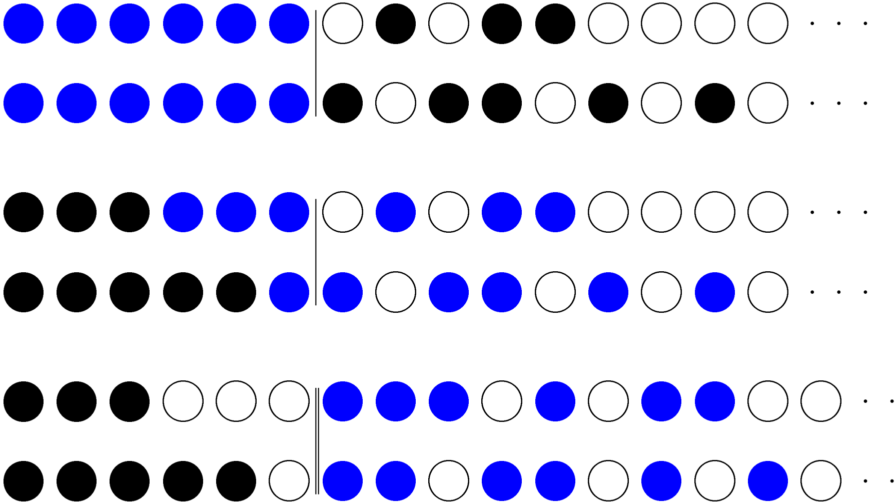
When $\Delta = 0$, we have

- $|H_{ij}^{\ell}(S)| = |H_{ij}^{\ell}(Q)|$, $H_{ij}^{\ell}(C) = \emptyset$, for all $\ell \geq 0$.

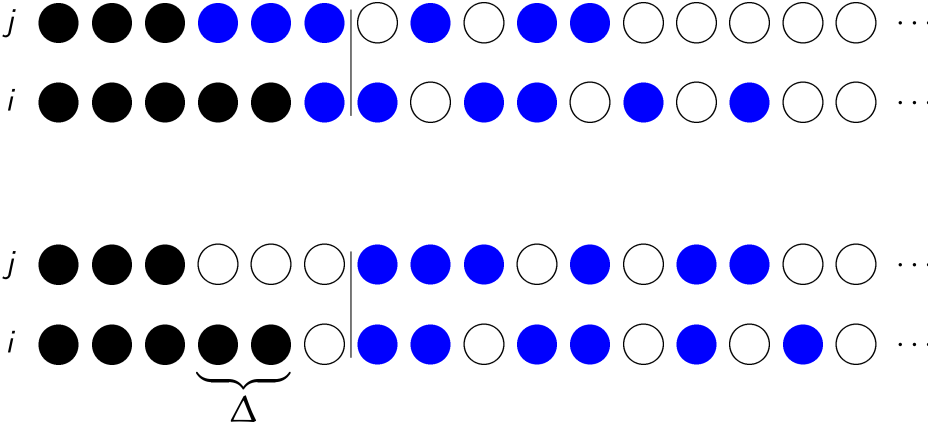


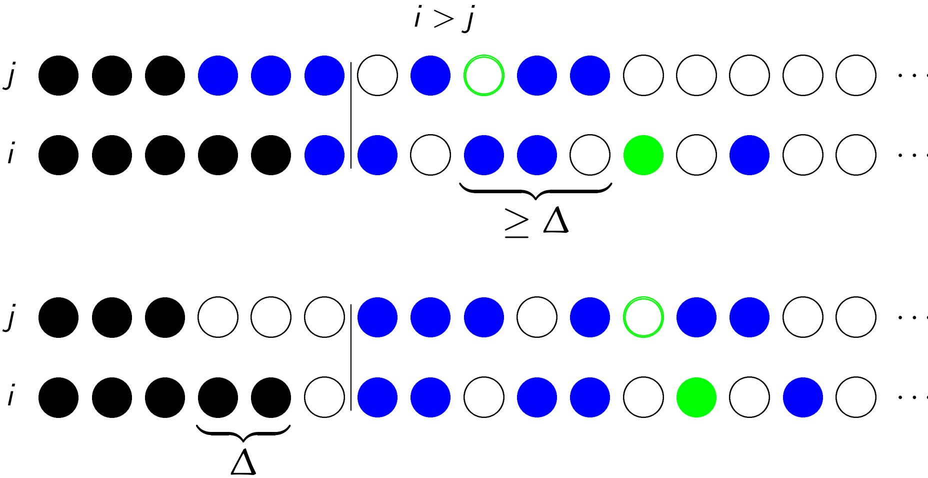


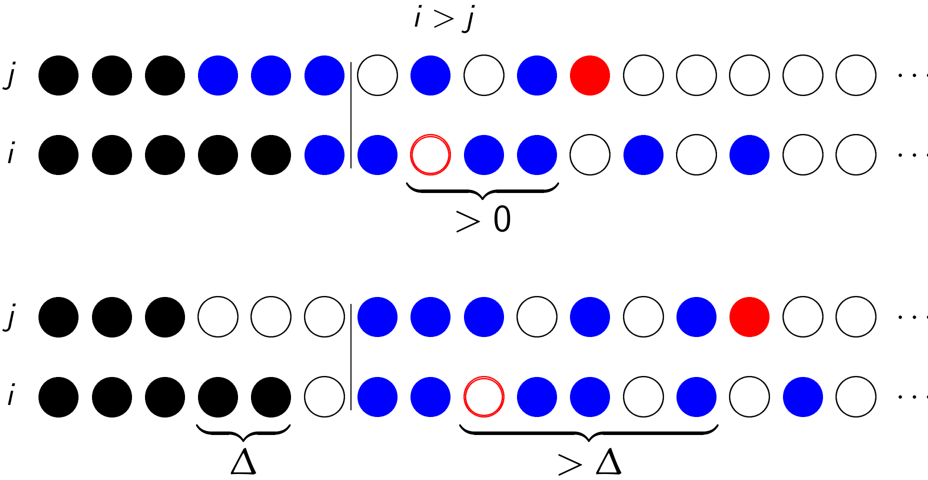




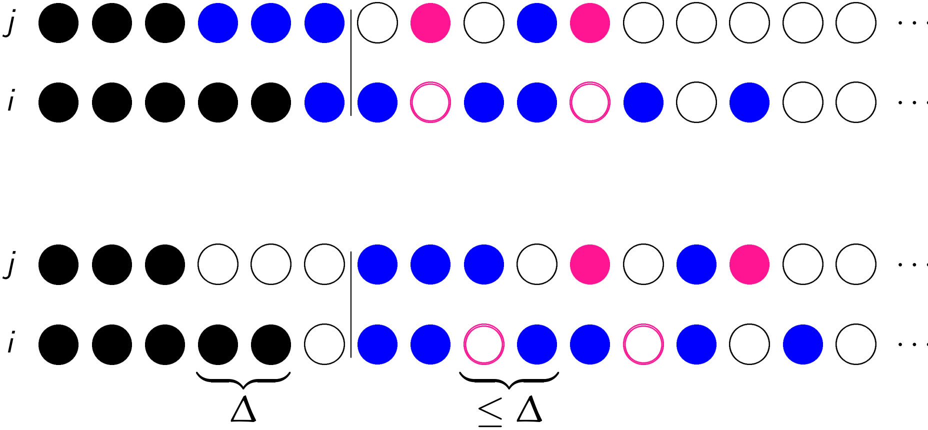
$i > j$

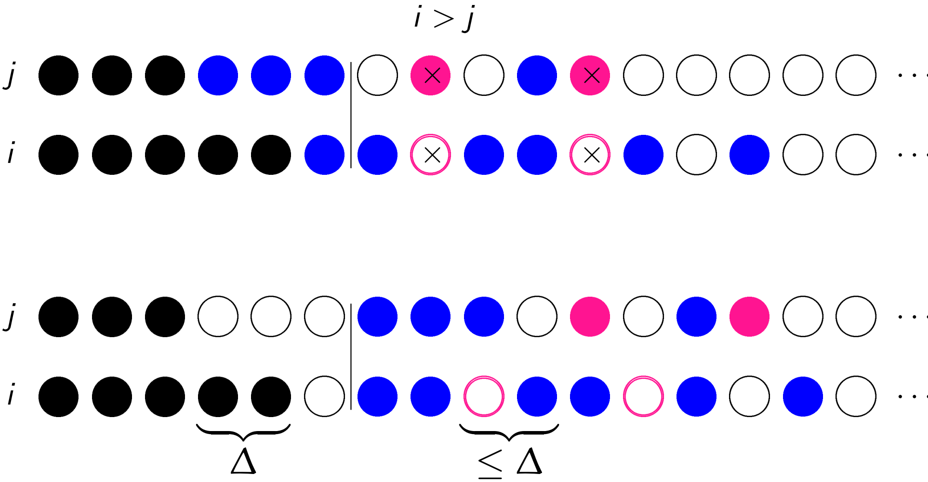



$i > j$
j
i
$\geq \Delta$
j
i
Δ

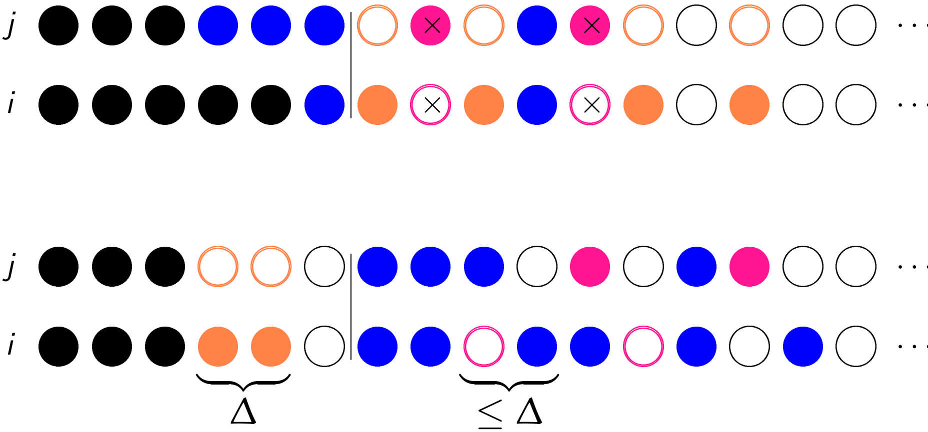

$i > j$
j
i
> 0
j
i
Δ
$> \Delta$

$i > j$




$i > j$
j
i
j
i
Δ
$\leq \Delta$

$i > j$



Let $H = \{(a, b, i, j) \mid a \geq b \text{ and } i > j \text{ if } a = b\}$.

Consider (generalized) hook length functions $h : H \to \mathbb{R}$ s.t.
the value $h(a, b, i, j)$ depends only on $\ell = a - b$, i and j.

Let $H = \{(a, b, i, j) \mid a \geq b \text{ and } i > j \text{ if } a = b\}$.

Consider (generalized) hook length functions $h : H \to \mathbb{R}$ s.t. **the value $h(a, b, i, j)$ depends only on $\ell = a - b$, i and j.**

Important hook length functions for d-symbols:
A d-hook data tuple is a $(d+1)$-tuple $\delta = (c_0, c_1, \ldots, c_{d-1}; k)$ of real numbers, $k \geq 0$.
We define the δ-length of $(a, b, i, j) \in H$ to be

$$h^\delta(a, b, i, j) = k(a - b) + c_i - c_j \,.$$

For any d-symbol S, we denote the multiset of generalized hook lengths by

$$\mathcal{H}^\delta(S) = \{h^\delta(a, b, i, j) \mid (a, b, i, j) \in H(S)\}.$$

Important special choices for applications:

- $\delta = (0, 1, \ldots, d-1; d)$ the partition d-hook data tuple.

  Then the δ-length of a hook of S equals the usual hook length $a - b$ of the corresponding hook (a, b) of X.

- $\delta = (0, 0, \ldots, 0; 1)$ the minimal d-hook data tuple.

  Then the δ-length of long hooks ($a > b$) in S coincides with the hook length in symbols as defined by Malle, and the short hooks ($a = b$) have δ-length 0.

Theorem (The Meta-Theorem)

Let $S = (X_0, X_1, \ldots, X_{d-1})$ be a d-symbol, $x_i = |X_i|$. Let $Q = Q(S)$ be its balanced quotient, $C = C(S)$ its core. Let $\delta = (c_0, c_1, \ldots, c_{d-1}; k)$ be a d-hook data tuple. Then with $\delta_S = (c_0 + x_0 k, c_1 + x_1 k, \ldots, c_{d-1} + x_{d-1} k; k)$, we have the multiset equality

$$\mathcal{H}^{\delta}(S) = \overline{\mathcal{H}}^{\delta_S}(Q) \cup \mathcal{H}^{\delta}(C),$$

where $\overline{\mathcal{H}}^{\delta_S}(Q)$ is the multiset of all modified hook lengths $\overline{h}^{\delta_S}(z)$, $z \in H(Q)$.

Modified hook lengths

We assume that i, j are such that $\Delta = x_i - x_j \geq 0$.
Let $H_{ij}^{\ell} = \{(a,b,i,j) \in H \mid a - b = \ell\}$.
Then for $z \in H_{\{ij\}}$ we define

$$\overline{h}^{\delta_S}(z) = \begin{cases} h^{\delta_S}(z) & \text{if } z \in H_{ij} \cup H_{ji}^{>\Delta}, \text{ or } z \in H_{ji}^{\Delta} \text{ if } i < j \\ -h^{\delta_S}(z) & \text{otherwise} \end{cases}$$

Crucial property w.r.t. the universal bijection ω_S:

$$h^{\delta}(z) = \begin{cases} h^{\delta}(\omega_S(z)) & \text{if} \quad \omega_S(z) \in H(C) \\ \overline{h}^{\delta_S}(\omega_S(z)) & \text{if} \quad \omega_S(z) \in H(Q) \end{cases}$$

Theorem

Let $d \in \mathbb{N}$, λ a partition, X a β-set for λ, $x_i = |X_i^{(d)}|$. Let $q_d(X)$ be the d-quotient partition of X.
For $z \in H(q_d(X))$, let $\overline{h}(z) = h(z) + (x_i - x_j)d$,
if z has hand and foot d-residue i and $j+1$, respectively.
Let $\overline{\mathcal{H}}(q_d(X))$ be the multiset of all $\overline{h}(z)$, $z \in H(q_d(X))$.
Then we have the multiset equality

$$\mathcal{H}(\lambda) = \mathcal{H}(\lambda_{(d)}) \cup \mathrm{abs}(\overline{\mathcal{H}}(q_d(X))$$

where $\mathrm{abs}(\overline{\mathcal{H}}(q_d(X)) = \{|h| \mid h \in \overline{\mathcal{H}}(q_d(X))\}$.

Corollary Generalization of the Malle-Navarro formula.
In particular, the Malle-Navarro formula **is** the hook formula!

Example

(cont.)

$\lambda = (7, 5, 4, 1)$, $X = \{11, 8, 6, 2, 0\}$, $d = 3$.

$S = (\{2, 0\}, \emptyset, \{3, 2, 0\})$, $q_3(X) = (3, 2, 2, 2)$, $\lambda_{(3)} = (4, 2, 1, 1)$.

We take $\delta = (0, 1, 2; 3)$, the partition data tuple.

As $(x_0, x_1, x_2) = (2, 0, 3)$, $\delta^S = (6, 1, 11; 3)$.

Hook diagrams for λ, $\lambda_{(3)}$, $q_3(X)$:

10	8	7	6	4	2	1
7	5	4	3	1		
5	3	2	1			
1						

7	4	2	1
4	1		
2			
1			

6	5	1
4	3	
3	2	
2	1	

Hook diagrams for λ, $\lambda_{(3)}$, $q_3(X)$:

10	8	7	6	4	2	1
7	5	4	3	1		
5	3	2	1			
1						

7	4	2	1
4	1		
2			
1			

6	5	1
4	3	
3	2	
2	1	

Consider the 3-residue diagram; we need to modify the hook lengths of $q_3(X)$ by $d(x_i - x_j)$ according to residues i and $j+1$ at the end of row and column. Finally, take absolute values!

Hook diagrams for λ, $\lambda_{(3)}$, $q_3(X)$:

10	8	7	6	4	2	1
7	5	4	3	1		
5	3	2	1			
1						

7	4	2	1
4	1		
2			
1			

6	5	1
4	3	
3	2	
2	1	

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0	1	2
2	0	
1	2	
0	1	

Hook diagrams for λ, $\lambda_{(3)}$, $q_3(X)$:

10	8	7	6	4	2	1
7	5	4	3	1		
5	3	2	1			
1						

7	4	2	1
4	1		
2			
1			

6	5	1
4	3	
3	2	
2	1	

Consider the 3-residue diagram; we need to modify the hook lengths of $q_3(X)$ by $d(x_i - x_j)$ according to residues i and $j+1$ at the end of row and column. Finally, take absolute values!

0	1	2
2	0	
1	2	
0	1	

	-9	-6	0
9	6	5	1
6	4	3	
9	3	2	
0	2	1	

Hook diagrams for λ, $\lambda_{(3)}$, $q_3(X)$:

10	8	7	6	4	2	1
7	5	4	3	1		
5	3	2	1			
1						

7	4	2	1
4	1		
2			
1			

6	5	1
4	3	
3	2	
2	1	

Consider the 3-residue diagram; we need to modify the hook lengths of $q_3(X)$ by $d(x_i - x_j)$ according to residues i and $j+1$ at the end of row and column. Finally, take absolute values!

0	1	2
2	0	
1	2	
0	1	

	−9	−6	0
9	6	5	1
6	4	3	
9	3	2	
0	2	1	

$\rightarrow$

6	8	10
1	3	
3	5	
−7	−5	

Hook diagrams for λ, $\lambda_{(3)}$, $q_3(X)$:

10	8	7	6	4	2	1
7	5	4	3	1		
5	3	2	1			
1						

7	4	2	1
4	1		
2			
1			

6	5	1
4	3	
3	2	
2	1	

Consider the 3-residue diagram; we need to modify the hook lengths of $q_3(X)$ by $d(x_i - x_j)$ according to residues i and $j+1$ at the end of row and column. Finally, take absolute values!

0	1	2
2	0	
1	2	
0	1	

	-9	-6	0
9	6	5	1
6	4	3	
9	3	2	
0	2	1	

$\rightarrow$

6	8	10
1	3	
3	5	
-7	-5	

$\rightarrow$

6	8	10
1	3	
3	5	
7	5	

More general: combinatorics towards a relative hook formula for unipotent characters of classical groups ...

Theorem

Let $S = (X_0, X_1, \ldots, X_{d-1})$ *be a* d*-symbol,* $\delta = (0, \ldots, 0; 1)$ *the minimal* d*-hook data tuple,* $\ell \in \mathbb{N}$*. Let* C *be the* ℓ*-core and* Q *the balanced* ℓ*-quotient of* S*. Then*

$$\mathcal{H}^{\delta}(S) = \mathcal{H}^{\delta}(C) \cup abs(\mathcal{H}^{\delta_{\ell,s}}(Q))$$

where $abs(\mathcal{H}^{\delta_{\ell,s}}(Q))$ *is the multiset of all* $|h^{\delta_{\ell,s}}(z)|$*,* $z \in H(Q)$*,* $\delta_{\ell,S}$ *a modified* $d\ell$*-hook data tuple.*

Theorem

Let $S = (X_0, X_1, \ldots, X_{d-1})$ *be a* d*-symbol,* $\delta = (0, \ldots, 0; 1)$ *the minimal* d*-hook data tuple,* $\ell \in \mathbb{N}$, $e \in \{0, \ldots, d-1\}$*. Let* C *be the* (ℓ, e)*-core of* S*,* Q *the balanced* (ℓ, e)*-quotient of* S*. With* $\delta' = \delta_{\ell,\sigma(S)}$ *we have*

$$\mathcal{H}^{\delta}_{>0}(S) = \mathcal{H}^{\delta}_{>0}(C) \cup abs(\mathcal{H}^{\delta'}_{>0}(Q)) \,,$$

where $abs(\mathcal{H}^{\delta'}_{>0}(Q))$ *is the multiset of all non-zero* $|h^{\delta'}(z)|$*,* $z \in H(Q)$*.*