Poset fiber theorem and some applications

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Outline

1. The original problem: noncrossing partitions and injective words
2. Tools: poset fiber theorems – a classical and a new one
3. Applications

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2. Tools: poset fiber theorems – a classical and a new one
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Noncrossing partitions: What is known?

W finite Coxeter group

$\mathrm{NC}(W)$ poset of *noncrossing partitions*

- $\mathrm{NC}(W)$ is a graded, (locally) self-dual lattice (Bessis, Brady, Watt, 2003).
- $\mathrm{NC}(S_n)$ is EL-shellable (Björner, Edelmann, 1980).
- $\mathrm{NC}(B_n)$ is EL-shellable (Reiner, 2002).
- case-free proof of EL-shellability of $\mathrm{NC}(W)$ (Athanasiadis, Brady, Watt, 2007).

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What else can be said about topological properties of $\mathrm{NC}(W)$*?*

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Injective words: What is known?

I_n poset of *injective words* on $[n] = \{1, \ldots, n\}$

Γ_n Boolean cell complex with face poset I_n

- Γ_n is homotopy equivalent to a wedge of spheres (Farmer, 1978).
- Γ_n is CL-shellable (Björner, Wachs, 1983).
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Cohen-Macaulay complexes

Δ simplicial complex

For $F \in \Delta$: $\mathrm{lk}_\Delta(F) = \{G \in \Delta \ : \ G \cap F = \emptyset; \ G \cup F \in \Delta\}$ *link of* F

Δ *homotopy Cohen-Macaulay* (HCM) $\Leftrightarrow$

$\mathrm{lk}_\Delta(F)$ is $(\dim(\mathrm{lk}_\Delta(F)) - 1)$-connected for all $F \in \Delta$.

Δ *doubly homotopy Cohen-Macaulay* $\Leftrightarrow$

For all $v \in \Delta$ the deletion $\Delta - \{v\} = \{F \in \Delta \ : \ v \notin F\}$ is HCM of the same dimension as Δ.

Hierarchy:

shellable $\Rightarrow$ constructible $\Rightarrow$ homotopy Cohen-Macaulay $\Rightarrow$ Cohen-Macaulay

A poset P is called shellable/HCM, ... if $\Delta(P)$ has this property.

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A classical poset fiber theorem (Quillen)

P, Q graded posets

$f: P \to Q$ *poset map* if $x \leq_P y \Rightarrow f(x) \leq_Q f(y)$.

$f: P \to Q$ is *rank-preserving* if $\mathrm{rank}(p) = \mathrm{rank} f(p)$ for all $p \in P$.

For $p \in P$: $\langle p \rangle = \{u \in p : u \leq_P p\}$.

Theorem (Quillen)

Let P be a graded and Q be a HCM poset.
Let $f: P \to Q$ be a surjective rank-preserving poset map.
Assume that for every $q \in Q$ the fiber $f^{-1}(\langle q \rangle) = \{f^{-1}(u) : u \leq q\}$ is HCM. Then P is HCM.

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(i) *for every $q \in Q$ the fiber $f^{-1}(\langle q \rangle)$ is HCM.*

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W finite Coxeter group with set of reflections T

$\leq_T$ absolute order, c Coxeter element

$$\mathrm{NC}(W,c) = [e,c] = \{w \in W \ : \ e \leq_T w \leq_T c\}$$

poset of *noncrossing partitions*

- independent (up to isomorphism) of c
- $\mathrm{NC}(S_n)$ is isomorphic to the poset of classical noncrossing partitions (Kreweras).

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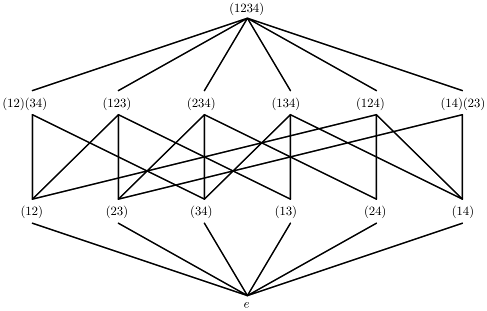

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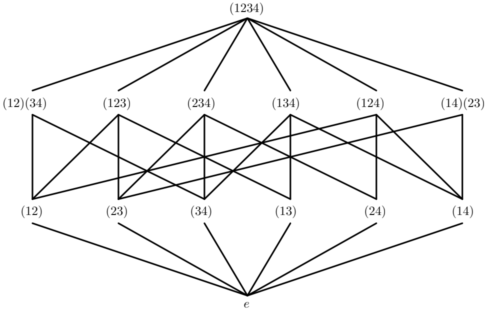

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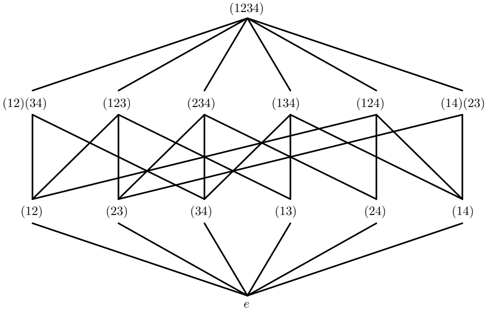

The map for $\mathrm{NC}(S_n)$

Let $w \in S_n$.

Let $\pi(w)$ be obtained from w by deleting n from its cycle decomposition.

- For $n = 7$ and $w = (146)(275)(3)$ we have $\pi(w) = (146)(25)(3)$.

Let

$$f : \mathrm{Abs}(S_n) \to \mathrm{Abs}(S_{n-1}) \times \{\hat{0}, \hat{1}\}$$

$$w \mapsto \begin{cases} (\pi(w), \hat{0}), & \text{if} \quad w(n) = n \\ (\pi(w), \hat{1}), & \text{if} \quad w(n) \neq n \end{cases}$$

- For $w = (146)(275)(3)$ we get $f(w) = ((146)(25)(3), \hat{1})$.
- For $w = (14)(23)(56)(7)$ we get $f(w) = ((14)(23)(56), \hat{0})$.

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The map for $\mathrm{NC}(S_n)$

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Goal: Show that $\mathrm{NC}(S_n) - \{x\}$ is HCM.

Key observation: By self-duality of $\mathrm{NC}(S_n)$ assume that x has a fixed point.

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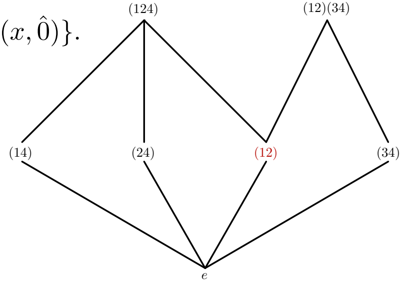

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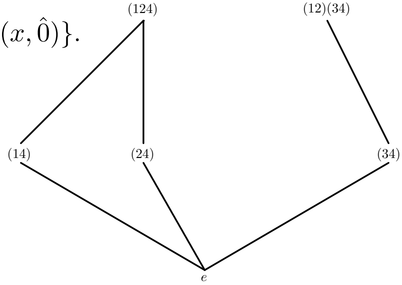

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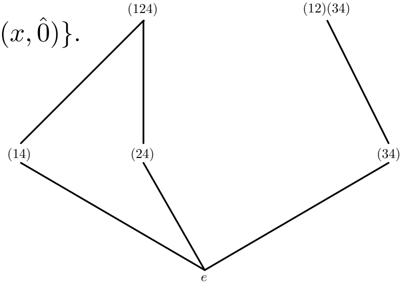

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The result: noncrossing partitions

Theorem

Let W be a Coxeter group of type A or type B. Then:
$\mathrm{NC}(W)$ *is doubly HCM.*

For type B:

- Reduce to elements with a fixed point (self-duality + Kreweras complement).
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Outline

Injective words

A finite alphabet, $w = w_1 \cdots w_s$ word over A

w is *injective* if no letter appears more than once.

Order words via subword containment: $v_1 \cdots v_r \preceq w_1 \cdots w_s$
$\Leftrightarrow$ There exist $1 \leq j_1 \leq \cdots \leq j_r$ such that $w_{j_1} \cdots w_{j_r} = v_1 \cdots v_r$.

E.g.: $5137 \preceq 5413276$

I_n poset of *injective words* on $[n]$
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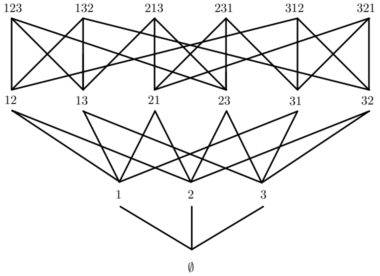

The map for I_n

Let $w \in \mathrm{I}_n$.

Let $\pi(w)$ be the word obtained from w by deleting the letter n.

- For $n = 7$ we have $\pi(146275) = 14625$ and $\pi(264) = 264$.

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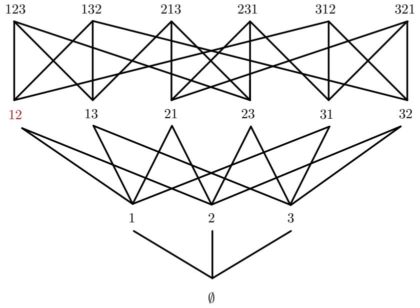

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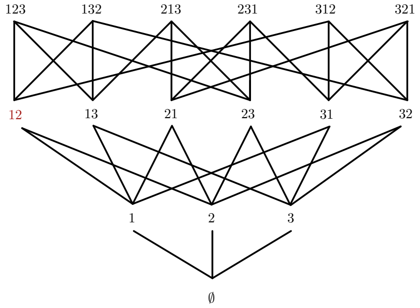

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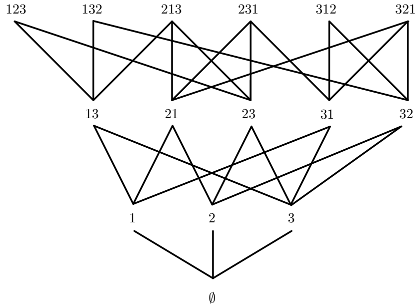

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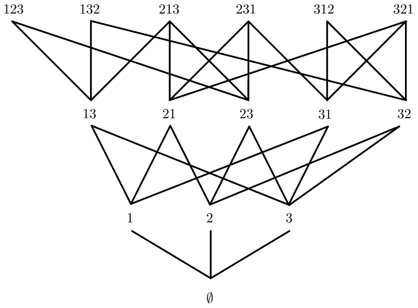

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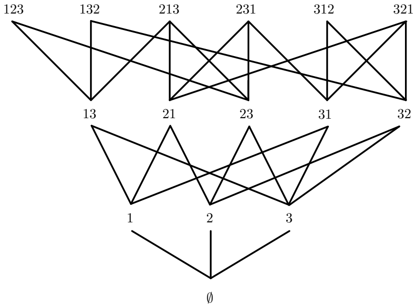

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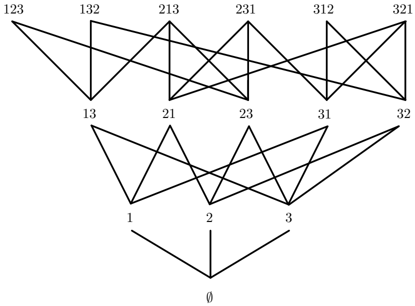
$$f: \; \mathrm{I}_n - \{x\} \to \big(\mathrm{I}_{n-1} \times \{\hat{0}, \hat{1}\}\big) - \{(x, \hat{0})\}.$$

By induction:

$\big(\mathrm{I}_{n-1} \times \{\hat{0}, \hat{1}\}\big) - \{(x, \hat{0})\}$ is HCM.

But: Some fibers $f^{-1}(\langle q \rangle)$ are not.

$\Rightarrow$ We cannot apply the poset fiber theorem by Quillen.



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What we can do instead

Let $x = 1 \cdots k$ for a certain $1 \leq k \leq n-1$ and set $q_0 = (x, \hat{0})$.

Consider

$$f : \mathrm{I}_n \to \mathrm{I}_{n-1} \times \{\hat{0}, \hat{1}\}$$
$$w \mapsto \begin{cases} (\pi(w), \hat{0}), & \text{if} \quad n \not\preceq w \\ (\pi(w), \hat{1}), & \text{if} \quad n \preceq w \end{cases}$$

- The fibers $f^{-1}(\langle q \rangle)$ are HCM.
- $f^{-1}(q_0) = \{x\}$ and $(\mathrm{I}_{n-1} \times \{\hat{0}, \hat{1}\}) - \{(x, \hat{0})\}$ is HCM.
- For $q > q_0$ and $p \in f^{-1}(q)$ the ideal $\langle p \rangle$ is isomorphic to a Boolean algebra. Therefore, $\langle p \rangle - \{x\}$ is HCM.

$\Rightarrow$ The new poset fiber theorem applies: I_n is doubly HCM.

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Outline

1. The original problem: noncrossing partitions and injective words
2. Tools: poset fiber theorems – a classical and a new one
3. Applications
   - Noncrossing partitions: type A and B
   - Injective words
   - Complexes of injective words

I_n poset of injective words on $[n]$

Γ_n Boolean cell complex whose face poset is I_n

$(P, \leq_P)$ poset on $[n]$

Δ simplicial complex on vertex set $[n]$

$$\Gamma(\Delta, P) = \{w = w_1 \cdots w_s \in \mathrm{I}_n \ : \ \{w_1, \ldots, w_s\} \in \Delta \text{ and } w_i <_P w_j \Rightarrow i < j\}$$

Example

(i) *If P is a total order, then $\Gamma(\Delta, P) \cong \Delta$.*

(ii) *If P is an antichain, then $\Gamma(\Delta, P) = \{w_1 \cdots w_s \in \mathrm{I}_n \ : \ \{w_1, \ldots, w_s\} \in \Delta\}$.*

(iii) *Let P be the poset on $[3]$ with $1 < 3$ and $2 < 3$.*
Let $\Delta = 2^{[3]}$ be the 2-simplex. Then
$\Gamma(\Delta, P) = \{123, 213, 12, 21, 13, 23, 1, 2, 3, \emptyset\}$.

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Theorem (Jonsson, Welker)

Let P be a poset on $[n]$ and Δ be a HCM simplicial complex. Then $\Gamma(\Delta, P)$ is HCM.

Sketch of the proof:
Consider the map

$$\begin{aligned} f : \Gamma(\Delta, P) &\rightarrow \Delta \\ w_1 \cdots w_s &\mapsto \{w_1, \ldots, w_s\}. \end{aligned}$$

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Let P be a poset on $[n]$ and Δ be a doubly HCM simplicial complex. Then $\Gamma(\Delta, P)$ is doubly HCM.

Proof:

Apply the new poset fiber theorem to the map used by Jonsson and Welker □

Remark

The same method can be used to show that the complex $\Gamma/G\,(\Delta)$ is doubly HCM if Δ is.

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