Diagonalization of the Matrices of the Multinomial Descent and Multinomial Inversion Statistics on the Symmetric Group

Hery Randriamaro

Universität Marburg

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Notation

$[n]$: $\{1, \ldots, n\}$

$\mathcal{S}_n$: symmetric group

$I_{n!}$: identity matrix of $\mathbb{R}^{n! \times n!}$

$Sp(\mathrm{A})$: spectrum of $\mathrm{A} \in \mathbb{R}^{n \times n}$

$V_{\mathrm{A}}(a)$: multiplicity of $a \in Sp(\mathrm{A})$

$E_{\mathrm{A}}(a)$: eigenspace of $a \in Sp(\mathrm{A})$

$< v >$: subspace generated by $v \in \mathbb{R}^n$

$\mathbb{R}[X_1, \ldots, X_k]$: polynomial ring in X_i

Contents

Presentation

The descents set

Let $n \geq 1$ and $\sigma \in \mathcal{S}_n$:

$$DES(\sigma) := \{k \in [n-1] \mid \sigma(k) > \sigma(k+1)\}.$$

The statistic $\mathtt{des}_{\mathtt{X}}$

Let $n \geq 1$:

$$\begin{array}{rccc} \mathtt{des}_{\mathtt{X}}: & \mathcal{S}_n & \to & \mathbb{R}[X_1, \ldots, X_{n-1}] \\ & \sigma & \mapsto & \mathtt{des}_{\mathtt{X}}(\sigma) := \sum_{i \in DES(\sigma)} X_i \end{array}$$

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Example

Let $\sigma = 5\,\bar{9}\,\bar{8}\,3\,\bar{7}\,\bar{4}\,1\,2\,6$. Then $\mathtt{des}_{\mathtt{X}}(\sigma) = X_2 + X_3 + X_5 + X_6$.

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Presentation

The inversions set

Let $n \geq 1$ and $\sigma \in \mathcal{S}_n$:

$$INV(\sigma) := \{(i,j) \mid i < j,\, \sigma(i) > \sigma(j)\}.$$

The statistic $\mathtt{inv}_{\mathtt{X}}$

Let $n \geq 1$:

$$\begin{array}{rccc} \mathtt{inv}_{\mathtt{X}}: & \mathcal{S}_n & \to & \mathbb{R}[X_{1,2}, \dots, X_{n-1,n}] \\ & \sigma & \mapsto & \mathtt{inv}_{\mathtt{X}}(\sigma) := \sum_{(i,j) \in INV(\sigma)} X_{i,j} \end{array}$$

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Let $n \geq 1$ and $\sigma \in \mathcal{S}_n$:

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Example

Let $\sigma = 2\,3\,5\,1\,4$. Then $\mathtt{des}_{\mathtt{X}}(\sigma) = X_{1,4} + X_{2,4} + X_{3,4} + X_{3,5}$.

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Let $n \geq 1$ and $\sigma \in \mathcal{S}_n$:

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Let $\sigma = 2\,3\,5\,1\,4$. Then $\mathtt{des}_{\mathtt{X}}(\sigma) = X_{1,4} + X_{2,4} + X_{3,4} + X_{3,5}$.

Presentation

The matrix $\mathfrak{D}_{\mathfrak{n}}$

Let $n \geq 1$. The matrix representation of the multiplication $\sum_{\sigma \in \mathcal{S}_n} \mathtt{des}_{\mathtt{X}}(\sigma)\sigma$ on $\mathbb{R}[X_1, \dots, X_{n-1}][\mathcal{S}_n]$ is:

$$\mathfrak{D}_{\mathfrak{n}} := \left(\mathtt{des}_{\mathtt{X}}(\pi\tau^{-1})\right)_{\pi,\tau \in \mathcal{S}_n}.$$

Example

$$\mathfrak{D}_3 = \begin{matrix} 0 & X_2 & X_1 & X_1 & X_2 & X_1+X_2 \\ X_2 & 0 & X_1 & X_1 & X_1+X_2 & X_2 \\ X_1 & X_2 & 0 & X_1+X_2 & X_2 & X_1 \\ X_2 & X_1 & X_1+X_2 & 0 & X_1 & X_2 \\ X_1 & X_1+X_2 & X_2 & X_2 & 0 & X_1 \\ X_1+X_2 & X_1 & X_2 & X_2 & X_1 & 0 \end{matrix}$$

Presentation

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The matrix $\mathfrak{I}_{\mathfrak{n}}$

Let $n \geq 1$. The matrix representation of the multiplication $\sum_{\sigma \in \mathcal{S}_n} \mathtt{inv}_{\mathtt{X}}(\sigma)\sigma$ on $\mathbb{R}[X_{1,2}, \ldots, X_{n-1,n}][\mathcal{S}_n]$ is:

$$\mathfrak{I}_{\mathfrak{n}} := \left(\mathtt{inv}_{\mathtt{X}}(\pi\tau^{-1})\right)_{\pi,\tau \in \mathcal{S}_n}.$$

Example

$$\mathfrak{I}_{3_{\pi,\tau\in\{123,213,132\}}} = \begin{matrix} 0 & X_{1,2} & X_{2,3} \\ X_{1,2} & 0 & X_{1,3} + X_{2,3} \\ X_{2,3} & X_{1,2} + X_{1,3} & 0 \end{matrix}$$

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Let $n \geq 1$. The matrix representation of the multiplication $\sum_{\sigma \in \mathcal{S}_n} \mathtt{inv}_{\mathtt{X}}(\sigma)\sigma$ on $\mathbb{R}[X_{1,2}, \ldots, X_{n-1,n}][\mathcal{S}_n]$ is:

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Presentation

With simple calculation we get:

1. $Sp(\mathfrak{D}_1) = \{0\}$ and $V_{\mathfrak{D}_1}(0) = 1$.
2. $Sp(\mathfrak{D}_2) = \{X_1, -X_1\}$ and $V_{\mathfrak{D}_2}(X_1) = 1$, $V_{\mathfrak{D}_2}(-X_1) = 1$.

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Presentation

Theorem 1

Let $n \geq 3$. Then $\mathfrak{D}_{\mathfrak{n}}$ is diagonalizable and:

$$Sp(\mathfrak{D}_{\mathfrak{n}}) = \{\frac{n!}{2}\sum_{k=1}^{n-1} X_k,\ -(n-2)!\sum_{k=1}^{n-1} X_k,\ 0\}$$

with:

- $V_{\mathfrak{D}_{\mathfrak{n}}}\big(\frac{n!}{2}\sum_{k=1}^{n-1} X_k\big) = 1$,
- $V_{\mathfrak{D}_{\mathfrak{n}}}\big(-(n-2)!\sum_{k=1}^{n-1} X_k\big) = \binom{n}{2}$,
- $V_{\mathfrak{D}_{\mathfrak{n}}}(0) = n! - \binom{n}{2} - 1$.

Presentation

Example

Diagonalized form of $\mathfrak{D}_3$:

$$\begin{matrix} 3(X_1+X_2) & 0 & 0 & 0 & 0 & 0 \\ 0 & -(X_1+X_2) & 0 & 0 & 0 & 0 \\ 0 & 0 & -(X_1+X_2) & 0 & 0 & 0 \\ 0 & 0 & 0 & -(X_1+X_2) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

Presentation

With simple calculation we get:

1. $Sp(\mathfrak{I}_1) = \{0\}$ and $V_{\mathfrak{I}_1}(0) = 1$.
2. $Sp(\mathfrak{I}_2) = \{X_{1,2}, -X_{1,2}\}$ and $V_{\mathfrak{I}_2}(X_{1,2}) = 1$, $V_{\mathfrak{I}_2}(-X_{1,2}) = 1$.

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With simple calculation we get:

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3. $Sp(\mathfrak{I}_3) = \{3X_{1,2} + 3X_{1,3} + 3X_{2,3},\ -X_{1,2} - 2X_{1,3} - X_{2,3},\ -X_{1,2} + X_{1,3} - X_{2,3},\ 0\}$ and
   - $V_{\mathfrak{I}_3}(3X_{1,2} + 3X_{1,3} + 3X_{2,3}) = 1$,
   - $V_{\mathfrak{I}_3}(-X_{1,2} - 2X_{1,3} - X_{2,3}) = 2$,
   - $V_{\mathfrak{I}_3}(-X_{1,2} + X_{1,3} - X_{2,3}) = 1$,
   - $V_{\mathfrak{I}_3}(0) = 2$.

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1. $Sp(\mathfrak{I}_1) = \{0\}$ and $V_{\mathfrak{I}_1}(0) = 1$.
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    - $V_{\mathfrak{I}_3}(-X_{1,2} + X_{1,3} - X_{2,3}) = 1$,
    - $V_{\mathfrak{I}_3}(0) = 2$.

Presentation

Theorem 2

Let $n \geq 4$. Then $\mathfrak{I}_n$ is diagonalizable and:

$$Sp(\mathfrak{I}_n) = \{\frac{n!}{2} \sum_{\{(i,j)\in[n]^2 \ | \ i<j\}} X_{i,j},\ -(n-2)! \sum_{\{(i,j)\in[n]^2 \ | \ i<j\}} (j-i)X_{i,j},$$

$$-(n-3)! \sum_{\{(i,j)\in[n]^2 \ | \ i<j\}} \Big(n-2(j-i)\Big)X_{i,j},\ 0\}$$

with

- $V_{\mathfrak{I}_n}\big(\frac{n!}{2}\sum_{\{(i,j)\in[n]^2 \ | \ i<j\}} X_{i,j}\big) = 1$,
- $V_{\mathfrak{I}_n}\big(-(n-2)!\sum_{\{(i,j)\in[n]^2 \ | \ i<j\}}(j-i)X_{i,j}\big) = n-1$,
- $V_{\mathfrak{I}_n}\big(-(n-3)!\sum_{\{(i,j)\in[n]^2 \ | \ i<j\}}\Big(n-2(j-i)\Big)X_{i,j}\big) = \binom{n-1}{2}$,
- $V_{\mathfrak{I}_n}(0) = n! - \binom{n}{2} - n$.

Presentation

Example

Diagonalized form of $\mathfrak{I}_3$:

$$
\begin{matrix}
3(X_{1,2}+X_{1,3}+X_{2,3}) & 0 & 0 & 0 \\
0 & -X_{1,2}-2X_{1,3}-X_{2,3} & 0 & 0 \\
0 & 0 & -X_{1,2}-2X_{1,3}-X_{2,3} & 0 \\
0 & 0 & 0 & -X_{1,2}+X_{1,3}-X_{2,3} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{matrix}
$$

Proof

Minimal polynomial of $\mathfrak{D}_{\mathfrak{n}}$

For $n \geq 3$, the minimal polynomial of $\mathfrak{D}_{\mathfrak{n}}$ is

$$X\left(X - \frac{n!}{2}\sum_{k=1}^{n-1} X_k\right)\left(X + (n-2)!\sum_{k=1}^{n-1} X_k\right).$$

Consequences

- $\mathfrak{D}_{\mathfrak{n}}$ is diagonalizable.

Proof

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Consequences

- $\mathfrak{D}_{\mathfrak{n}}$ is diagonalizable.
- $Sp(\mathfrak{D}_{\mathfrak{n}}) = \{\frac{n!}{2}\sum_{k=1}^{n-1} X_k,\ 0,\ -(n-2)!\sum_{k=1}^{n-1} X_k\}$

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For $n \geq 3$, the minimal polynomial of $\mathfrak{D}_{\mathfrak{n}}$ is

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Proof

Multinomial version of Theorem of Perron-Frobenius

Let $n \geq 2$ and $\mathrm{P_n} = \big(P_{i,j}\big)_{i,j \in [n]}$ be a $n \times n$-matrix of polynomial $P_{i,j} \in \mathbb{R}[X_1, \ldots, X_k]$ such that:

(a) $P_{i,j} \neq 0$ and $(P_{i,j}\,,\, X_1^{i_1} \ldots X_k^{i_k}) \geq 0$,

(b) for any $i', i'' \in [n]$,

$$\sum_{j=1}^{n} P_{i',j} = \sum_{j=1}^{n} P_{i'',j} = P_n.$$

Then $P_n \in Sp(\mathrm{P_n})$ and $E_{\mathrm{P_n}}(P_n) =< \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} >$.

Proof

Consequences

- Multinomial version and minimal polynomial:

$$V_{\mathfrak{D}_n}\Big(\frac{n!}{2}\sum_{k=1}^{n-1} X_k\Big) = 1$$

- The trace of $\mathfrak{D}_n$ is 0:

$$V_{\mathfrak{D}_n}\Big(-(n-2)!\sum_{k=1}^{n-1} X_k\Big) = \binom{n}{2}$$

Proof

Consequences

- Multinomial version and minimal polynomial:

$$V_{\mathfrak{D}_{\mathfrak{n}}}\left(\frac{n!}{2}\sum_{k=1}^{n-1} X_k\right) = 1$$

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- The dimension of $\mathfrak{D}_{\mathfrak{n}}$ is $n!$:

$$V_{\mathfrak{D}_{\mathfrak{n}}}(0) = n! - \binom{n}{2} - 1$$

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Proof

Minimal polynomial of $\mathfrak{I}_\mathfrak{n}$

Let $n \geq 4$. We write

$$\Omega = \frac{n!}{2} \sum_{\{(i,j)\in[n]^2 \ | \ i<j\}} X_{i,j},$$

$$\Lambda = (n-2)! \sum_{\{(i,j)\in[n]^2 \ | \ i<j\}} (j-i)X_{i,j},$$

$$\Delta = (n-3)! \sum_{\{(i,j)\in[n]^2 \ | \ i<j\}} \Big(n-2(j-i)\Big)X_{i,j}.$$

Then the minimal polynomial of $\mathfrak{I}_\mathfrak{n}$ is $X\big(X+\Lambda\big)\big(X+\Delta\big)\big(X-\Omega\big)$.

Consequences

- $\mathfrak{I}_\mathfrak{n}$ is diagonalizable.

Proof

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Then the minimal polynomial of $\mathfrak{I}_\mathfrak{n}$ is $X\,(X+\Lambda)\,(X+\Delta)\,(X-\Omega)$.

Consequences

- $\mathfrak{I}_\mathfrak{n}$ is diagonalizable.
- $Sp(\mathfrak{I}_\mathfrak{n}) = \{\Omega, -\Lambda, -\Delta, 0\}$

Proof

Minimal polynomial of $\mathfrak{I}_n$

Let $n \geq 4$. We write

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Then the minimal polynomial of $\mathfrak{I}_n$ is $X(X+\Lambda)(X+\Delta)(X-\Omega)$.

Consequences

- $\mathfrak{I}_n$ is diagonalizable.
- $Sp(\mathfrak{I}_n) = \{\Omega, -\Lambda, -\Delta, 0\}$

Proof

Consequences

- Multinomial version and minimal polynomial:

$$V_{\mathfrak{I}_n}(\Omega) = 1$$

- The trace of $\mathfrak{I}_n$ is 0:

$$V_{\mathfrak{I}_n}(\Lambda) = n - 1 \text{ and } V_{\mathfrak{I}_n}(\Delta) = \binom{n-1}{2}$$

Proof

Consequences

- Multinomial version and minimal polynomial:

$$V_{\mathfrak{I}_n}(\Omega) = 1$$

- The trace of $\mathfrak{I}_n$ is 0:

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- The dimension of $\mathfrak{I}_n$ is $n!$:

$$V_{\mathfrak{I}_n}(0) = n! - \binom{n}{2} - 1$$

Proof

Consequences

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Applications and new problems

The statistic *des*

Let $n \geq 1$:

$$des: \mathcal{S}_n \to \mathbb{R}, \quad \sigma \mapsto des(\sigma) := \#DES(\sigma)$$

Example

Let $\sigma = 5\,\bar{9}\,\bar{8}\,3\,\bar{7}\,\bar{4}\,1\,2\,6$. Then $des(\sigma) = 4$.

Applications and new problems

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Applications and new problems

The matrix $\mathrm{D_n}$

Let $n \geq 1$:

$$\mathrm{D_n} := \big(des(\pi\tau^{-1})\big)_{\pi,\tau\in\mathcal{S}_n}.$$

Example

$$\mathrm{D_3} = \begin{matrix} 0 & 1 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 & 1 & 1 \\ 1 & 2 & 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 1 & 1 & 0 \end{matrix}$$

Applications and new problems

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Applications and new problems

Corollary

$\mathrm{D_n}$ is diagonalizable and:

1. If $n = 1$ then $Sp(\mathrm{D}_1) = \{0\}$ and $V_{\mathrm{D}_1}(0) = 1$.
2. If $n = 2$ then $Sp(\mathrm{D}_2) = \{1, -1\}$ and $V_{\mathrm{D}_2}(1) = 1$, $V_{\mathrm{D}_2}(-1) = 1$.

Applications and new problems

Corollary

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3. If $n \geq 3$ then $Sp(\mathrm{D_n}) = \{\binom{n}{2}(n-1)!,\ 0,\ -(n-1)!\}$ and
   - $V_{\mathrm{D_n}}(\binom{n}{2}(n-1)!) = 1$,
   - $V_{\mathrm{D_n}}(-(n-1)!) = \binom{n}{2}$,
   - $V_{\mathrm{D_n}}(0) = n! - \binom{n}{2} - 1$.

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3. If $n \geq 3$ then $Sp(\mathrm{D_n}) = \{\binom{n}{2}(n-1)!,\ 0,\ -(n-1)!\}$ and
   - $V_{\mathrm{D_n}}(\binom{n}{2}(n-1)!) = 1$,
   - $V_{\mathrm{D_n}}\big(-(n-1)!\big) = \binom{n}{2}$,
   - $V_{\mathrm{D_n}}(0) = n! - \binom{n}{2} - 1$.

Applications and new problems

Example

Diagonalized form of D_3:

$$\begin{matrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

Applications and new problems

The statistic *maj*

Let $n \geq 1$:

$$\begin{array}{rccl} maj: & \mathcal{S}_n & \to & \mathbb{R} \\ & \sigma & \mapsto & maj(\sigma) := \sum_{i \in DES(\sigma)} i \end{array}$$

Example

Let $\sigma = 5\,\bar{9}\,\bar{8}\,3\,\bar{7}\,\bar{4}\,1\,2\,6$. Then $maj(\sigma) = 2 + 3 + 5 + 6 = 16$.

Applications and new problems

The statistic *maj*

Let $n \geq 1$:

$$maj : \begin{array}{ccc} \mathcal{S}_n & \to & \mathbb{R} \\ \sigma & \mapsto & maj(\sigma) := \sum_{i \in DES(\sigma)} i \end{array}$$

Example

Let $\sigma = 5\,\bar{9}\,\bar{8}\,3\,\bar{7}\,\bar{4}\,1\,2\,6$. Then $maj(\sigma) = 2 + 3 + 5 + 6 = 16$.

Applications and new problems

The matrix $\mathrm{M_n}$

Let $n \geq 1$:

$$\mathrm{M_n} := \big(maj(\pi\tau^{-1})\big)_{\pi,\tau\in\mathcal{S}_n}.$$

Example

$$\mathrm{M_3} = \begin{matrix} 0 & 2 & 1 & 1 & 2 & 3 \\ 2 & 0 & 1 & 1 & 3 & 2 \\ 1 & 2 & 0 & 3 & 2 & 1 \\ 2 & 1 & 3 & 0 & 1 & 2 \\ 1 & 3 & 2 & 2 & 0 & 1 \\ 3 & 1 & 2 & 2 & 1 & 0 \end{matrix}$$

Applications and new problems

The matrix M_n

Let $n \geq 1$:

$$\mathrm{M}_n := \left(maj(\pi\tau^{-1})\right)_{\pi,\tau\in\mathcal{S}_n}.$$

Example

$$\mathrm{M}_3 = \begin{matrix} 0 & 2 & 1 & 1 & 2 & 3 \\ 2 & 0 & 1 & 1 & 3 & 2 \\ 1 & 2 & 0 & 3 & 2 & 1 \\ 2 & 1 & 3 & 0 & 1 & 2 \\ 1 & 3 & 2 & 2 & 0 & 1 \\ 3 & 1 & 2 & 2 & 1 & 0 \end{matrix}$$

Applications and new problems

Corollary

$\mathrm{M_n}$ diagonalizable and:

1. If $n = 1$ then $Sp(\mathrm{M}_1) = \{0\}$ and $V_{\mathrm{M}_1}(0) = 1$.
2. If $n = 2$ then $Sp(\mathrm{M}_2) = \{1, -1\}$ and $V_{\mathrm{M}_2}(1) = 1$, $V_{\mathrm{M}_2}(-1) = 1$.

Applications and new problems

Corollary

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1. If $n = 1$ then $Sp(\mathrm{M}_1) = \{0\}$ and $V_{\mathrm{M}_1}(0) = 1$.
2. If $n = 2$ then $Sp(\mathrm{M}_2) = \{1, -1\}$ and $V_{\mathrm{M}_2}(1) = 1$, $V_{\mathrm{M}_2}(-1) = 1$.
3. If $n \geq 3$ then $Sp(\mathrm{M_n}) = \{\binom{n}{2}\frac{n!}{2},\ 0,\ -\frac{n!}{2}\}$ and
   - $V_{\mathrm{M_n}}(\binom{n}{2}\frac{n!}{2}) = 1$,
   - $V_{\mathrm{M_n}}(-\frac{n!}{2}) = \binom{n}{2}$,
   - $V_{\mathrm{M_n}}(0) = n! - \binom{n}{2} - 1$.

Applications and new problems

Corollary

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   - $V_{\mathrm{M_n}}(-\frac{n!}{2}) = \binom{n}{2}$,
   - $V_{\mathrm{M_n}}(0) = n! - \binom{n}{2} - 1$.

Applications and new problems

Example

Diagonalized form of M_3:

$$\begin{matrix} 9 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

Applications and new problems

The statistic *thi* of Thibon

Let $n \geq 1$:

$$\begin{array}{cccc} thi: & \mathcal{S}_n & \rightarrow & \mathbb{R}[X] \\ & \sigma & \mapsto & thi(\sigma) := \prod_{i \in DES(\sigma)} X^i \end{array}$$

Example

Let $\sigma = 5\,\bar{9}\,\bar{8}\,3\,\bar{7}\,\bar{4}\,1\,2\,6$. Then $thi(\sigma) = X^{16}$.

Applications and new problems

The statistic *thi* of Thibon

Let $n \geq 1$:

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Example

Let $\sigma = 5\,\bar{9}\,\bar{8}\,3\,\bar{7}\,\bar{4}\,1\,2\,6$. Then $thi(\sigma) = X^{16}$.

Applications and new problems

The matrix T_n

Let $n \geq 1$:

$$\mathrm{T_n} := \big(thi(\pi\tau^{-1})\big)_{\pi,\tau \in \mathcal{S}_n}.$$

Example

$$\mathrm{T_3} = \begin{pmatrix} 1 & X^2 & X & X & X^2 & X^3 \\ X^2 & 1 & X & X & X^3 & X^2 \\ X & X^2 & 1 & X^3 & X^2 & X \\ X^2 & X & X^3 & 1 & X & X^2 \\ X & X^3 & X^2 & X^2 & 1 & X \\ X^3 & X & X^2 & X^2 & X & 1 \end{pmatrix}$$

Applications and new problems

The matrix $\mathrm{T_n}$

Let $n \geq 1$:

$$\mathrm{T_n} := \big(thi(\pi\tau^{-1})\big)_{\pi,\tau\in\mathcal{S}_n}.$$

Example

$$\mathrm{T_3} = \begin{matrix} 1 & X^2 & X & X & X^2 & X^3 \\ X^2 & 1 & X & X & X^3 & X^2 \\ X & X^2 & 1 & X^3 & X^2 & X \\ X^2 & X & X^3 & 1 & X & X^2 \\ X & X^3 & X^2 & X^2 & 1 & X \\ X^3 & X & X^2 & X^2 & X & 1 \end{matrix}$$

Applications and new problems

Theorem of Thibon

Let $n \geq 1$. Then the eigenvalues of $\mathrm{T_n}$ are

$$\frac{(X;X)_n}{\prod_{i\geq 1}(1-X^{\mu_i})}$$

with multiplicities

$$\frac{n!}{1^{m_1}m_1!2^{m_2}m_2!}\cdots$$

where $\mu = (\mu_1, \mu_2, \dots)$ varies through all partitions of n and m_i is the number of occurences of i in the partition μ.

Applications and new problems

A new statistic thi_X

Let $n \geq 1$:

$$thi_X : \quad \mathcal{S}_n \rightarrow \mathbb{R}[X_1, \ldots, X_{n-1}]$$
$$\sigma \mapsto thi_X(\sigma) := \prod_{i \in DES(\sigma)} X_i$$

Example

Let $\sigma = 5\,\bar{9}\,\bar{8}\,3\,\bar{7}\,\bar{4}\,1\,2\,6$. Then $thi_X(\sigma) = X_2 X_3 X_5 X_6$.

Applications and new problems

A new statistic thi_X

Let $n \geq 1$:

$$\begin{array}{cccc} thi_X : & \mathcal{S}_n & \rightarrow & \mathbb{R}[X_1, \ldots, X_{n-1}] \\ & \sigma & \mapsto & thi_X(\sigma) := \prod_{i \in DES(\sigma)} X_i \end{array}$$

Example

Let $\sigma = 5\,\bar{9}\,\bar{8}\,3\,\bar{7}\,\bar{4}\,1\,2\,6$. Then $thi_X(\sigma) = X_2X_3X_5X_6$.

Applications and new problems

The matrix $\mathfrak{T}_{\mathfrak{n}}$

Let $n \geq 1$. The matrix representation of the multiplication $\sum_{\sigma \in \mathcal{S}_n} \mathtt{thi}_{\mathtt{X}}(\sigma)\sigma$ on $\mathbb{R}[X_1, \ldots, X_{n-1}][\mathcal{S}_n]$ is:

$$\mathfrak{T}_{\mathfrak{n}} := \left(\mathtt{thi}_{\mathtt{X}}(\pi\tau^{-1})\right)_{\pi,\tau \in \mathcal{S}_n}.$$

Example

$$\mathfrak{T}_3 = \begin{matrix} 1 & X_2 & X_1 & X_1 & X_2 & X_1X_2 \\ X_2 & 1 & X_1 & X_1 & X_1X_2 & X_2 \\ X_1 & X_2 & 1 & X_1X_2 & X_2 & X_1 \\ X_2 & X_1 & X_1X_2 & 1 & X_1 & X_2 \\ X_1 & X_1X_2 & X_2 & X_2 & 1 & X_1 \\ X_1X_2 & X_1 & X_2 & X_2 & X_1 & 1 \end{matrix}$$

Applications and new problems

The matrix $\mathfrak{T}_\mathfrak{n}$

Let $n \geq 1$. The matrix representation of the multiplication $\sum_{\sigma \in \mathcal{S}_n} \mathtt{thi}_\mathtt{X}(\sigma)\sigma$ on $\mathbb{R}[X_1, \dots, X_{n-1}][\mathcal{S}_n]$ is:

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Example

$$\mathfrak{T}_3 = \begin{matrix} 1 & X_2 & X_1 & X_1 & X_2 & X_1X_2 \\ X_2 & 1 & X_1 & X_1 & X_1X_2 & X_2 \\ X_1 & X_2 & 1 & X_1X_2 & X_2 & X_1 \\ X_2 & X_1 & X_1X_2 & 1 & X_1 & X_2 \\ X_1 & X_1X_2 & X_2 & X_2 & 1 & X_1 \\ X_1X_2 & X_1 & X_2 & X_2 & X_1 & 1 \end{matrix}$$

Applications and new problems

An open problem

The determinant of $\mathfrak{T}_\mathfrak{n}$ or the spectrum of $\mathfrak{T}_\mathfrak{n}$ with the multiplicities of his elements.

Applications and new problems

The statistic *inv*

Let $n \geq 1$:

$$\begin{array}{rccc} inv: & \mathcal{S}_n & \rightarrow & \mathbb{R} \\ & \sigma & \mapsto & inv(\sigma) := \#INV(\sigma) \end{array}$$

Example

Let $\sigma = 2\,3\,5\,1\,4$. Then $inv(\sigma) = 4$.

Applications and new problems

The statistic *inv*

Let $n \geq 1$:

$$\begin{array}{rccc} inv: & \mathcal{S}_n & \rightarrow & \mathbb{R} \\ & \sigma & \mapsto & inv(\sigma) := \#INV(\sigma) \end{array}$$

Example

Let $\sigma = 2\,3\,5\,1\,4$. Then $inv(\sigma) = 4$.

Applications and new problems

The matrix $\mathrm{I_n}$

Let $n \geq 1$:

$$\mathrm{I_n} := \big(inv(\pi\tau^{-1})\big)_{\pi,\tau\in\mathcal{S}_n}.$$

Example

$$\mathrm{I_3} = \begin{matrix} 0 & 1 & 1 & 2 & 3 & 2 \\ 1 & 0 & 2 & 3 & 2 & 1 \\ 1 & 2 & 0 & 1 & 2 & 3 \\ 2 & 3 & 1 & 0 & 1 & 2 \\ 3 & 2 & 2 & 1 & 0 & 1 \\ 2 & 1 & 3 & 2 & 1 & 0 \end{matrix}$$

Applications and new problems

The matrix $\mathrm{I_n}$

Let $n \geq 1$:

$$\mathrm{I_n} := \left(inv(\pi\tau^{-1})\right)_{\pi,\tau\in\mathcal{S}_n}.$$

Example

$$\mathrm{I_3} = \begin{matrix} 0 & 1 & 1 & 2 & 3 & 2 \\ 1 & 0 & 2 & 3 & 2 & 1 \\ 1 & 2 & 0 & 1 & 2 & 3 \\ 2 & 3 & 1 & 0 & 1 & 2 \\ 3 & 2 & 2 & 1 & 0 & 1 \\ 2 & 1 & 3 & 2 & 1 & 0 \end{matrix}$$

Applications and new problems

Inspiration from Rainer-Saliola-Welker

I_n is diagonalizable and $Sp(\mathsf{I}_n) \subset \mathbb{N}$.

Corollary

I_n is diagonalizable and:

1. If $n = 1$ then $Sp(\mathsf{I}_1) = \{0\}$ and $V_{\mathsf{I}_1}(0) = 1$.

Applications and new problems

Inspiration from Rainer-Saliola-Welker

$\mathrm{I_n}$ is diagonalizable and $Sp(\mathrm{I_n}) \subset \mathbb{N}$.

Corollary

$\mathrm{I_n}$ is diagonalizable and:

1. If $n = 1$ then $Sp(\mathrm{I}_1) = \{0\}$ and $V_{\mathrm{I}_1}(0) = 1$.
2. If $n = 2$ then $Sp(\mathrm{I}_2) = \{1, -1\}$ and $V_{\mathrm{I}_2}(1) = 1$, $V_{\mathrm{I}_2}(-1) = 1$.

Applications and new problems

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$\mathrm{I_n}$ is diagonalizable and:

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2. If $n = 2$ then $Sp(\mathrm{I_2}) = \{1, -1\}$ and $V_{\mathrm{I_2}}(1) = 1$, $V_{\mathrm{I_2}}(-1) = 1$.
3. $Sp(\mathfrak{I}_3) = \{9, \ -4, \ -1, \ 0\}$ and
   - $V_{\mathfrak{I}_3}(9) = 1$,
   - $V_{\mathfrak{I}_3}(-4) = 2$,
   - $V_{\mathfrak{I}_3}(-1) = 1$,
   - $V_{\mathfrak{I}_3}(0) = 2$.

Applications and new problems

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1. If $n = 1$ then $Sp(\mathrm{I}_1) = \{0\}$ and $V_{\mathrm{I}_1}(0) = 1$.
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3. $Sp(\mathfrak{I}_3) = \{9,\ -4,\ -1,\ 0\}$ and
   - $V_{\mathfrak{I}_3}(9) = 1$,
   - $V_{\mathfrak{I}_3}(-4) = 2$,
   - $V_{\mathfrak{I}_3}(-1) = 1$,
   - $V_{\mathfrak{I}_3}(0) = 2$.
4. If $n \geq 4$ then
   $$Sp(\mathrm{I}_n) = \{\frac{n!}{2}\binom{n}{2},\ -\frac{(n+1)!}{6},\ -\frac{n!}{6},\ 0\}$$
   and
   - $V_{\mathrm{I}_n}(\frac{n!}{2}\binom{n}{2}) = 1$,
   - $V_{\mathrm{I}_n}(\frac{(n+1)!}{6}) = n - 1$,
   - $V_{\mathrm{I}_n}(\frac{n!}{6}) = \binom{n}{2}$,
   - $V_{\mathrm{I}_n}(0) = n! - \binom{n}{2} - n$.

Applications and new problems

Inspiration from Rainer-Saliola-Welker

I_n is diagonalizable and $Sp(\mathrm{I}_n) \subset \mathbb{N}$.

Corollary

I_n is diagonalizable and:

1. If $n = 1$ then $Sp(\mathrm{I}_1) = \{0\}$ and $V_{\mathrm{I}_1}(0) = 1$.
2. If $n = 2$ then $Sp(\mathrm{I}_2) = \{1, -1\}$ and $V_{\mathrm{I}_2}(1) = 1$, $V_{\mathrm{I}_2}(-1) = 1$.
3. $Sp(\mathfrak{I}_3) = \{9,\ -4,\ -1,\ 0\}$ and
   - $V_{\mathfrak{I}_3}(9) = 1$,
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$$Sp(\mathrm{I}_n) = \{\frac{n!}{2}\binom{n}{2},\ -\frac{(n+1)!}{6},\ -\frac{n!}{6},\ 0\}$$

   and
   - $V_{\mathrm{I}_n}(\frac{n!}{2}\binom{n}{2}) = 1$,
   - $V_{\mathrm{I}_n}(\frac{(n+1)!}{6}) = n - 1$,
   - $V_{\mathrm{I}_n}(\frac{n!}{6}) = \binom{n}{2}$,
   - $V_{\mathrm{I}_n}(0) = n! - \binom{n}{2} - n$.

Applications and new problems

Example

Diagonalized form of I_3:

$$\begin{matrix} 9 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

Applications and new problems

The statistic var_X of Varchenko

Let $n \geq 1$:

$$\begin{array}{rccc} var_X : & \mathcal{S}_n & \rightarrow & \mathbb{R}[X_{1,2}, \dots, X_{n-1,n}] \\ & \sigma & \mapsto & var_X(\sigma) := \prod_{(i,j) \in \mathtt{INV}(\sigma)} X_{i,j} \end{array}$$

Example

Let $\sigma = 2\,3\,5\,1\,4$. Then $var_X(\sigma) = X_{1,4}X_{2,4}X_{3,4}X_{3,5}$.

Applications and new problems

The statistic var_X of Varchenko

Let $n \geq 1$:

$$\begin{array}{rccc} var_X : & \mathcal{S}_n & \to & \mathbb{R}[X_{1,2}, \ldots, X_{n-1,n}] \\ & \sigma & \mapsto & var_X(\sigma) := \prod_{(i,j) \in \mathtt{INV}(\sigma)} X_{i,j} \end{array}$$

Example

Let $\sigma = 2\,3\,5\,1\,4$. Then $var_X(\sigma) = X_{1,4} X_{2,4} X_{3,4} X_{3,5}$.

Applications and new problems

The matrix $\mathfrak{V}_\mathfrak{n}$

Let $n \geq 1$:

$$\mathfrak{V}_\mathfrak{n} := \left(var_X(\pi\tau^{-1})\right)_{\pi,\tau\in\mathcal{S}_n}.$$

Example

$$\mathfrak{V}_{3_{\pi,\tau\in\{123,213,132\}}} = \begin{matrix} 0 & X_{1,2} & X_{2,3} \\ X_{1,2} & 0 & X_{1,3}X_{2,3} \\ X_{2,3} & X_{1,2}X_{1,3} & 0 \end{matrix}$$

Applications and new problems

The matrix $\mathfrak{V}_{\mathfrak{n}}$

Let $n \geq 1$:

$$\mathfrak{V}_{\mathfrak{n}} := \left(var_X(\pi\tau^{-1})\right)_{\pi,\tau \in \mathcal{S}_n}.$$

Example

$$\mathfrak{V}_{3_{\pi,\tau\in\{123,213,132\}}} = \begin{matrix} 0 & X_{1,2} & X_{2,3} \\ X_{1,2} & 0 & X_{1,3}X_{2,3} \\ X_{2,3} & X_{1,2}X_{1,3} & 0 \end{matrix}$$

Applications and new problems

Theorem of Varchenko

Let $n \geq 1$:

$$det(\mathfrak{V}_{\mathfrak{n}}) = \prod_{L \subseteq 2^{\binom{[n]}{2}}} (1 - a(L)^2)^{l(L)}$$

where $a(L) = \prod_{i,j \in L} X_{i,j}$ is the weight of L and $l(L)$ is the multiplicity of L.

Applications and new problems

An open problem

The spectrum of $\mathfrak{V}_n$ and the multiplicities of his elements.