Symmetric group characters and generating functions for Kronecker and reduced Kronecker coefficients

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Symmetric functions

Symmetric functions of indeterminates $x = (x_1, x_2, \ldots, x_n)$

- Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition of length $\ell(\lambda) \leq n$
- Let $\delta = (n-1, n-2, \ldots, 1, 0)$
- Schur functions: $s_\lambda(x) = \dfrac{a_{\lambda+\delta}(x)}{a_\delta(x)} = \dfrac{|\, x_i^{\lambda_j+n-j} \,|}{|\, x_i^{n-j} \,|}$
- Special cases: $s_{(m)}(x) = h_m(x)$ and $s_{(1^m)}(x) = e_m(x)$
- Power sums: $p_k(x) = x_1^k + x_2^k + \cdots + x_n^k$ for $k = 1, 2, \ldots$
- Let $\rho = (1^{\alpha_1}, 2^{\alpha_2}, \cdots, n^{\alpha_n})$ and $z_\rho = \prod_{k=1}^{n} k^{\alpha_k} \alpha_k!$
- Power sum functions: $p_\rho(x) = p_1^{\alpha_1}(x)\, p_2^{\alpha_2}(x) \,\cdots\, p_n^{\alpha_n}(x)$

Symmetric functions

Ring of symmetric functions $\Lambda_n(x) = z[x_1, x_2, \ldots, x_n]^{S_n}$

- Schur function basis: $s_\lambda(x)$
- Products: $s_\mu(x)\, s_\nu(x) = \sum_\lambda c^\lambda_{\mu\nu}\, s_\lambda(x)$
- Littlewood-Richardson coefficients: $c^\lambda_{\mu\nu} \in \mathbb{N}$
- Scalar product: $\langle s_\mu(x)\,,\, s_\nu(x)\rangle = \delta_{\mu\nu}$

- Power sum function basis: $p_\rho(x)$
- Products: $p_\sigma(x)\, p_\tau(x) = p_{\sigma\cup\tau}(x)$
- Scalar product: $\langle p_\sigma(x)\,,\, p_\tau(x)\rangle = z_\sigma\, \delta_{\sigma\tau}$

Characters of the symmetric group S_n

- Irreps V^λ specified by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \vdash n$
- Classes C_ρ specified by $\rho = (1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n}) \vdash n$
- Characters: $\chi^\lambda_\rho = \mathrm{ch}\, V^\lambda(\pi)$ for $\pi \in C_\rho$

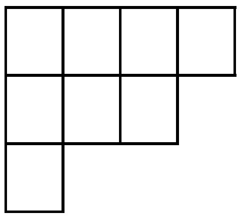
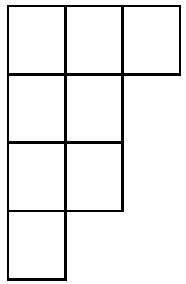
Frobenius $$p_\rho(x) = \sum_{\lambda \vdash n} \chi^\lambda_\rho \, s_\lambda(x) \qquad s_\lambda(x) = \sum_{\rho \vdash n} z_\rho^{-1} \, \chi^\lambda_\rho \, p_\rho(x)$$

- Orthogonality $$\sum_{\rho \vdash n} z_\rho^{-1} \, \chi^\lambda_\rho \, \chi^\mu_\rho = \delta_{\lambda\mu} \qquad \sum_{\lambda \vdash n} z_\rho^{-1} \, \chi^\lambda_\rho \, \chi^\lambda_\sigma = \delta_{\rho\sigma}$$
- Character formula $$\chi^\lambda_\rho = \langle s_\lambda \, , \, p_\rho \rangle = [x^{\lambda+\delta}] \, a_\delta(x) \, p_\rho(x)$$
  - $x^\kappa = x_1^{\kappa_1} \, x_2^{\kappa_2} \, \cdots \, x_n^{\kappa_n}$ for all $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n)$
  - $[x^\kappa] \, P(x)$ is the coefficient of $x_1^{\kappa_1} x_2^{\kappa_2} \cdots x_n^{\kappa_n}$ in $P(x)$

S_n character formulae

Special cases

- Identity rep $\chi_\rho^{(n)} = 1$ for all ρ
  - $\chi_\rho^{(n)}\,\chi_\rho^\lambda = \chi_\rho^\lambda$
- Sign rep $\chi_\rho^{(1^n)} = \epsilon(\pi)$ for any $\pi \in C_\rho$
  - with $\epsilon(\pi) = (-1)^{\alpha_2+\alpha_4+\cdots} = (-1)^{n-\ell(\rho)}$
  - $\chi_\rho^{(1^n)}\,\chi_\rho^\lambda = \chi_\rho^{\lambda'}$ where λ' is the conjugate of λ
  - Ex: If $\lambda = (4,3,1)$ then $\lambda' = (3,2,2,1)$

$F^{431} =$  $F^{3221} =$ 

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Schur function products

Products

- Outer product $s_\lambda \; s_\mu = \sum_\nu \; c^\nu_{\lambda\mu} \; s_\nu$
- Inner product $s_\lambda * s_\mu = \sum_\nu \; g_{\lambda\mu\nu} \; s_\nu$
- Reduced inner product $\langle s_\rho \rangle * \langle s_\sigma \rangle = \sum_\tau \; \overline{g}_{\rho\sigma\tau} \; \langle s_\tau \rangle$

  where $\langle s_\rho \rangle = \sum_{n\in\mathbb{Z}} \; s_{(n-r,\rho)}$ with $r = |\rho|$

Coprodults

- Let $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$
- Then $x + y = (x_1, \ldots, x_m, y_1, \ldots, y_n)$
  and $x \, y = (x_1 y_1, x_1 y_2, \ldots, x_m y_n)$
- Outer coproduct $s_\lambda(x+y) = \sum_{\mu,\nu} \; c^\lambda_{\mu\nu} \; s_\mu(x) \; s_\nu(y)$
- Inner coproduct $s_\lambda(x\,y) = \sum_{\mu,\nu} \; g_{\lambda\mu\nu} \; s_\mu(x) \; s_\nu(y)$

Kronecker coefficients

- Characters χ^λ_ρ form an orthogonal basis of the space of class functions of S_n.
- Product: $\chi^\mu_\rho \, \chi^\nu_\rho = \sum_\lambda g_{\lambda\mu\nu} \, \chi^\lambda_\rho$ with $\lambda, \mu, \nu, \rho \vdash n$
- Kronecker coefficients: $g_{\lambda\mu\nu} = \sum_\rho z_\rho^{-1} \, \chi^\lambda_\rho \, \chi^\mu_\rho \, \chi^\nu_\rho$
- Proof

$$\sum_\rho z_\rho^{-1} \, \chi^\lambda_\rho \, \chi^\mu_\rho \, \chi^\nu_\rho = \sum_\rho z_\rho^{-1} \, \chi^\lambda_\rho \sum_\zeta g_{\zeta\mu\nu} \, \chi^\zeta_\rho$$

$$= \sum_\zeta g_{\zeta\mu\nu} \sum_\rho z_\rho^{-1} \, \chi^\lambda_\rho \, \chi^\zeta_\rho = \sum_\zeta g_{\zeta\mu\nu} \, \delta_{\zeta\lambda} = g_{\lambda\mu\nu}$$

- Note: $g_{\lambda\mu\nu} \in \mathbb{Z}_{\geq 0}$ is symmetric in λ, μ, ν

Stability of Kronecker coeficients

Frobenius map $\chi^\lambda \;\mapsto\; s_\lambda$

- defines the inner product $s_\lambda * s_\mu = \sum_\lambda g_{\lambda\mu\nu}\, s_\nu$
- with $g_{\lambda\mu\nu} = \langle s_\lambda * s_\mu \,,\, s_\nu \rangle = \langle s_\lambda * s_\mu * s_\nu \,,\, s_{(n)} \rangle$

Reduced notation and stability Murnaghan, 1938

- $\lambda = (n-r, \rho) = \langle \rho \rangle$, $\mu = (n-s, \sigma) = \langle \sigma \rangle$, $\nu = (n-t, \tau) = \langle \tau \rangle$ with $\rho \vdash r$, $\sigma \vdash s$, $\tau \vdash t$ so that $g_{\lambda\mu\nu} = g_{\langle\rho\rangle\langle\sigma\rangle\langle\tau\rangle}$
- There exists $N \in \mathbb{N}$ and $\overline{g}_{\rho\sigma\tau}$ such that

$$g_{\lambda\mu\nu} = g_{\langle\rho\rangle\langle\sigma\rangle\langle\tau\rangle} = \overline{g}_{\rho\sigma\tau} \quad \text{for all } n \geq N$$

Reduced inner product Thibon, 1991

- Let $\langle s_\rho \rangle = \sum_{n\in\mathbb{Z}} s_{(n-r,\rho)}$ with $\rho \vdash r$
- Then $\langle s_\rho \rangle * \langle s_\sigma \rangle = \sum_\tau \overline{g}_{\rho\sigma\tau} \langle s_\tau \rangle$

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Generating functions

- Let $x = (x_1, \ldots, x_p)$, $y = (y_1, \ldots, y_q)$, $z = (z_1, \ldots, z_r)$.
- Complete homogeneous $\prod_{i=1}^{p} (1 - q\, x_i)^{-1} = \sum_{m \geq 0} q^m\, s_{(m)}(x)$
- Cauchy identity $\prod_{i,j=1}^{p,q} (1 - q\, x_i y_j)^{-1} = \sum_{\lambda \vdash m \geq 0} q^m\, s_\lambda(x) s_\lambda(y)$
- Stanley Enum Comb Vol 2, Exercise 7.78 (f and g).

$$\prod_{i,j,k=1}^{p,q,r} (1 - q\, x_i y_j z_k)^{-1} = \sum_{\lambda,\mu,\nu \vdash m \geq 0} q^m\, g_{\lambda\mu\nu}\, s_\lambda(x) s_\mu(y) s_\nu(z)$$

- $$\prod_{i,j,\ldots,k=1}^{p,q,\ldots,r} (1 - q\, x_i y_j \ldots z_k)^{-1} = \sum_{\lambda,\mu,\ldots,\nu \vdash m \geq 0} q^m\, g_{\lambda\mu\ldots\nu}\, s_\lambda(x) s_\mu(y) \ldots s_\nu(z)$$

where $g_{\lambda\mu\ldots\nu} = \sum_{\rho \vdash m} z_\rho^{-1} \chi_\rho^\lambda \chi_\rho^\mu \cdots \chi_\rho^\nu = \langle s_\lambda * s_\mu * \cdots * s_\nu, s_{(m)} \rangle$

Proof

- Let $M = \prod_{i,j,k=1}^{p,q,r} (1 - q\, x_i\, y_j\, z_k)^{-1}$

- Then $M = \sum_{m \geq 0} q^m\, s_{(m)}(xyz) = \sum_{\rho \vdash m \geq 0} q^m\, z_\rho^{-1}\, \chi_\rho^{(m)}\, p_\rho(xyz)$

  with $\chi_\rho^{(m)} = 1$ for all $\rho \vdash m \geq 0$

  and $p_\rho(xyz) = p_\rho(x)\, p_\rho(y)\, p_\rho(z)$ for all ρ

  and $p_\rho(x) = \sum_{\lambda \vdash m} \chi_\rho^\lambda\, s_\lambda(x)$

- Hence $M = \sum_{m \geq 0} q^m \sum_{\rho,\lambda,\mu,\nu \vdash m} z_\rho^{-1}\, \chi_\rho^\lambda\, \chi_\rho^\mu\, \chi_\rho^\nu\; s_\lambda(x)\, s_\mu(y)\, s_\nu(z)$

  $$= \sum_{\lambda,\mu,\nu \vdash m \geq 0} q^m\, g_{\lambda\mu\nu}\, s_\lambda(x) s_\mu(y) s_\nu(z)$$

- Note $g_{\lambda\mu\nu} = g_{\lambda'\mu'\nu} = g_{\lambda'\mu\nu'} = g_{\lambda\mu'\nu'}$

Grand generating function

Corollary Let $\lambda,\mu,\nu \vdash m$ have lengths p,q,r then

$$g_{\lambda\mu\nu} = [q^m\, x^\lambda\, y^\mu\, z^\nu]\ (GGF(q,xyz)) = [x^\lambda\, y^\mu\, z^\nu]\ (GGF(1,xyz))$$

with $$GGF(q,xyz) = \prod_{i,j,k=1}^{p,q,r} (1 - q\, x_i\, y_j\, z_k)^{-1}$$

$$\times \prod_{1\le i<j\le p} (1 - x_j/x_i) \prod_{1\le i<j\le q} (1 - y_j/y_i) \prod_{1\le i<j\le r} (1 - z_j/z_i)$$

Proof: $$\prod_{i,j,k=1}^{p,q,r} (1 - q\, x_i\, y_j\, z_k)^{-1} = \sum_{\lambda,\mu,\nu\vdash m\ge 0} q^m\, g_{\lambda\mu\nu}\, s_\lambda(x)\, s_\mu(y)\, s_\nu(z)$$

with $$s_\lambda(x) = a_{\lambda+\delta}(x)/a_\delta(x)$$

$$a_\delta(x) = \prod_{1\le i<j\le p} (x_i - x_j) = x^\delta \prod_{1\le i<j\le p} (1 - x_j/x_i)$$

$$a_{\lambda+\delta}(x) = \sum_{\pi\in S_p} \epsilon(\pi)\, x^{\pi(\lambda+\delta)} = x^\delta \sum_{\pi\in S_p} \epsilon(\pi)\, x^{\pi(\lambda+\delta)-\delta} = x^\delta \left(x^\lambda + nst\right)$$

Evaluation of Kronecker coefficients

- Simply pick out coefficients of $[x^\lambda y^\mu z^\nu]$ in $GGF(1, xyz)$
- Ex: $\lambda = (m-6,6)$, $\mu = (m-7,5)$, $\nu = (m-6,4,2)$ for various m

m	λ	μ	ν	$g_{\lambda\mu\nu}$
12	66	75	642	0
13	76	85	742	2
14	86	95	842	3
15	96	10, 5	942	4
16	10, 6	11, 5	10, 42	4

Simplified generating function

- Let $\lambda = (m - |\rho|, \rho)$, $\mu = (m - |\sigma|, \sigma)$, $\nu = (m - |\tau|, \tau)$
- Let $x \mapsto (1, u)$, $y \mapsto (1, v)$, $z \mapsto (1, w)$
- Under this map let $GGF(q, xyz) \mapsto SGF(q, uvw)$
- Then $g_{\lambda\mu\nu} = [q^m u^\rho v^\sigma w^\tau] \, (SGF(q, uvw))$
- Let $G_{\rho\sigma\tau}(q) = [u^\rho v^\sigma w^\tau] \, (SGF(q, uvw)) = \sum_{m \geq 0} \, g_{\lambda\mu\nu} \, q^m$
- Ex: $G_{6,5,42}(q)$

$$= \frac{1}{1-q} \, \left(-q^7 + q^9 - q^{10} + q^{11} + 2q^{13} + q^{14} + q^{15}\right)$$

$$= -q^7 - q^8 - q^{10} + 2q^{13} + 3q^{14} + 4q^{15} + 4q^{16} + 4q^{17} + 4q^{18} + \cdots$$

- Note: Stable limit $g_{\lambda\mu\nu} = \overline{g}_{\rho\sigma\tau} = 4$ for all $m \geq 15$

Evaluation of inner products

Consider the problem of evaluating $g_{\lambda\mu\nu}$ in the case $\lambda = (\lambda_1, \lambda_2)$, $\mu = (\mu_1, \mu_2)$, $\nu = (\nu_1, \nu_2, \nu_3)$

- We know that $g_{\lambda\mu\nu} = [x^\lambda y^\mu z^\nu] \left(GGF(1, xyz)\right)$
- We can restrict x, y, z to (x_1, x_2), (y_1, y_2), (z_1, z_2, z_3)
- But the expansion of $GGF(1, xyz)$ involves many terms $x^\xi y^\eta z^\zeta$ in which ξ, η, ζ are not all partitions
- In principle we can exploit the Andrews, Paule, Riese package Omega to tackle this.
- For example, given $H(x_1, x_2) = \sum_{m,n} c_{m,n} x_1^m x_2^n$ then
- Omega applied to $H(ax_1, x_2/a)$ returns $\sum_{m \geq n} c_{m,n} x_1^m x_2^n$

Example

In the case $\lambda = (\lambda_1, \lambda_2)$, $\mu = (\mu_1, \mu_2)$, $\nu = (\nu_1, \nu_2, \nu_3)$

- Let $N = (1 - \frac{x_2}{x_1})(1 - \frac{y_2}{y_1})(1 - \frac{z_2}{z_1})(1 - \frac{z_3}{z_1})(1 - \frac{z_3}{z_2})$
- Let $$D = (1 - x_1y_1z_1)(1 - x_1y_2z_1)(1 - x_2y_1z_1)(1 - x_2y_2z_1)$$ $$(1 - x_1y_1z_2)(1 - x_1y_2z_2)(1 - x_2y_1z_2)(1 - x_2y_2z_2)$$ $$(1 - x_1y_1z_3)(1 - x_1y_2z_3)(1 - x_2y_1z_3)(1 - x_2y_2z_3)$$
- Let $G := N/D$ with $x_1 \mapsto ax_1$, $x_2 \mapsto x_2/a$
  $y_1 \mapsto by_1$, $y_2 \mapsto y_2/b$
  $z_1 \mapsto cz_1$, $z_2 \mapsto dz_2/c$, $z_3 \mapsto z_3/d$
- Call Omega2.m then evaluate $\mathbf{OR}[G, \{a, b, c, d\}]$

Output

- Obtain new generating function $PS = N/D$ with
- $N = 1 + x^{21}y^{21}z^{21} + x^{32}y^{32}z^{221}$
  $-2x^{42}y^{42}z^{321} + x^{33}y^{42}z^{321} + x^{42}y^{33}z^{321}$
  $-x^{43}y^{52}z^{431} - x^{52}y^{43}z^{421} - x^{53}y^{53}z^{422} - x^{53}y^{53}z^{431}$
  $-x^{54}y^{63}z^{432} - x^{63}y^{54}z^{432} + x^{64}y^{73}z^{532} + x^{73}y^{64}z^{532} - 2x^{64}y^{64}z^{532}$
  $+x^{74}y^{74}z^{632} + x^{85}y^{85}z^{643} + x^{10,6}y^{10,6}z^{853}$
- $D = (1 - x^1y^1z^1)(1 - x^{11}y^{11}z^2)(1 - x^{11}y^2z^{11})(1 - x^2y^{11}z^{11})$
  $(1 - x^{21}y^{21}z^{111})(1 - x^{22}y^{22}z^{22})(1 - x^{22}y^{31}z^{211})(1 - x^{31}y^{22}z^{211})$
  $(1 - x^{33}y^{33}z^{222})$
- Expanding this gives $PS = \sum_{\lambda,\mu,\nu} g_{\lambda\mu\nu}\, x^\lambda\, y^\mu\, z^\nu$ with λ, μ, ν partitions of lengths $2, 2, 3$, respectively.
- First obtained by Patera and Sharp, 1980

Inner products in two-rowed case

- To evaluate $s_\lambda * s_\mu = \sum_\nu g_{\lambda\mu\nu}\, s_\nu$ in full we must take $p = \ell(\lambda), q = \ell(\mu), r = pq \geq \ell(\nu)$
- In the case $p = q = 2$ and $r = 4$ it is necessary to include in the PS formula one extra denominatior factor $(1 - x^{22}y^{22}z^{1111})$
- Then $g_{\lambda\mu\nu} = [x^\lambda y^\mu z^\nu](N/(D(1 - x^{22}y^{22}z^{1111}))$ with N and D as in the PS formula
- This was also obtained by Patera and Sharp, 1980
- An explicit formula for $g_{\lambda\mu\nu}$ in this two-rowed inner product case was first given by Remmel and Whitehead, 1994, with a combinatorial rule for their calculation supplied by Rosas, 2001.

Inner product $s_{dd} * s_{dd}$

- Of interest in complexity theory and the study of qubits
- To include all terms it is necessary to use $(p, q, r) = (2, 2, 4)$ in our GGF
- To restrict to terms of the form $x^{(dd)} y^{(dd)}$ use
  - $x_1 \mapsto a x_1$, $x_2 \mapsto x_2 / a$
  - $y_1 \mapsto b y_1$, $y_2 \mapsto y_2 / b$
  - and the Omega command $\mathbf{OEqR}[G, \{a, b\}]$
- To restrict to terms of the form z^ν with $\nu \vdash 2d$ use
  - $z_1 \mapsto e z_1$, $z_2 \mapsto f z_2 / e$, $z_3 \mapsto g z_3 / f$, $z_4 \mapsto z_4 / g$
  - and the Omega command $\mathbf{OR}[G, \{e, f, g\}]$

Inner product $s_{dd} * s_{dd}$ contd.

- Resulting generating function

$$\frac{1}{(1-x^{11}y^{11}z^{2})(1-x^{22}y^{22}z^{22})(1-x^{33}y^{33}z^{222})(1-x^{22}y^{22}z^{1111})}$$

- Expanding this gives

$$s_{dd} * s_{dd} = \sum_{\nu \vdash 2d} s_{\nu}$$

  The sum is over partitions $\nu = (\nu_1, \nu_2, \nu_3, \nu_4)$ whose 4 parts, including zeros, are either all even or all odd.

- Obtained by Garsia, Wallach, Xin and Zabrocki, 2010

Inner product $s_{dd} * s_{d+k,d-k}$

- In this case the generating function is $G = N/D$ with
  - $N = (1 + x^{33}y^{42}z^{321})$
  - $D = (1 - x^{11}y^{11}z^{2})(1 - x^{11}y^{2}z^{11})(1 - x^{22}y^{22}z^{22})$
    $(1 - x^{22}y^{31}z^{211})(1 - x^{33}y^{33}z^{222})(1 - x^{22}y^{22}z^{1111})$
- Let H be our previous generating function for $s_{dd} * s_{dd}$
- Then

$$G = H \frac{(1 + x^{33}y^{42}z^{321})}{(1 - x^{11}y^{2}z^{11})(1 - x^{22}y^{31}z^{211})}$$

- To identify terms contributing to $s_{dd} * s_{d+k,d-k}$ we require

$$[p^k] \left(\sum_{a,b,c} p^{a+b+c} (x^{33}y^{42}z^{321})^a \, (x^{11}y^{2}z^{11})^b \, (x^{22}y^{31}z^{211})^c \right)$$

Inner product $s_{dd} * s_{d+k,d-k}$ contd.

- It is only necessary to consider the terms in z, that is

$$[p^k]\left(\sum_{a,b,c} p^{a+b+c}\ z^{a(321)}\ z^{b(11)}\ z^{c(211)}\right)$$

  with $a \in \{0,1\}$, $b, c \in \{0,1,\ldots,k\}$ and $a+b+c=k$

- This reduces to

$$\sum_{a=0}^{1} \sum_{b=0}^{k-a} x^{(k+b+2a,k+a,b+a)}$$

- Hence $s_{dd} * s_{d+k,d-k} =$

$$\sum_{\delta}\left(\sum_{i=0}^{k} s_{\delta+(k+i,k,i)} + \sum_{i=1}^{k} s_{\delta+(1,1,1,1)+(k+i,k+1,i)}\right)$$

  where δ is summed over partitions of length ≤ 4 whose parts are all even

- Obtained by Brown, Willigenburg and Zabrocki, 2008

An application to higher order products

Ex: Coefficients $g_{(m-r_1,r_1)(m-r_2,r_2)\cdots(m-r_k,r_k)}$

- Generating function $GGF(q, x(1), x(2), \ldots, x(k))$
- Specialisation $x(i) = (1, w_i)$ for $i = 1, 2, \ldots, k$
- Numerator maps to $\prod_{i=1}^{k}(1 - w_i)$
- Denominator maps to $\prod_{S\subseteq[1,k]}(1 - q\prod_{i\in S} w_i)$
- Hence $g_{(m-r_1,r_1)(m-r_2,r_2)\cdots(m-r_k,r_k)} = [q^m w^r]SGF(q, w)$ with $w^r = w_1^{r_1} w_2^{r_2} \cdots w_k^{r_k}$ and

$$SGF(q,w) = \frac{\prod_{i=1}^{k}(1 - w_i)}{\prod_{S\subseteq[1,k]}(1 - q\prod_{i\in S} w_i)}$$

- Case $k = 3$: Welsh, 2009. All $k \geq 2$: Garsia, Wallach, Xin and Zabrocki, 2010, with a proof provided by Thibon

Special case of relevance to k-qubit systems

- Each qubit is a two-state system on which $SL(2,\mathbb{C})$ acts
- k-qubit system subject to action of $SL(2,\mathbb{C})^{\otimes k}$
- Number of invariants $g_{(dd)(dd)\cdots(dd)}$
- Required generating function $W_k(q) = \sum_{d\geq 0} q^{2d}\, g_{(dd)(dd)\cdots(dd)}$
- However $g_{(dd)(dd)\cdots(dd)} = [q^{2d}\, w^d] SGF(q,w)$ Hence:
- $$W_k(t) = [q^0 w_1^0 \cdots w_k^0]\; SGF(q,w) \frac{1}{1 - t^2/q^2\, w_1 \cdots w_k}$$
$$= [q^0 w_1^0 \cdots w_k^0]\; \frac{\prod_{i=1}^{k}(1-w_i)}{\prod_{S\subseteq[1,k]}(1 - q\prod_{i\in S} w_i)}\; \frac{1}{1 - t^2/q^2\, w_1 \cdots w_k}$$
- An equivalent $k=4$ generating function is due to Briand, Luque and Thibon 2003

Contents

- Symmetric functions and the symmetric group
  - Schur functions $s_\lambda(x)$ and power sum functions $p_\rho(x)$
  - Symmetric group characters χ^λ_ρ
- Products
  - Outer – Littlewood-Richardson coefficients $c^\lambda_{\mu\nu}$
  - Inner – Kronecker coefficients $g_{\lambda\mu\nu}$
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- Generating functions
  - Kronecker coefficients $g_{\lambda\mu\nu}$
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Reduced Kronecker coefficients

- Let $\lambda = (m - |\rho|, \rho)$, $\mu = (m - |\sigma|, \sigma)$, $\nu = (m - |\tau|, \tau)$
- Reduced inner product $\langle s_\rho \rangle * \langle s_\sigma \rangle = \sum_\tau \overline{g}_{\rho\sigma\tau} \langle s_\tau \rangle$
- Under the map $x \mapsto (1, u)$, $y \mapsto (1, v)$, $z \mapsto (1, w)$
  - $GGF(q, xyz) \mapsto SGF(q, uvw)$
  - $(1 - q\, x_1 y_1 z_1)^{-1} \mapsto 1/(1 - q)$
  - $SGF(q, uvw) = \dfrac{PGF(q, uvw)}{1 - q}$
- Now set $RGF(uvw) = PGF(1, uvw)$, so that

  in the stable limit $\overline{g}_{\rho\sigma\tau} = [u^\rho v^\sigma w^\tau]\ RGF(uvw)$
- Thus $RGF(uvw)$ is the generating function for reduced Kronecker coefficients

Evaluation of reduced inner products

Consider the case $\rho = (r)$, $\sigma = (s)$, $\tau = (\tau_1, \tau_2, \tau_3)$

- The expansion of $RGF(uvw)$ involves many terms $u^r v^s w^\zeta$ in which ζ is not a partition
- Once again we can exploit the package Omega to remedy this
- We find the generating function

$$G = \frac{(1 - u^2 v^2 w^{21})}{(1-u\,v)(1-u\,w^1)(1-v\,w^1)(1-u\,v\,w^1)} \\ (1-u\,v\,w^{11})(1-u^2\,v\,w^{11})(1-u\,v^2\,w^{11})(1-u^2\,v^2\,w^{111})$$

- Then evaluate $[u^r\,v^s]\,G$

Example $\langle s_6 \rangle * \langle s_5 \rangle = \cdots + 4\langle s_{42} \rangle + \cdots$

- This gives

$$w^{332} + w^{432} + w^{522} + w^{322} + w^{422} + 2w^{611} + w^{711} + w^{441} + w^{631} + w^{541} + 2w^{411} + w^{721} + 2w^{531} + w^{811} + 3w^{521} + w^{331} + w^{211} + 2w^{621} + 2w^{321} + w^{221} + w^{311} + 3w^{431} + 3w^{421} + 2w^{511} + 2w^{64} + 4w^{63} + 2w^{33} + 5w^{53} + 2w^{44} + 4w^{72} + 4w^{42} + 2w^{82} + w^{83} + 3w^{32} + 6w^{52} + 4w^{43} + 5w^{62} + w^{10,1} + 4w^{41} + w^{92} + w^{74} + w^{55} + 3w^{54} + 2w^{21} + w^{65} + 5w^{51} + 3w^{31} + 5w^{61} + 4w^{71} + w^{22} + 2w^{73} + w^{11} + 3w^{81} + 2w^{91} + 2w^{3} + 3w^{5} + 3w^{7} + w^{11,0} + w^{10,0} + 2w^{8} + 2w^{4} + 2w^{9} + w^{2} + w^{1} + 3w^{6}$$

- The term $4w^{42}$ signifies that $\overline{g}_{(6)(5)(42)} = 4$ as before

Selected references

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Postscript

Stretched coefficients postponed to a later date!

- Kronecker coefficients $g_{\lambda\mu\nu}$
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