Enumeration with "catalytic" parameters

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Four equations

$$F(x) = 1 + txF(x) + t\,\frac{F(x) - F(0)}{x}$$

$$F(x) = 1 + tx^2F(x)^2 + t\,\frac{xF(x) - F(1)}{x - 1}$$

$$F(x,y) = 1 + tyF(x,y) + t\,\frac{F(x,y) - F(0,y)}{x} + tx\,\frac{F(x,y) - F(x,0)}{y}$$

$$F(x,y) = xq(q-1) + \frac{xyt}{q}F(1,y)F(x,y) + xt\,\frac{F(x,y) - F(x,0)}{y} - x^2yt\,\frac{F(x,y) - F(1,y)}{x - 1}$$

Four equations

- Where do they come from?

- Do we really have to solve them?

- Do they have relatives?

- How can we solve... polynomial equations with catalytic variables?

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Enumerative combinatorics and generating functions

Let $\mathcal{A}$ be a set of discrete objects equipped with a size:

$$\begin{aligned} size : \mathcal{A} &\rightarrow \mathbb{N} \\ a &\mapsto |a| \end{aligned}$$

Assume that for all n,

$$\mathcal{A}_n := \{a \in \mathcal{A} : |a| = n\} \text{ is finite.}$$

Let $a(n) = |\mathcal{A}_n|$.

Objective: Determine $a(n)$, or the generating function of the objects of $\mathcal{A}$:

$$\begin{aligned} A(t) &:= \sum_{n \geq 0} a(n) t^n \\ &= \sum_{a \in \mathcal{A}} t^{|a|}. \end{aligned}$$

Multivariate enumeration:

$$A(t;x) := \sum_{n,k \geq 0} a(n,k) t^n x^k$$

Applications: probability, algebra, computer science (analysis of algorithms), statistical physics... and curiosity

Why generating functions?

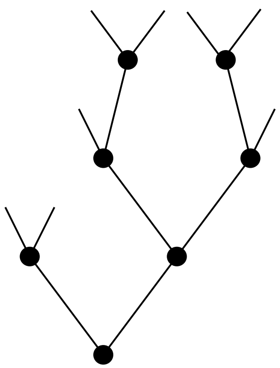
$$A(t) := \sum_{n\geq 0} a(n)t^n$$

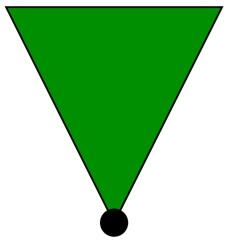
- Encode the sequence $a(n)$
- Write recurrence relations on $a(n)$ as functional equations on $A(t)$
- Use all kinds of tools developped for functions and functional equations

Combinatorial constructions and operations on series: A dictionary

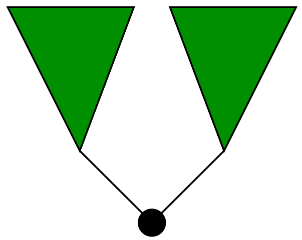
Construction	Numbers	Generating function
Union $\mathcal{A} = \mathcal{B} \sqcup \mathcal{C}$	$a(n) = b(n) + c(n)$	$A(t) = B(t) + C(t)$
Product $\mathcal{A} = \mathcal{B} \times \mathcal{C}$ $\lvert(\beta,\gamma)\rvert = \lvert\beta\rvert + \lvert\gamma\rvert$	$a(n) = b(0)c(n) + \cdots + b(n)c(0)$	$A(t) = B(t) \cdot C(t)$

Example: binary trees $\qquad A(t) = 1 + tA(t)^2$





$= \qquad \{\varepsilon\} \qquad \sqcup$



A hierarchy of formal power series

• The formal power series $A(t)$ is rational if it can be written

$$A(t) = \frac{P(t)}{Q(t)}$$

where $P(t)$ and $Q(t)$ are polynomials in t.

• The formal power series $A(t)$ is algebraic if it satisfies a polynomial equation:

$$P(t, A(t)) = 0.$$

• The formal power series $A(t)$ is D-finite if it satisfies a linear differential equation:

$$P_k(t)A^{(k)}(t) + \cdots + P_1(t)A'(t) + P_0(t)A(t) = 0.$$

• The formal power series $A(t)$ is D-algebraic if it satisfies an algebraic-differential equation:

$$P\left(t, A^{(k)}(t), \ldots, A'(t), A(t)\right) = 0$$

for some polynomial P.

Some charms of rational and algebraic series

• Closure properties ($+, \times, /$, derivatives, composition...)

• "Easy" to handle (partial fraction decomposition, Puiseux expansions, elimination, resultants, Gröbner bases...)

• Algebraicity can be guessed from the first coefficients (GFUN)

• The coefficients can be computed in a linear number of operations.

• (Almost) automatic asymptotics of the coefficients: in general,

$$a(n) \sim \frac{\kappa}{\Gamma(d+1)} \mu^n n^d,$$

where κ and μ are algebraic over $\mathbb{Q}$ and $d \in \mathbb{Q} \setminus \{-1, -2, \ldots\}$.

• Algebraicity suggests that plane trees are lurking around (cf. $A(t) = 1 + tA(t)^2$)

Some charms of D-finite series

- Closure properties ($+$, $\times$, derivatives, composition with algebraic series...)

- "Easy" to handle (GFUN)

- D-finiteness can be guessed from the first coefficients (GFUN)

- The coefficients can be computed in a linear number of operations.

- (Almost) automatic asymptotics of the coefficients

A closer look at our four equations

$$F(x) = 1 + txF(x) + t\,\frac{F(x) - F(0)}{x}$$

$$F(x) = 1 + tx^2F(x)^2 + t\,\frac{xF(x) - F(1)}{x-1}$$

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A closer look at our four equations

$$\boxed{F(x) = 1 + txF(x) + t\,\frac{F(x) - F(0)}{x}}$$

- Where is t? $F(x)$ stands for $F(t;x)$
- Linear (i.e., degree 1) in F
- The divided difference

$$\frac{F(x) - F(0)}{x}$$

is what makes life interesting. We say that the variable x is catalytic: no x, no equation!

- Is $F(0)$ (and $F(x)$) rational? algebraic? D-finite?

A closer look at our four equations

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- The divided difference is taken around $x = 1$
- Quadratic in F

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- Again linear in F, but...

- Two divided differences, w.r.t. x and y: two catalytic variables

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- Quadratic in F, two catalytic variables

Outline of the talks

	One catalytic variable	Several catalytic variables
Linear	$F(x) = 1 + txF(x) + t\,\frac{F(x)-F(0)}{x}$ always algebraic	$F(x,y) = 1 + tyF(x,y)$ $+t\,\frac{F(x,y)-F(0,y)}{x} + tx\,\frac{F(x,y)-F(x,0)}{y}$ this one: D-finite
Non-linear	$F(x) = 1 + tx^2F(x)^2 + t\,\frac{xF(x)-F(1)}{x-1}$ always algebraic	$F(x,y) = xq(q-1) + \frac{xyt}{q}F(1,y)F(x,y)$ $+xt\,\frac{F(x,y)-F(x,0)}{y} - x^2yt\,\frac{F(x,y)-F(1,y)}{x-1}$ this one: D-algebraic

In each case: a prototype, plus (attempts at) a general approach

I. Linear equations with one catalytic variable

$$F(x) = 1 + txF(x) + t\,\frac{F(x) - F(0)}{x}$$

Where does it come from? Walks on a half-line

• Count walks on the half-line $\mathbb{N}$, starting from 0, by their length (variable t) and the position of their endpoint (variable x):

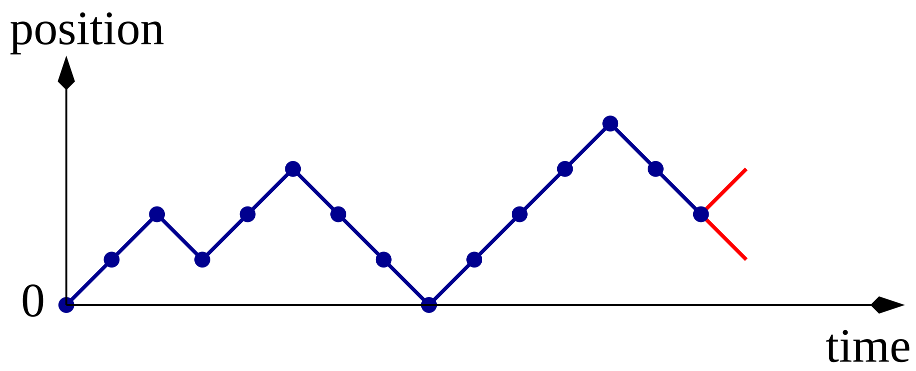
$$F(t;x) \equiv F(x) = \sum_w t^{\ell(w)} x^{e(w)}$$

In particular:

- $\circ$ $F(t;0) \equiv F(0)$ counts walks ending at 0 (Dyck paths),
- $\circ$ $F(x) - F(0)$ those ending at a positive height.

• A step by step construction:

$$F(x) = 1 + txF(x) + \frac{t}{x}(F(x) - F(0))$$



[Knuth, The Art of Computer programming, Vol. 2, 1972]

Do we really need this equation?

Maybe not...

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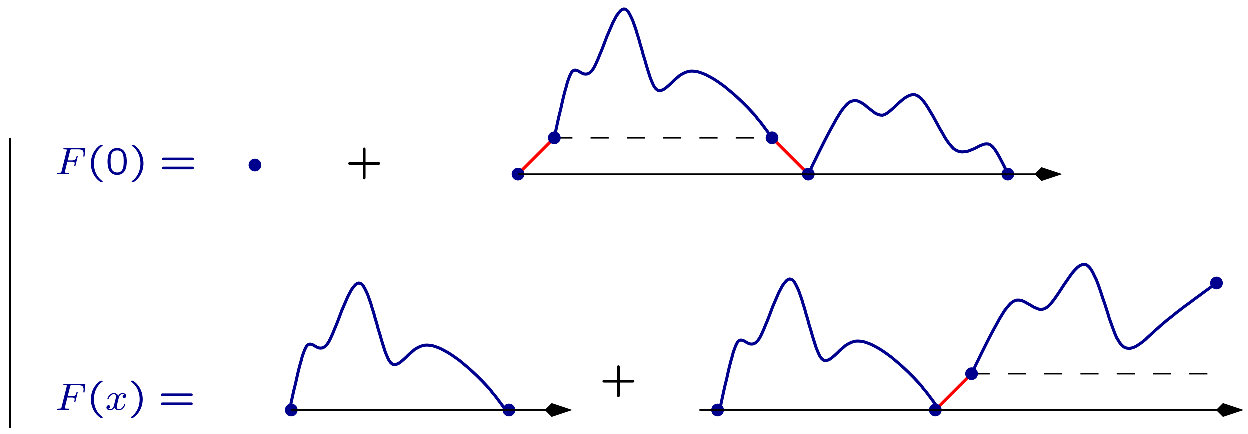
Maybe not...

- we can write directly algebraic equations:

$$F(0) = 1 + t^2 F(0)^2$$

$$F(x) = F(0) + txF(0)F(x)$$



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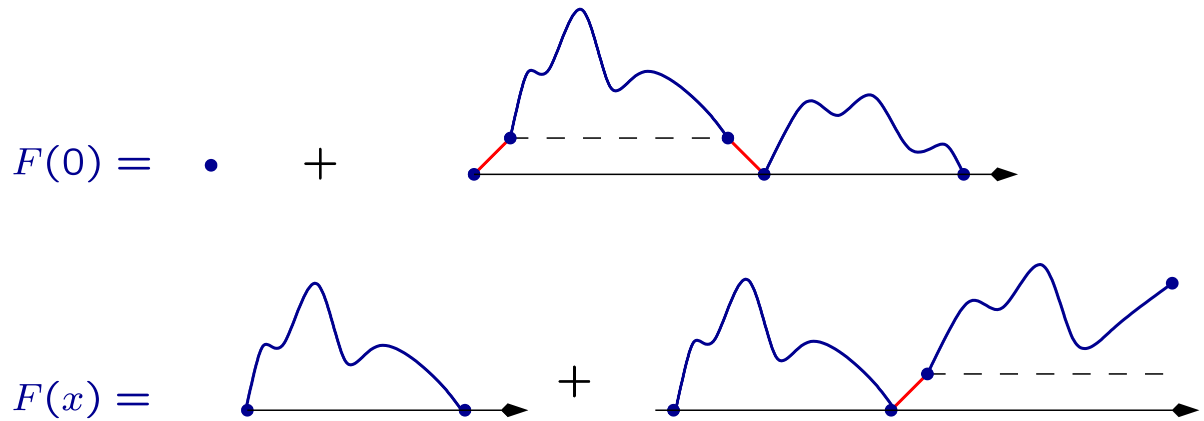

and solve them:

$$F(x) = \frac{1 - 2\,tx - \sqrt{1 - 4t^2}}{2t\left(t - x + tx^2\right)}$$

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Maybe not...

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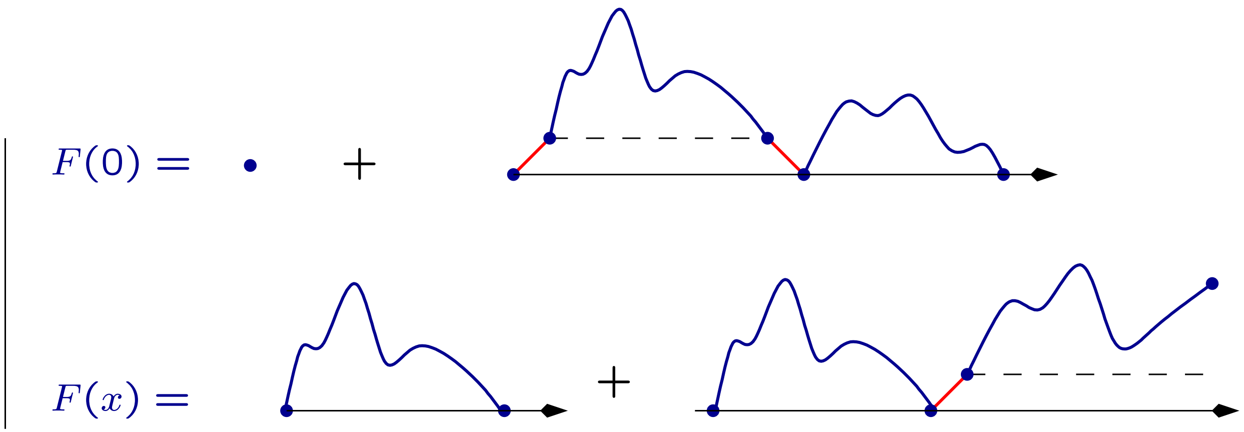

and solve them:

$$F(x) = \frac{1 - 2\,tx - \sqrt{1-4t^2}}{2t\left(t - x + tx^2\right)}$$

- or we can use the reflection principle

OK... but...

What if the steps are $+3$ and -2?

- One can still write algebraic equations for $F_0 \equiv F(0)$ (and $F(x)$):

$$\left\{ \begin{array}{lcllcl} F_0 & = & 1 + L_1 R_1 + L_2 R_2 & L_1 & = & L_2 R_1 + L_3 R_2 \\ R_1 & = & L_1 R_2 & L_2 & = & L_3 R_1 \\ R_2 & = & t F_0 & L_3 & = & t F_0 \end{array} \right.$$

[Duchon 98, Labelle-Yeh 90...]

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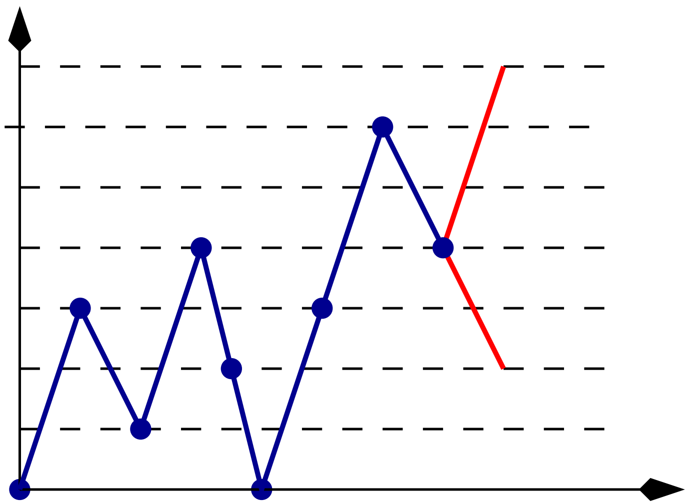
$$\left\{\begin{array}{lllllll} F_0 &=& 1+L_1R_1+L_2R_2 & \quad & L_1 &=& L_2R_1+L_3R_2 \\ R_1 &=& L_1R_2 & & L_2 &=& L_3R_1 \\ R_2 &=& tF_0 & & L_3 &=& tF_0 \end{array}\right.$$

[Duchon 98, Labelle-Yeh 90...]

- But the step-by-step approach gives a single easier equation:

$$F(x) = 1 + tx^3F(x) + \frac{t}{x^2}\left(F(x) - F_0 - xF_1\right)$$

where $F_i = [x^i]F(x)$ is the generating function of walks ending at position i.



Our prototype has many relatives

- Walks on a half-line with steps $+3$ and -2
- Walks on a half-line with steps in any prescribed finite set $\mathcal{S}$
- Permutations with no ascending sequence of length 3
- Families of column-convex polyominoes [Temperley 56], [Feretic-Svrtan 93], [MBM 96]
- Lots and lots of problems that are equivalent to (possibly weighted) 1D walks [Prodinger 04, De Mier-Noy 03]...

$$F(x) = 1 + txF(x) + t\,\frac{F(x) - F(0)}{x}$$

Solving our prototype: The kernel method

$$F(x) = 1 + txF(x) + \frac{t}{x}(F(x) - F(0))$$

Equivalently,

$$(1 - t(x + 1/x))F(x) = 1 - tF(0)/x$$

- Let $X \equiv X(t)$ be the unique formal power series in t that cancels the kernel $1 - t(x + 1/x)$:

$$X(t) = \frac{1 - \sqrt{1 - 4t^2}}{2t} = t + t^3 + O(t^5)$$

- Eliminate $F(x)$ by setting $x = X(t)$:

$$0 = 1 - tF(0)/X \quad \Rightarrow \quad F(0) = X/t = \frac{1 - \sqrt{1 - 4t^2}}{2t^2}$$

- In particular, $F(0)$ (and $F(x)$) are algebraic

[Knuth, The Art of Computer programming, Vol. 2, 1972]

Our prototype has many relatives

- Walks on a half-line with steps $+3$ and -2

- Walks on a half-line with steps in any prescribed finite set $\mathcal{S}$

- Permutations with no ascending sequence of length 3

- Lots and lots of problems that are equivalent to (possibly weighted) 1D walks...

The kernel method solves them all and $F(x)$ is always algebraic

Walks with steps $+3, -2$: The kernel method

$$\left(1 - t(x^3 + 1/x^2)\right) F(x) = 1 - tF_0/x^2 - tF_1/x$$

• There exists two fractional series in t, denoted $X_{1,2} \equiv X_{1,2}(t)$ that cancel the kernel $1 - t(x^3 + 1/x^2)$. Equivalently,

$$X_i^2 = t(X_i^5 + 1)$$

Their expansions can be computed using Gfun (Maple)

• Eliminate $F(x)$ by setting $x = X_i(t)$:

$$0 = 1 - tF_0/X_i^2 - tF_1/X_i \qquad \text{for } i = 1, 2$$

• We have two equations with two unknowns F_0 and F_1. Solving for F_0 gives

$$F_0 = -\frac{X_1 X_2}{t}$$

• If needed, the elimination of X_1 and X_2 gives

$$F_0 = 1 + 2\,t^5 {F_0}^5 - t^5 {F_0}^6 + t^5 {F_0}^7 + t^{10} {F_0}^{10}.$$

The roots of the kernel: the Newton-Puiseux theorem

Let $\mathbb{L}$ be an algebraically closed field of characteristic 0. Let $K(t;x) \in \mathbb{L}[t,x]$, of degree d in x. For instance,

$$K(t;x) = x^2 - t(1+x^5) \qquad\qquad (d=5)$$

- The equation (in x) $K(t;x)=0$ has d roots, which are Puiseux series in t :

$$X = \sum_{n \geq n_0} a_n t^{n/q}, \quad n_0 \in \mathbb{Z}, \quad q \in \mathbb{N} \setminus \{0\}.$$

- The number of roots that are finite at $t=0$ (that is, such that $n_0 \geq 0$) is

$$d_0 = \deg K(0;x) \qquad\qquad (d_0 = 2).$$

Example: walks with steps $+3, -2$

- The equation $x^2 - t(1+x^5) = 0$ has 5 roots, 2 of which are finite at $t=0$:

$$X_1 = \sqrt{t} + \frac{1}{2}\sqrt{t}^6 + \frac{9}{8}\sqrt{t}^{11} + O(\sqrt{t}^{15})$$

$$X_2 = -\sqrt{t} + \frac{1}{2}\sqrt{t}^6 - \frac{9}{8}\sqrt{t}^{11} + O(\sqrt{t}^{15})$$

and

$$X_{3,4,5} = \frac{1}{z} - \frac{z^3}{3} - \frac{z^8}{3} + O(z^{14})$$

where z is one of the 3 cubic roots of t.

- GFun, command “algeqtoseries”

A generic example: Walks on a half-line

- $\mathcal{S} \subset \mathbb{Z}$: the (finite) set of allowed steps. Denote $a = \max \mathcal{S}$ and $-b = \min \mathcal{S}$.

- Proposition: Let $K(t;x) = x^b \left(1 - t \sum_{j \in \mathcal{S}} x^j\right)$.

It is a polynomial in x of degree $a+b$. Exactly, b of its roots, say $X_1, X_2, \ldots, X_b$, are finite at $t = 0$.

The generating function of walks on the half-line $\mathbb{N}$ starting and ending at 0 is:

$$F_0 = \frac{(-1)^{b+1}}{t} \prod_{i=1}^{b} X_i$$

- Corollary: F_0 is algebraic of degree (at most) $\binom{a+b}{b}$

Some references

- Knuth's historical example

  ◦ The Art of Computer programming, Vol. 2, Section 2.2.1, Ex. 4, 1972

- Walks on a half-line

  ◦ Linear recurrences with constant coefficients: the multivariate case, MBM & Petkovšek, Discrete Math. 225 (2000)

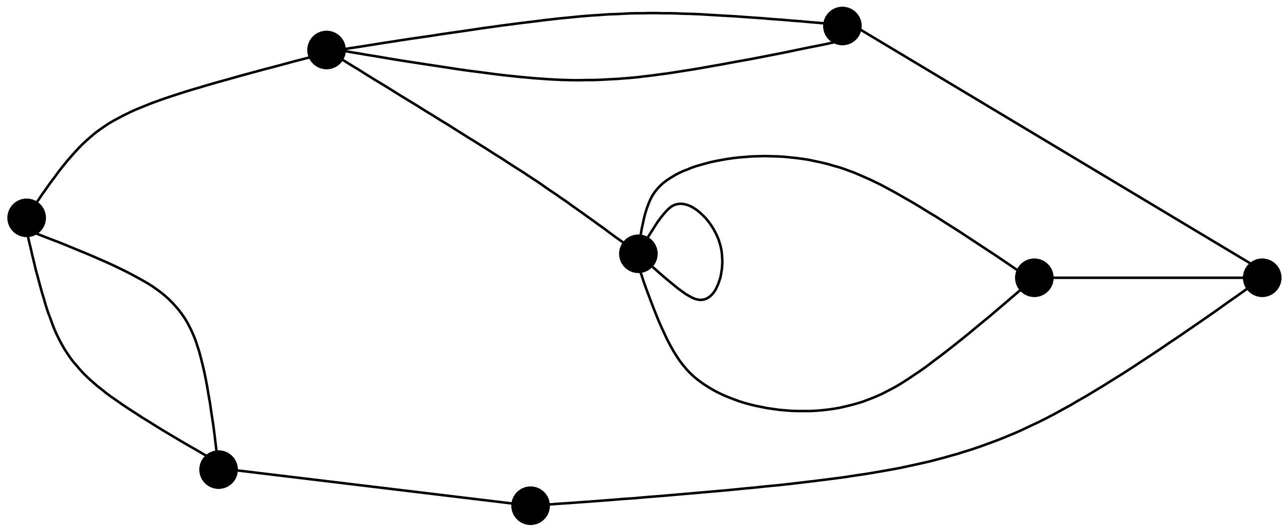
  ◦ Generating functions for generating trees, Banderier, MBM, Denise, Flajolet, Gardy, Gouyou-Beauchamps, Discrete Mathematics 246 (2002)

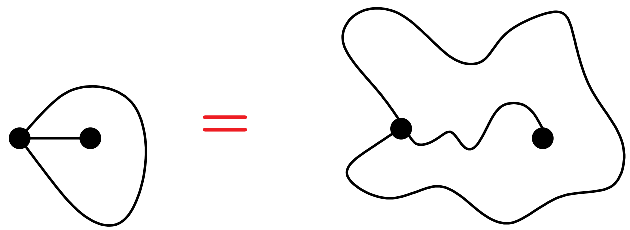
  ◦ Basic analytic combinatorics of directed lattice paths, Banderier & Flajolet, Theoret. Comput. Sci. 281 (2002)

II. Polynomial equations with one catalytic variable

$$F(x) = 1 + tx^2F(x)^2 + t\,\frac{xF(x) - F(1)}{x-1}$$

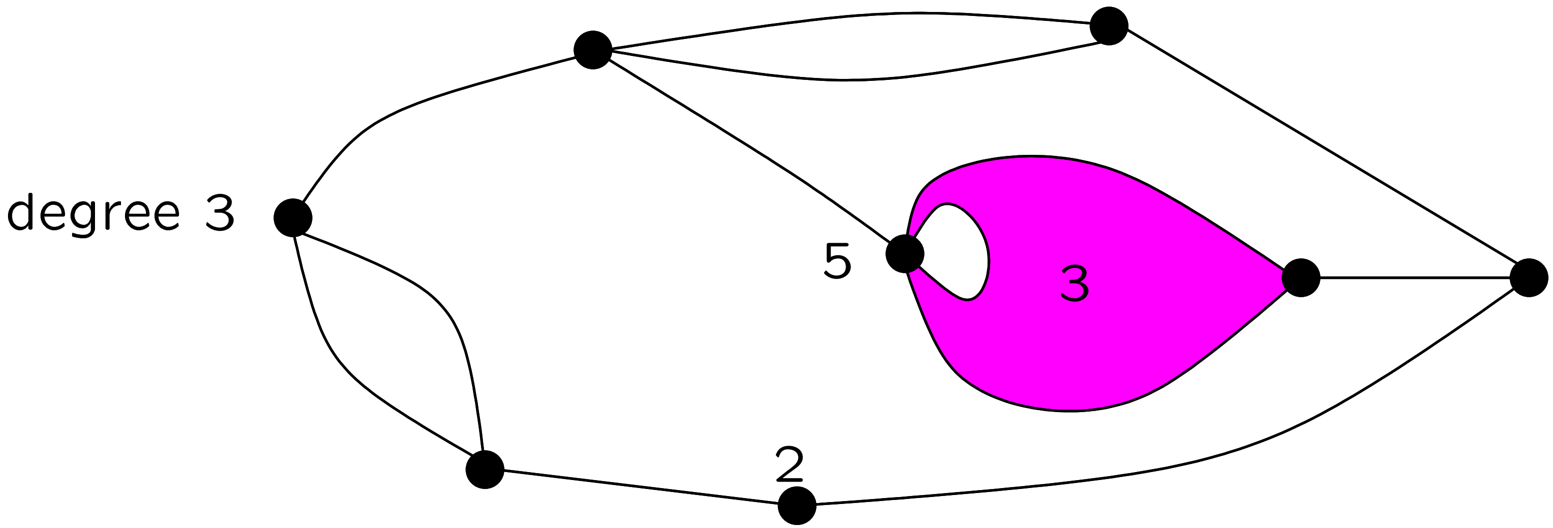
Where does it come from? Rooted planar maps





There are finitely many maps with n edges

Where does it come from? Rooted planar maps



- vertices $V(M)$
- edges $E(M)$
- and faces

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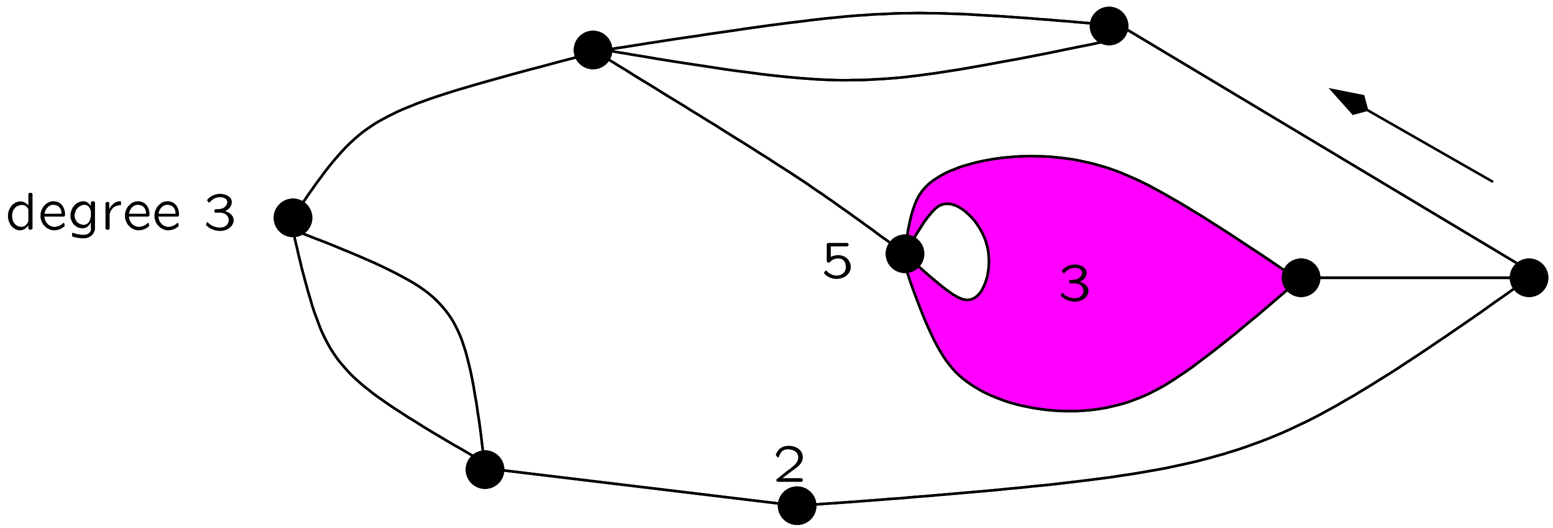


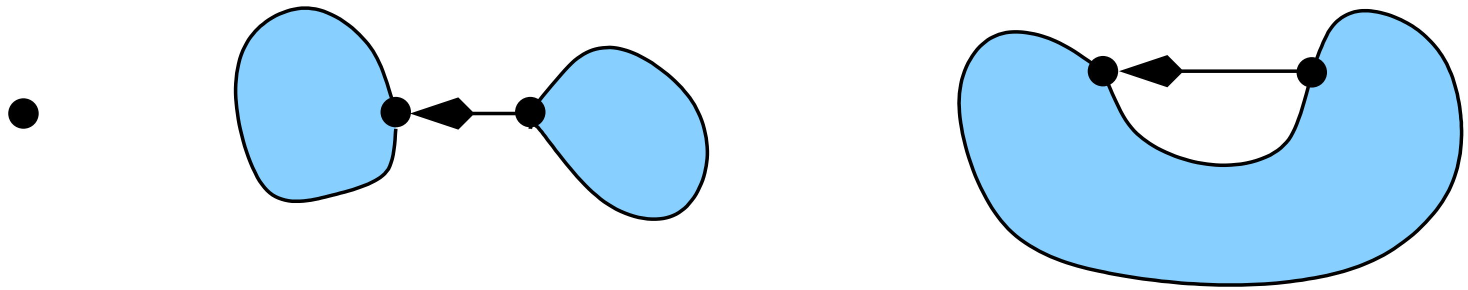
- vertices $V(M)$
- edges $E(M)$
- and faces

Recursive description of planar maps: deleting the root-edge

Let

$$F(t;x) \equiv F(x) = \sum_{M} t^{\mathsf{e}(M)} x^{\mathsf{df}(M)} = \sum_{d \geq 0} F_d(t) x^d$$

where $\mathsf{e}(M)$ is the number of edges and $\mathsf{df}(M)$ the degree of the outer face.



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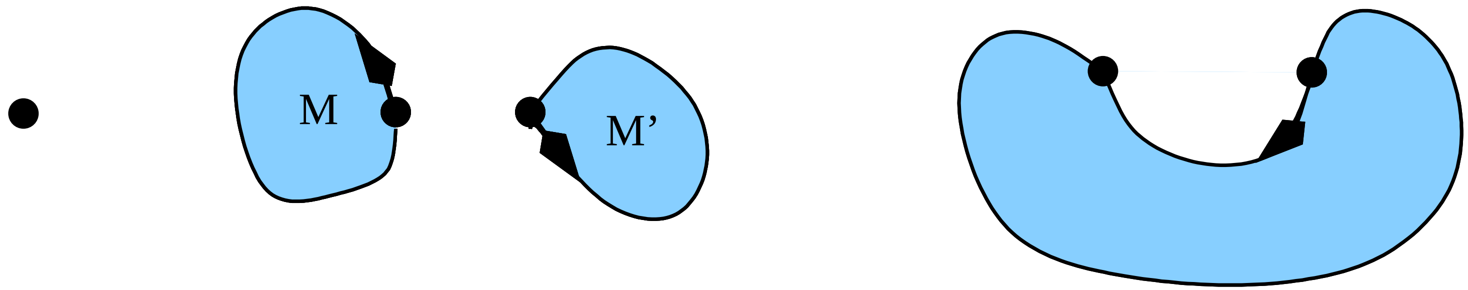

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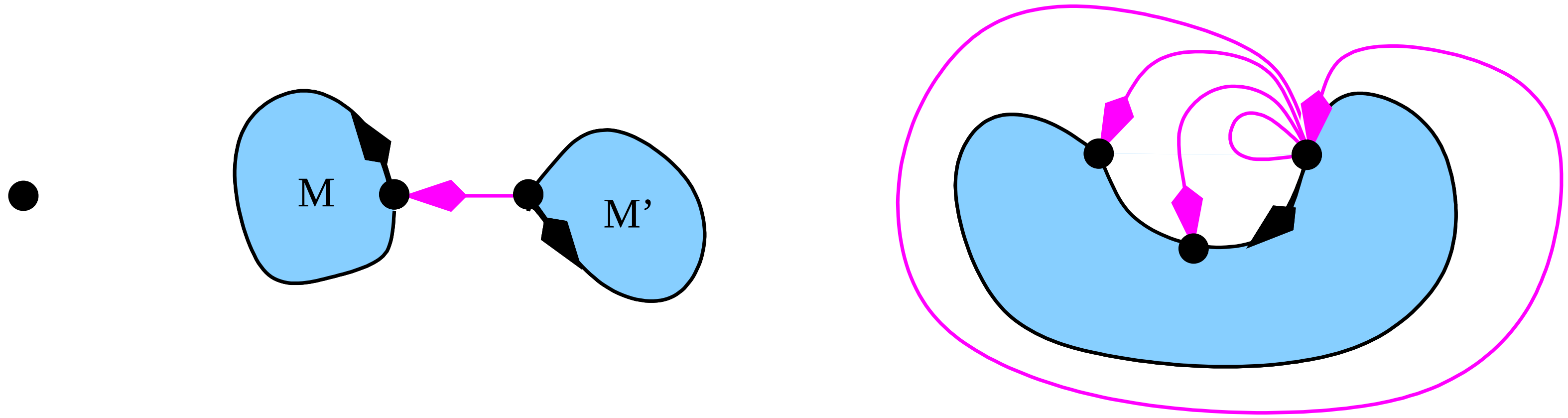

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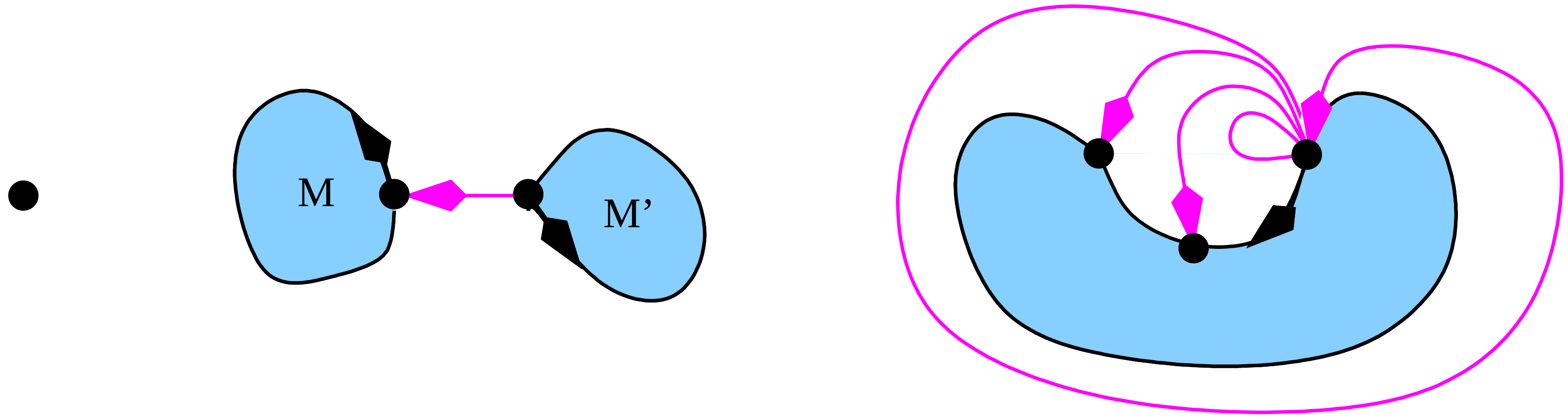

$$\begin{aligned} F(x) &= 1 + tx^2F(x)^2 + t\sum_{d\geq 0} F_d(t)\left(x^{d+1}+x^d+\cdots+x\right) \\ &= 1 + tx^2F(x)^2 + tx\frac{xF(x)-F(1)}{x-1} \end{aligned}$$

[Tutte 68] A quadratic equation with one catalytic variable, x

Do we really need this equation?

Maybe... From

$$F(x) = 1 + tx^2F(x)^2 + t\,\frac{xF(x) - F(1)}{x-1},$$

Tutte and Brown derived

$$F(t;1) = \frac{(1-12t)^{3/2} - 1 + 18\,t}{54t^2} = \sum_{n\geq 0} \frac{2\cdot 3^n}{n(n+1)}\binom{2n}{n}\,t^n.$$

But it took more than 10 years to find a combinatorial explanation of this formula [Cori-Vauquelin 81]

Moreover...

Our prototype has many relatives

• All kinds of maps (with prescribed degrees, non-separable, of higher genus, with hard particles...)
[Tutte, Brown, Bender & Canfield, Gao, Wanless & Wormald, MBM-Jehanne...]

$$F(x) = 1 + txF(x)^3 + tx(2F(x) + F(1))\frac{F(x) - F(1)}{x-1} + tx\frac{F(x) - F(1) - (x-1)F'(1)}{(x-1)^2}$$

• Two-stack sortable permutations [Zeilberger 92]

• Intervals in the Tamari lattices [Chapoton 06], [mbm, Fusy, Préville-Ratelle 11]

• ...

Polynomial equations with one catalytic variable [MBM-Jehanne 05]

- General framework

Assume

$$P(F(x), F_1, \ldots, F_k, t, x) = 0 \quad (1)$$

where $F(x) \equiv F(t;x)$ is a series in t with polynomial coefficients in x, and $F_i \equiv F_i(t)$ is (for instance) the coefficient of x^{i-1} in $F(t;x)$.

- Results

1. The solution of every well-founded equation of this type is algebraic.

2. A practical strategy allows to solve specific examples (that is, to derive from (1) an algebraic equation for $F(x)$, or $F_1, \ldots, F_k$).

(Includes and generalizes the kernel method and Brown's quadratic method.)

The general strategy: principle

Assume

$$P(F(x), F_1, \dots, F_k, t, x) = 0$$

where $P(x_0, x_1, \dots, x_k, x_{k+1}, x_{k+2})$ is a polynomial with coefficients in $\mathbb{K}$, $F(x)$ is a series in t with coefficients in $\mathbb{K}[x]$, and F_i a series in t with coefficients in $\mathbb{K}$ for all i.

For all series $X \equiv X(t)$ such that

- the series $F(X) \equiv F(t; X)$ is well-defined
- $\frac{\partial P}{\partial x_0}(F(X), F_1, \dots, F_k, t, X) = 0$,

one has

$$\frac{\partial P}{\partial x_{k+2}}(F(X), F_1, \dots, F_k, t, X) = 0.$$

(And of course

$$P(F(X), F_1, \dots, F_k, t, X) = 0.)$$

The general strategy: principle

Assume

$$P(F(x), F_1, \ldots, F_k, t, x) = 0 \tag{2}$$

where $P(x_0, x_1, \ldots, x_k, x_{k+1}, x_{k+2})$ is a polynomial with coefficients in $\mathbb{K}$,
$F(x)$ is a series in t with coefficients in $\mathbb{K}[x]$,
and F_i a series in t with coefficients in $\mathbb{K}$ for all i.

For all series $X \equiv X(t)$ such that

- the series $F(X) \equiv F(t, X)$ is well-defined
- $\frac{\partial P}{\partial x_0}(F(X), F_1, \ldots, F_k, t, X) = 0$,

one has

$$\frac{\partial P}{\partial x_{k+2}}(F(X), F_1, \ldots, F_k, t, X) = 0.$$

Proof: differentiate (2) with respect to x

$$F'(x) \frac{\partial P}{\partial x_0}(F(x), F_1, \ldots, F_k, t, x) + \frac{\partial P}{\partial x_{k+2}}(F(x), F_1, \ldots, F_k, t, x) = 0.$$

The general strategy: hope

• There exist k series $X_1, \ldots, X_k$ such that

$$\frac{\partial P}{\partial x_0}(F(X_i), F_1, \ldots, F_k, t, X_i) = 0.$$

In this case, for each X_i,

$$\frac{\partial P}{\partial x_{k+2}}(F(X_i), F_1, \ldots, F_k, t, X_i) = 0$$

and

$$P(F(X_i), F_1, \ldots, F_k, t, X_i) = 0.$$

• This system of $3k$ polynomial equations in $3k$ unknowns $F_1, \ldots, F_k$, $X_1, \ldots, X_k$, $F(X_1), \ldots, F(X_k)$ implies (together with the fact that the X_i are distinct) the algebraicity of the F_i.

The linear case: recovering the kernel method

- Assume

$$P(F(x), F_1, \ldots, F_k, t, x) = K(t; x)F(x) + Q(F_1, \ldots, F_k, t, x) = 0$$

for some polynomial Q.

The linear case: recovering the kernel method

- Assume

$$P(F(x), F_1, \ldots, F_k, t, x) = K(t; x)F(x) + Q(F_1, \ldots, F_k, t, x) = 0$$

for some polynomial Q.

- Then $\frac{\partial P}{\partial x_0}(F(X_i), F_1, \ldots, F_k, t, X_i) = 0$ reads

$$K(t; X_i) = 0.$$

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- Then $\frac{\partial P}{\partial x_0}(F(X_i), F_1, \ldots, F_k, t, X_i) = 0$ reads

$$K(t; X_i) = 0.$$

- Combined with $P(F(X_i), F_1, \ldots, F_k, t, X_i) = 0$, this implies

$$Q(F_1, \ldots, F_k, t, X_i) = 0$$

for $1 \leq i \leq k$: we have a system of $2k$ polynomial equations in $2k$ unknowns $F_1, \ldots, F_k,\ X_1, \ldots, X_k$.

The linear case: recovering the kernel method

- Assume

$$P(F(x), F_1, \ldots, F_k, t, x) = K(t; x)F(x) + Q(F_1, \ldots, F_k, t, x) = 0$$

for some polynomial Q.

- Then $\frac{\partial P}{\partial x_0}(F(X_i), F_1, \ldots, F_k, t, X_i) = 0$ reads

$$K(t; X_i) = 0.$$

- Combined with $P(F(X_i), F_1, \ldots, F_k, t, X_i) = 0$, this implies

$$Q(F_1, \ldots, F_k, t, X_i) = 0$$

for $1 \leq i \leq k$: we have a system of $2k$ polynomial equations in $2k$ unknowns $F_1, \ldots, F_k$, $X_1, \ldots, X_k$.

- The equations $\frac{\partial P}{\partial x_{k+2}}(F(X_i), F_1, \ldots, F_k, t, X_i) = 0$ are not needed unless we are interested in the series $F(X_i)$.

Solution of our prototype

- Planar maps [Tutte 68]

$$F(x) = 1 + tx^2F(x)^2 + t\,\frac{xF(x) - F(1)}{x-1}$$

$\Rightarrow P(F(x), F_1, t, x) = 0$ with $F_1 = F(1)$

- Existence of X

$$\begin{aligned} \frac{\partial P}{\partial x_0}(F(X), F_1, t, X) = 0 \;&\Leftrightarrow\; 1 = 2tX^2F(X) + \frac{tX}{X-1} \\ &\Leftrightarrow\; X = 1 + 2tX^2(X-1)F(X) + tX. \end{aligned}$$

$\Rightarrow$ There exists one series $X(t)$ such that

$$\frac{\partial P}{\partial x_0}(F(X), F_1, t, X) = 0,$$

$$\frac{\partial P}{\partial x_3}(F(X), F_1, t, X) = 0, \qquad P(F(X), F_1, t, X) = 0.$$

- Elimination of $F(X)$ and X

$$27t^2F_1^2 + F_1 - 1 - 18tF_1 + 16t = 0$$

Equivalently,

$$F_1 = F(t;1) = \frac{(1-12t)^{3/2} - 1 + 18\,t}{54t^2} = \sum_{n\geq 0} \frac{2\cdot 3^n}{n(n+1)}\binom{2n}{n}\,t^n$$

Polynomial equations with one catalytic variable

Thm. Let Q be a polynomial in $k+3$ variables. Let $F(t;x) \equiv F(x)$ be the unique formal power series in t (with polynomial coefficients in x) such that

$$F(x) = F_0(x) + t\, Q\Big(F(x), \Delta F(x), \Delta^{(2)} F(x), \ldots, \Delta^{(k)} F(x), t, x\Big),$$

where $F_0(x) \in \mathbb{C}[x]$,

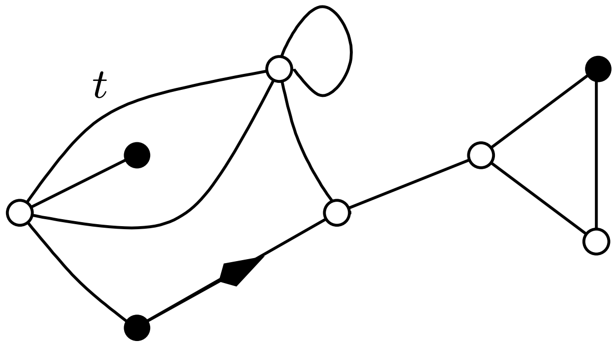
$$\Delta^{(i)} F(x) = \frac{F(x) - F_1 - xF_2 - \cdots - x^{i-1} F_i}{x^i}$$

and F_i is the coefficient of x^{i-1} in $F(x)$.

The above method works and $F(x)$ is algebraic (as well as all the F_i's).

[MBM-Jehanne 05]

Example: the hard-particle model on planar maps



$$\circ \quad F(x) = 1 + G(x) + tx^2F(x)^2 + \frac{tx\,(xF(x) - F(1))}{x-1}$$

$$\bullet \quad G(x) = txF(x) + txF(x)G(x) + \frac{tx\,(G(x) - G(1))}{x-1}$$

Proposition: Let $T \equiv T(t)$ be the unique series with constant term 0 such that

$$T(1-2T)(1-3T+3T^2) = t.$$

Then

$$t^2F(1) = T^2(1 - 7T + 16T^2 + T - 15\,T^3 + 4\,T^4).$$

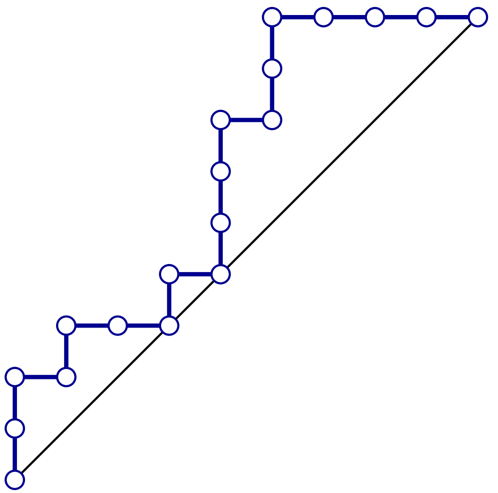
[MBM-Jehanne 05]

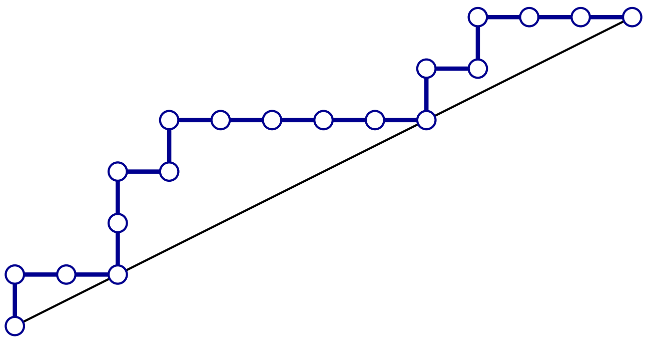
A generic example: intervals in the m-Tamari lattices

An m-ballot path of size n:

- starts at $(0, 0)$,
- ends at (mn, n),
- never goes below the line $\{x = my\}$.

Examples: $m = 1$ $m = 2$

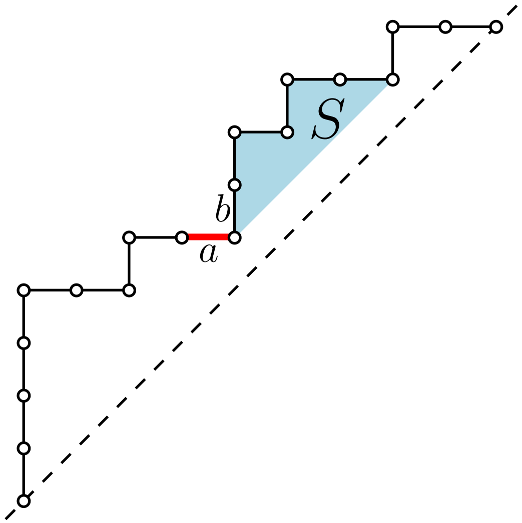




[mbm, Fusy, Préville-Ratelle 11]

$m = 1$: The (usual) Tamari lattice $\mathcal{T}_n$

Covering relation:



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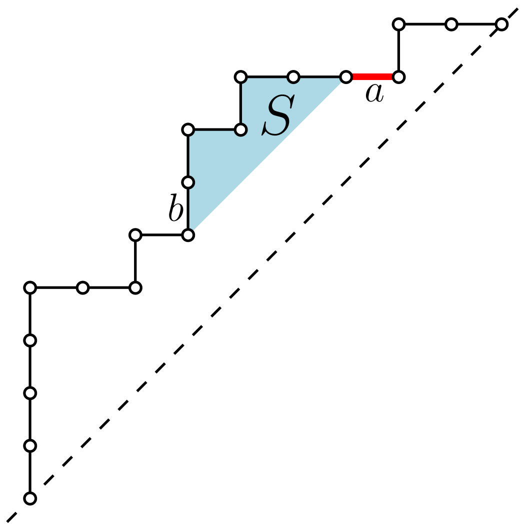
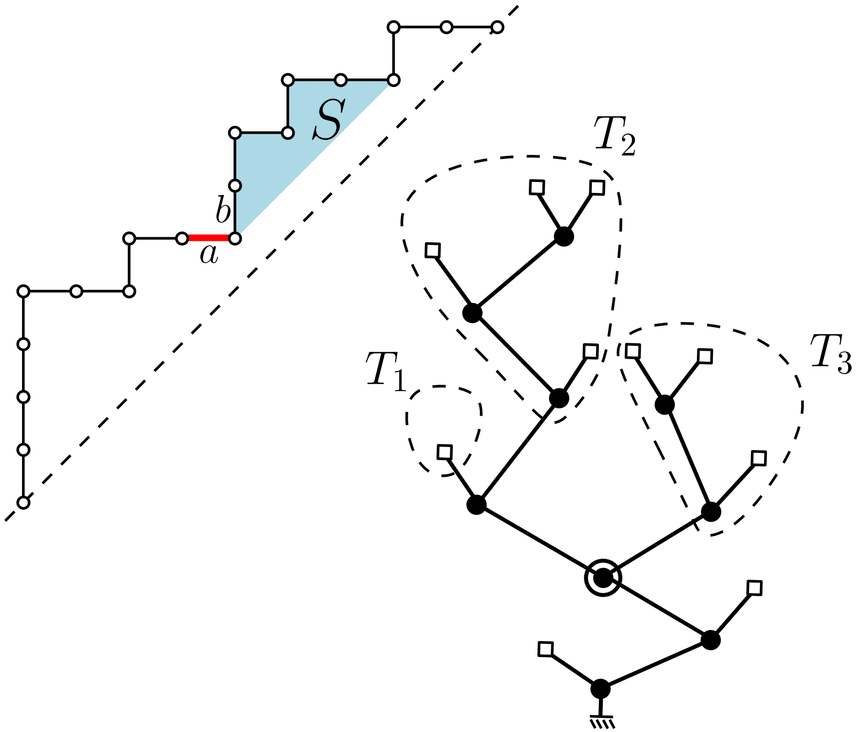


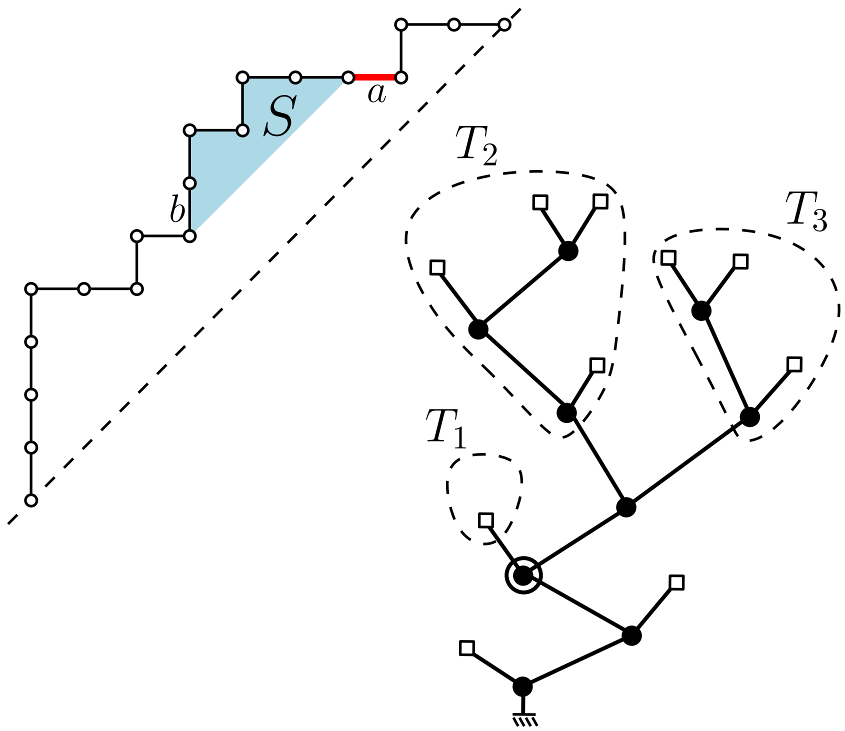
[Huang-Tamari 72]

$m = 1$: The (usual) Tamari lattice $\mathcal{T}_n$

Covering relation:



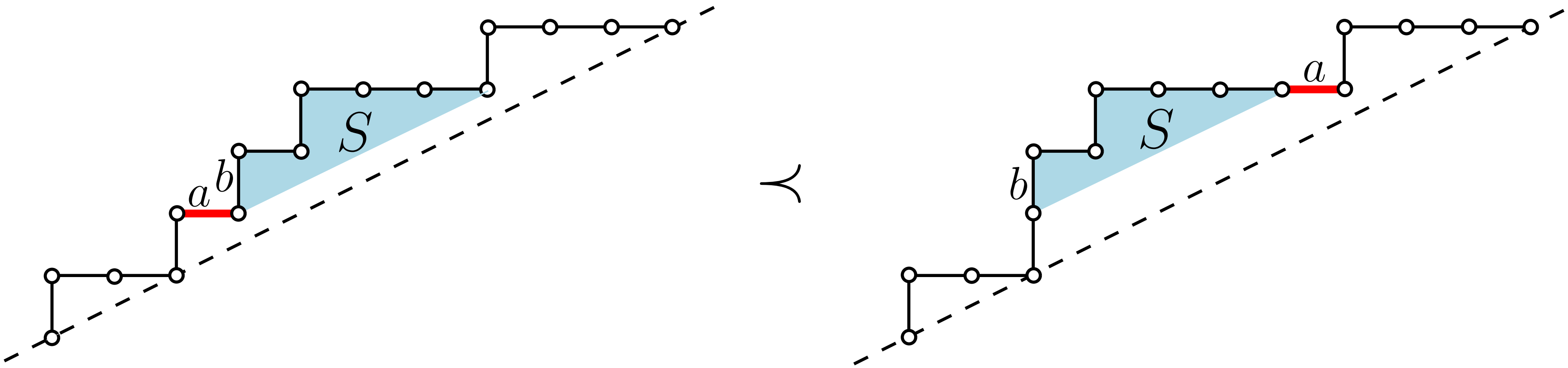
$\prec$



[Huang-Tamari 72]

The m-Tamari lattice $\mathcal{T}_n^{(m)}$

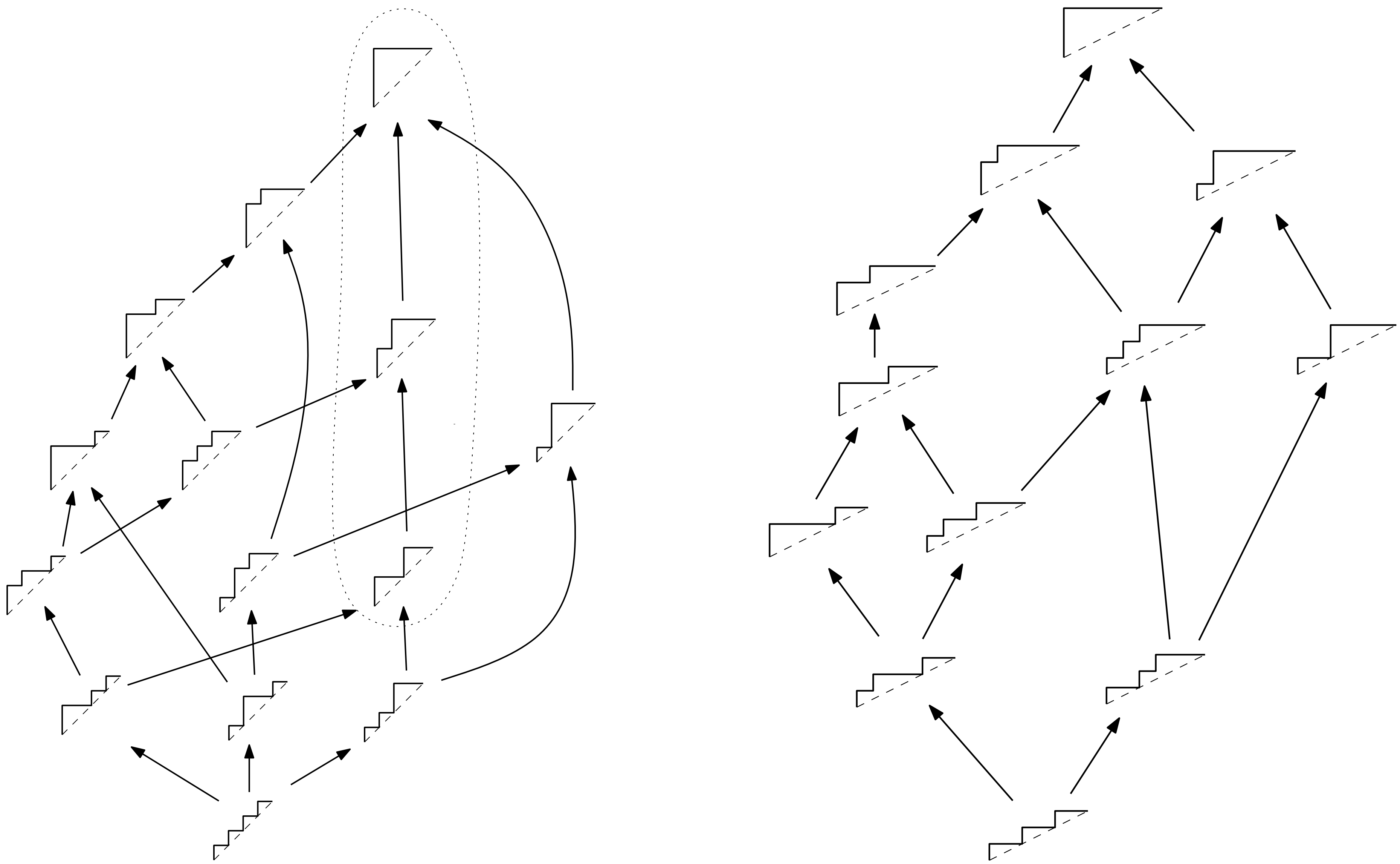
Covering relation:



[Bergeron 10]

Proposition: Defines a lattice

The m-Tamari lattice $\mathcal{T}_n^{(m)}$



$m = 1, n = 4$ $\qquad$ $m = 2, n = 3$

Bergeron's conjecture (2010)

Conjecture: Let $m \geq 1$ and $n \geq 1$. The number of intervals in the Tamari lattice $\mathcal{T}_n^{(m)}$ is

$$f_n^{(m)} = \frac{m+1}{n(mn+1)} \binom{n(m+1)^2+m}{n-1}$$

- Related to the study of coinvariant spaces of polynomials in 3 sets of variables

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- Related to the study of coinvariant spaces of polynomials in 3 sets of variables
- Map-like numbers!

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- Related to the study of coinvariant spaces of polynomials in 3 sets of variables
- Map-like numbers!
- When $m=1$: proved by [Chapoton 06]

$$f_n^{(1)} = \frac{2}{n(n+1)} \binom{4n+1}{n-1}.$$

This is also the number of 3-connected planar triangulations on $n+3$ vertices [Tutte 62]

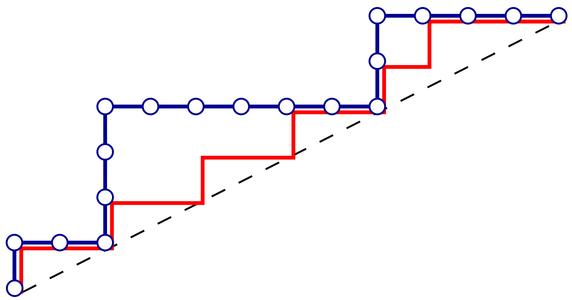
$\Rightarrow$ Bijection found by [Bernardi & Bonichon 09]

Generating functions

Let $I = [P, Q]$ be a Tamari interval. A contact of I is a contact of the lower path P with the x-axis.

The initial rise of I is the height of the first peak of the upper path Q.

We denote by $F^{(m)}(t; x, y)$ the generating function of m-Tamari intervals, where t counts the size, x the number of contacts and y the initial rise.



size 6

6 contacts

initial rise 1

[mbm, Fusy, Préville-Ratelle 11]

A functional equation

Proposition: For $m \geq 1$, let $F(x,y) \equiv F^{(m)}(t;x,y)$ be the generating function of m-Tamari intervals. Then

$$F(x,y) = x + xyt\,(F(x,1)\cdot\Delta)^{(m)}\,(F(x,y)),$$

where Δ is the divided difference operator

$$\Delta S(x) = \frac{S(x)-S(1)}{x-1},$$

and the power m means that the operator $G(x) \mapsto F(x)\cdot\Delta G(x)$ is applied m times.

Examples

1. When $m = 1$, the equation reads

$$F(x,y) \;=\; x + xytF(x,1)\,\frac{F(x,y)-F(1,y)}{x-1}.$$

When $y = 1$, we obtain a quadratic equation with one catalytic variable:

$$F(x) = x + xtF(x)\,\frac{F(x)-F(1)}{x-1}.$$

2. When $m = 2$,

$$F(x,y) \;=\; x + \frac{xyt}{x-1}\,F(x,1)\,\left(F(x,1)\,\frac{F(x,y)-F(1,y)}{x-1} - F(1,1)F'_x(1,y)\right),$$

When $y = 1$, we obtain a cubic equation with one catalytic variable:

$$F(x) = x + \frac{xt}{x-1}F(x)\,\left(F(x)\,\frac{F(x)-F(1)}{x-1} - F(1)F'(1)\right).$$

Solution of the functional equation

Proposition: Let z, u and v be three indeterminates, and set

$$t = z(1-z)^{m^2+2m}, \quad x = \frac{1+u}{(1+zu)^{m+1}}, \quad \text{and} \quad y = \frac{1+v}{(1+zv)^{m+1}}.$$

Then $F^{(m)}(t; x, y)$ becomes a formal power series in z with coefficients in $\mathbb{Q}[u, v]$, and this series is rational. More precisely,

$$yF^{(m)}(t; x, y) = \frac{(1+u)(1+zu)(1+v)(1+zv)}{(u-v)(1-zuv)(1-z)^{m+2}} \left(\frac{1+u}{(1+zu)^{m+1}} - \frac{1+v}{(1+zv)^{m+1}} \right).$$

In particular, $yF^{(m)}(t; x, y)$ is a symmetric series in x and y.

Proof: solve for small values of m, guess the general form, and check!

Bergeron's conjecture

The generating function $F^{(m)}(t;1,1)$ of m-Tamari intervals is

$$F(t;1,1) = \frac{1-(m+1)Z}{(1-Z)^{m+2}},$$

with

$$Z = \frac{t}{(1-Z)^{m^2+2m}}.$$

The Lagrange inversion formula gives the number of intervals of size mn as

$$f_n^{(m)} = \frac{m+1}{n(mn+1)}\binom{(m+1)^2n+m}{n-1}.$$

Combinatorial proof?

Some references

• Polynomial equations with one catalytic variable, algebraic series and map enumeration, MBM & Jehanne, J. Combin. Theory Ser. B 96 (2006)

• The number of intervals in the m-Tamari lattices, MBM, Fusy & Préville Ratelle, arxiv 1106.1498 (2011).

III. Linear equations with two (or more) catalytic variables

$$F(x,y) = 1 + tyF(x,y) + t\,\frac{F(x,y) - F(0,y)}{x} + tx\,\frac{F(x,y) - F(x,0)}{y}$$

Where does it come from? Walks in the quarter plane

• Count walks in the quarter plane $\mathbb{N}^2$, starting from $(0,0)$, consisting of steps N, W and SE, by their length (variable t) and the position of their endpoint (variables x and y):

$$F(t;x,y) \equiv F(x,y) = \sum_w t^{\ell(w)} x^{i(w)} y^{j(w)}$$

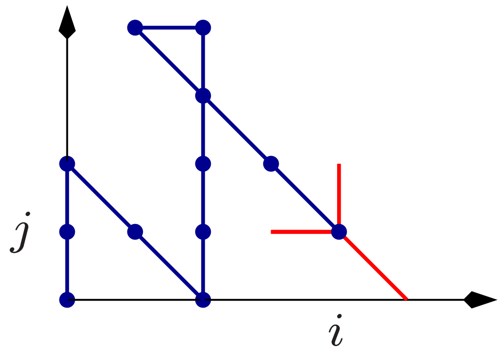
In particular:

- $\circ$ $F(t;0,y) \equiv F(0,y)$ counts walks ending on the y-axis,
- $\circ$ $F(x,y) - F(0,y)$ counts those ending at a positive abscissa.

• A step by step construction:

$$F(x,y) = 1 + tyF(x,y) + \frac{t}{x}\left(F(x,y) - F(0,y)\right)$$

$$+\frac{tx}{y}\left(F(x,y) - F(x,0)\right)$$



Do we really need this equation?

YES!

(pas de discussion)

Does it have relatives?

- Walks in the quarter plane taking their steps in any (finite) $\mathcal{S} \subset \mathbb{Z}^2$
[Kreweras 65], [Gessel 86], MBM, Mishna, Rechnitzer, Raschel, Kurkova, Kauers, Bostan, Zeilberger...

- Permutations with no ascending sequence of length 4

- Involutions with no descending sequence of length 5

- Baxter permutations

- Vexillary involutions

- Planar maps equipped with a bipolar orientation [Baxter 01]

- Planar maps equipped with a spanning tree

- ...

With an arbitrary number of catalytic variables: even more relatives

• Walks in $\mathbb{N}^d$ taking their steps in any (finite) $\mathcal{S} \subset \mathbb{Z}^d$
[$d = 3$: Bostan & Kauers 09]

$$F(x) = 1 + t \sum_{i=1}^{d} x_i F(x) + t \sum_{i=1}^{d} \frac{F(x) - F(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_d)}{x_i}$$

• Permutations with no ascending sequence of length m [Guibert 95, MBM 09]

• Involutions with no descending sequence of length m
[Guibert 95, Jaggard & Marincel 07, MBM 09]

$$B(x) = x_1 + t x_1 B(x) + t^2 x_1 \sum_{k=1}^{m} x_k x_{k+1} \frac{B(x) - B(x_1, \ldots, x_{k-1}, x_{k+1}, x_{k+1}, \ldots, x_m)}{x_k - x_{k+1}}$$

• Young tableaux, plane partitions, vicious walkers, osculating walkers...

Some bad news

- We have no general method that solves all such equations

- The solution is not always D-finite

[MBM-Petkovšek 03, Mishna-Rechnitzer 09]

$$F(x,y)=1+txyF(x,y)+ty\frac{F(x,y)-F(0,y)}{x}+tx\frac{F(x,y)-F(x,0)}{y}$$

- But it *is* sometimes D-finite... [MBM-Mishna 08]

$$F(x,y)=1+tyF(x,y)+t\,\frac{F(x,y)-F(0,y)}{x}+tx\,\frac{F(x,y)-F(x,0)}{y}$$

- ... or even algebraic [MBM 05, Bostan-Kauers 10]

$$F(x,y)=1+txyF(x,y)+t\,\frac{F(x,y)-F(0,y)}{x}+t\,\frac{F(x,y)-F(x,0)}{y}$$

Some bad news

Classification?

- The solution is not always D-finite

[MBM-Petkovšek 03, Mishna-Rechnitzer 09]

$$F(x,y) = 1 + txyF(x,y) + ty\frac{F(x,y)-F(0,y)}{x} + tx\frac{F(x,y)-F(x,0)}{y}$$

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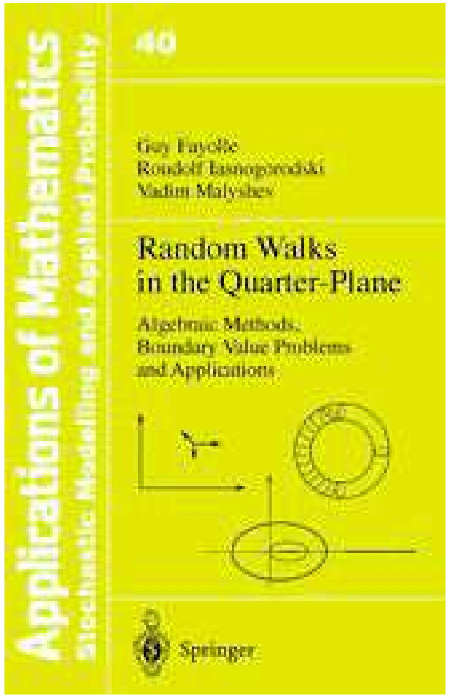
$$F(x,y) = 1 + txyF(x,y) + t\frac{F(x,y)-F(0,y)}{x} + t\frac{F(x,y)-F(x,0)}{y}$$

Some tools

- A key tool is a certain group associated with the kernel of the equation (the coefficient of $F(x,y)$) [Fayolle et al. 99]

- From examples:

it seems that $F(x,y)$ is D-finite if and only if the group is finite



Walks on the half-line: another solution

- The equation:

$$(1 - t(x + \bar{x}))\, xF(x) = x - F_0$$

with $\bar{x} = 1/x$.

- The kernel is unchanged when $x \mapsto \bar{x}$. Hence

$$(1 - t(x + \bar{x}))\bar{x}F(\bar{x}) = \bar{x} - F_0$$

- Eliminate F_0 (rather than $F(x)$) by taking the difference:

$$xF(x) - \bar{x}F(\bar{x}) = \frac{x - \bar{x}}{1 - t(x + \bar{x})}$$

- Extract the positive powers of x:

$$xF(x) = [x^{>0}]\ \frac{x - \bar{x}}{1 - t(x + \bar{x})}$$

- A group of order 2 is generated by $x \mapsto \bar{x}$

Our prototype: Where is the group?

$$F(x,y) = 1 + tyF(x,y) + t\,\frac{F(x,y) - F(0,y)}{x} + tx\,\frac{F(x,y) - F(x,0)}{y}$$

The kernel reads

$$K(x,y) = 1 - ty - \frac{t}{x} - \frac{tx}{y}$$

It is a Laurent polynomial in x (and y), of degree 1 and valuation -1.

Equivalently, $xyK(x,y)$ is a quadratic polynomial in x (and y)

Our prototype: Where is the group?

$$F(x,y) = 1 + tyF(x,y) + t\,\frac{F(x,y) - F(0,y)}{x} + tx\,\frac{F(x,y) - F(x,0)}{y}$$

The kernel reads

Φ

$$K(x,y) = 1 - ty - \frac{t}{x} - \frac{tx}{y}$$

Ψ

Observation: $K(x,y)$ is left invariant under the rational transformations

$$\Phi : (x,y) \mapsto \left(\frac{y}{x}, y\right) \quad \text{and} \quad \Psi : (x,y) \mapsto \left(x, \frac{x}{y}\right).$$

Moreover,

- Φ and Ψ are involutions
- They generate a (dihedral) group G

Our prototype: Where is the group?

• The transformations $\Phi : (x,y) \mapsto (\bar{x}y, y)$ and $\Psi : (x,y) \mapsto (x, x\bar{y})$ generate a group of order 6:

$$
\begin{tikzcd}
 & (\bar{x}y, y) \arrow[r, dash, "\Psi"] & (\bar{x}y, \bar{x}) \arrow[dr, dash, "\Phi"] & \\
(x,y) \arrow[ur, dash, "\Phi"] \arrow[dr, dash, "\Psi"'] & & & (\bar{y}, \bar{x}) \\
 & (x, x\bar{y}) \arrow[r, dash, "\Phi"'] & (\bar{y}, x\bar{y}) \arrow[ur, dash, "\Psi"'] &
\end{tikzcd}
$$

with $\bar{x} = 1/x$ and $\bar{y} = 1/y$

Our prototype: the role of the group

- The equation reads

$K(x,y)\ xyF(x,y) = xy - txF(x,0) - tyF(0,y)$ with $K(x,y) = 1 - t(y + \bar{x} + x\bar{y})$.

- The orbit of (x,y) under G is

$$(x,y) \overset{\Phi}{\longleftrightarrow} (\bar{x}y,y) \overset{\Psi}{\longleftrightarrow} (\bar{x}y,\bar{x}) \overset{\Phi}{\longleftrightarrow} (\bar{y},\bar{x}) \overset{\Psi}{\longleftrightarrow} (\bar{y},x\bar{y}) \overset{\Phi}{\longleftrightarrow} (x,x\bar{y}) \overset{\Psi}{\longleftrightarrow} (x,y).$$

- All transformations of G leave $K(x,y)$ invariant. Hence

$$\begin{array}{rclcccc} K(x,y)\ xyF(x,y) & = & xy & - & txF(x,0) & - & tyF(0,y) \\ K(x,y)\ \bar{x}y^2F(\bar{x}y,y) & = & \bar{x}y^2 & - & t\bar{x}yF(\bar{x}y,0) & - & tyF(0,y) \end{array}$$

Our prototype: the role of the group

- The equation reads

$$K(x,y)\ xyF(x,y) = xy - txF(x,0) - tyF(0,y) \qquad \text{with } K(x,y) = 1 - t(y + \bar{x} + x\bar{y}).$$

- The orbit of (x,y) under G is

$$(x,y) \overset{\Phi}{\longleftrightarrow} (\bar{x}y,y) \overset{\Psi}{\longleftrightarrow} (\bar{x}y,\bar{x}) \overset{\Phi}{\longleftrightarrow} (\bar{y},\bar{x}) \overset{\Psi}{\longleftrightarrow} (\bar{y},x\bar{y}) \overset{\Phi}{\longleftrightarrow} (x,x\bar{y}) \overset{\Psi}{\longleftrightarrow} (x,y).$$

- All transformations of G leave $K(x,y)$ invariant. Hence

$$\begin{array}{rclcccc}
K(x,y)\ xyF(x,y) & = & xy & - & txF(x,0) & - & tyF(0,y) \\
K(x,y)\ \bar{x}y^2F(\bar{x}y,y) & = & \bar{x}y^2 & - & t\bar{x}yF(\bar{x}y,0) & - & tyF(0,y) \\
K(x,y)\ \bar{x}^2yF(\bar{x}y,\bar{x}) & = & \bar{x}^2y & - & t\bar{x}yF(\bar{x}y,0) & - & t\bar{x}F(0,\bar{x}) \\
K(x,y)\ \bar{x}\bar{y}F(\bar{y},\bar{x}) & = & \bar{x}\bar{y} & - & t\bar{y}F(\bar{y},0) & - & t\bar{x}F(0,\bar{x}) \\
K(x,y)\ x^2\bar{y}F(x,x\bar{y}) & = & x^2\bar{y} & - & t\bar{y}F(\bar{y},0) & - & tx\bar{y}F(0,x\bar{y}) \\
K(x,y)\ x^2\bar{y}F(x,x\bar{y}) & = & x^2\bar{y} & - & txF(x,0) & - & tx\bar{y}F(0,x\bar{y})
\end{array}$$

Our prototype: the role of the group

- All transformations of G leave $K(x,y)$ invariant. Hence

$$\begin{array}{rclclcl}
K(x,y)\ xyF(x,y) & = & xy & - & txF(x,0) & - & tyF(0,y) \\
K(x,y)\ \bar{x}y^2F(\bar{x}y,y) & = & \bar{x}y^2 & - & t\bar{x}yF(\bar{x}y,0) & - & tyF(0,y) \\
K(x,y)\ \bar{x}^2yF(\bar{x}y,\bar{x}) & = & \bar{x}^2y & - & t\bar{x}yF(\bar{x}y,0) & - & t\bar{x}F(0,\bar{x}) \\
\cdots & = & \cdots & & & & \\
K(x,y)\ x^2\bar{y}F(x,x\bar{y}) & = & x^2\bar{y} & - & txF(x,0) & - & tx\bar{y}F(0,x\bar{y})
\end{array}$$

$\Rightarrow$ Form the alternating sum of the equation over all elements of the orbit: this eliminates all unknown series on the r.h.s.

$$K(x,y)\Big(xyF(x,y) - \bar{x}y^2F(\bar{x}y,y) + \bar{x}^2yF(\bar{x}y,\bar{x}) \\
- \bar{x}\bar{y}F(\bar{y},\bar{x}) + x\bar{y}^2F(\bar{y},x\bar{y}) - x^2\bar{y}F(x,x\bar{y})\Big) = \\
xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}.$$

The orbit sum

Our prototype: the role of the group

$$
\begin{aligned}
xyF(x,y) - \bar{x}y^2F(\bar{x}y,y) + \bar{x}^2yF(\bar{x}y,\bar{x}) \\
- \bar{x}\bar{y}F(\bar{y},\bar{x}) + x\bar{y}^2F(\bar{y},x\bar{y}) - x^2\bar{y}F(x,x\bar{y}) = \\
\frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}
\end{aligned}
$$

- Both sides are power series in t, with coefficients in $\mathbb{Q}[x,\bar{x},y,\bar{y}]$.

- Extract the part with positive powers of x and y:

$$
xyF(x,y) = [x^{>0}y^{>0}]\,\frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}
$$

is a D-finite series.

[Lipshitz 88]

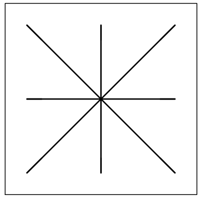
But but but...

This is just the reflection principle! [Gessel-Zeilberger 92]

True. But the reflection principle is performed here at the level of power series rather than at a combinatorial level. One can first perform on the equation all kinds of changes of variables, that do not necessarily have a combinatorial counterpart.

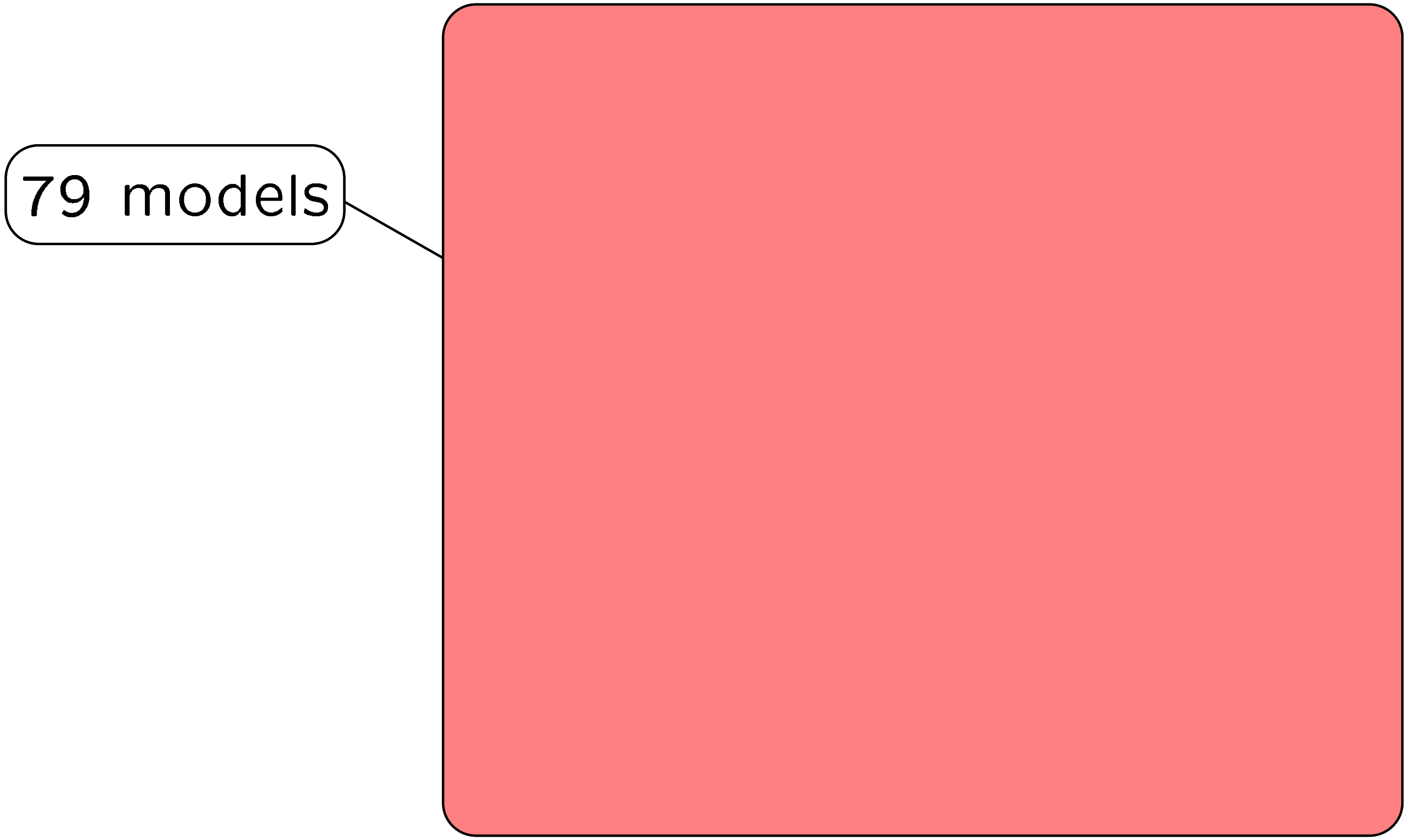
Two possible developments

• Classification of walks with small steps in the quarter plane ($\mathcal{S} \subset \{-1,0,1\}^2$)
[MBM & Mishna 08]
[Bostan & Kauers 10], [Kauers, Koutschan, Zeilberger 09]
[Mishna & Rechnitzer 09]

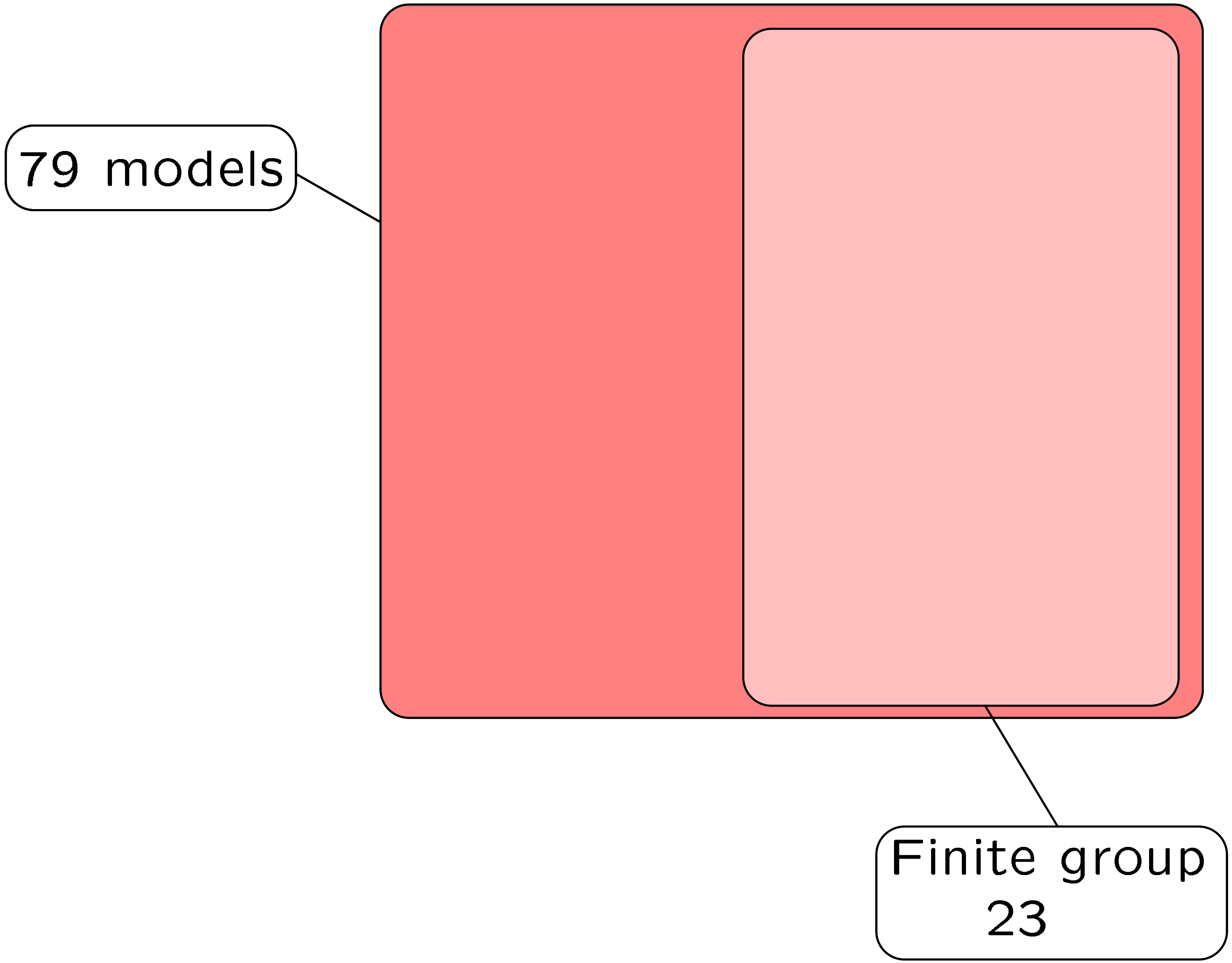


• Examples with arbitrarily many catalytic parameters
[MBM 10]

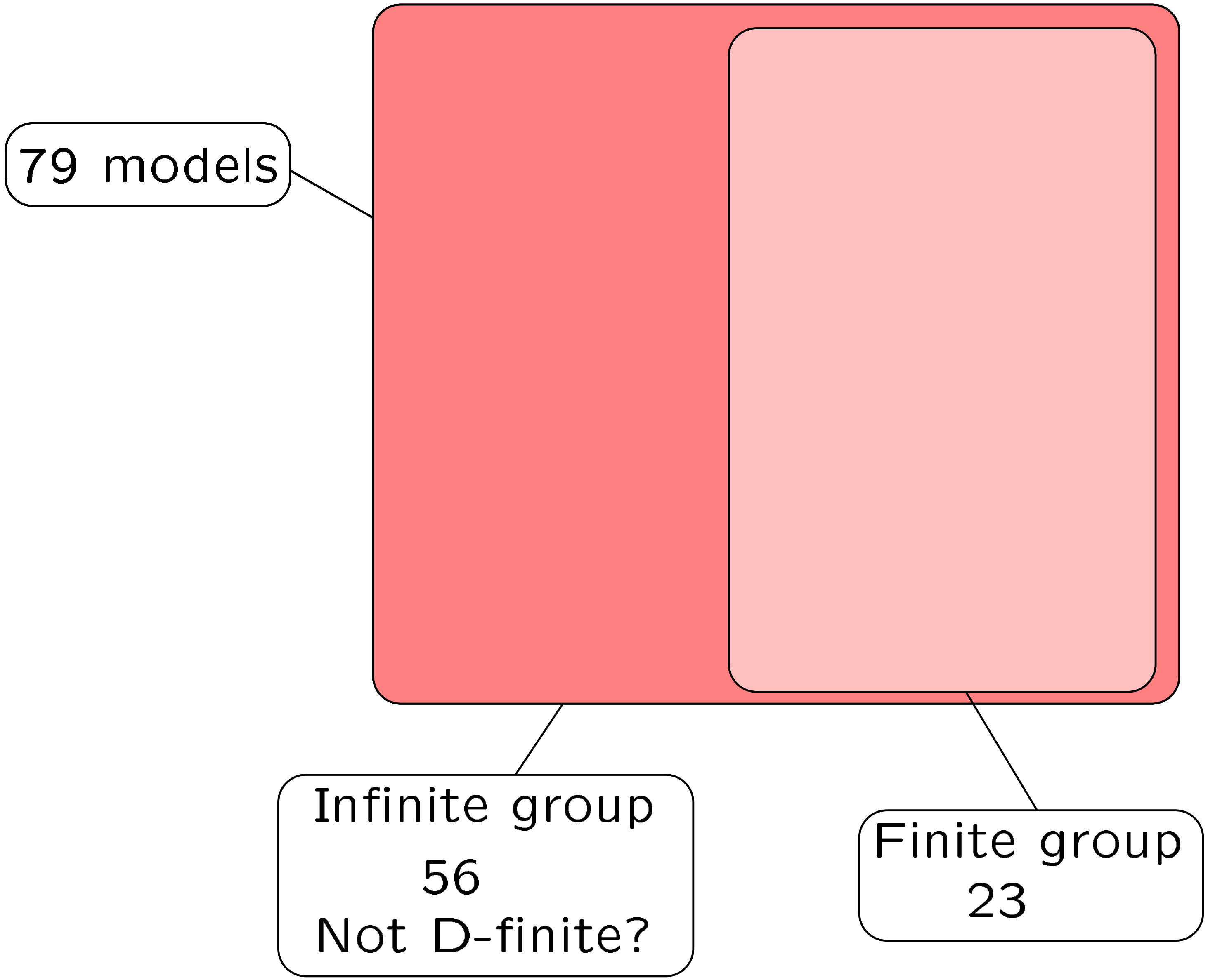
Walks with small steps in the quarter plane

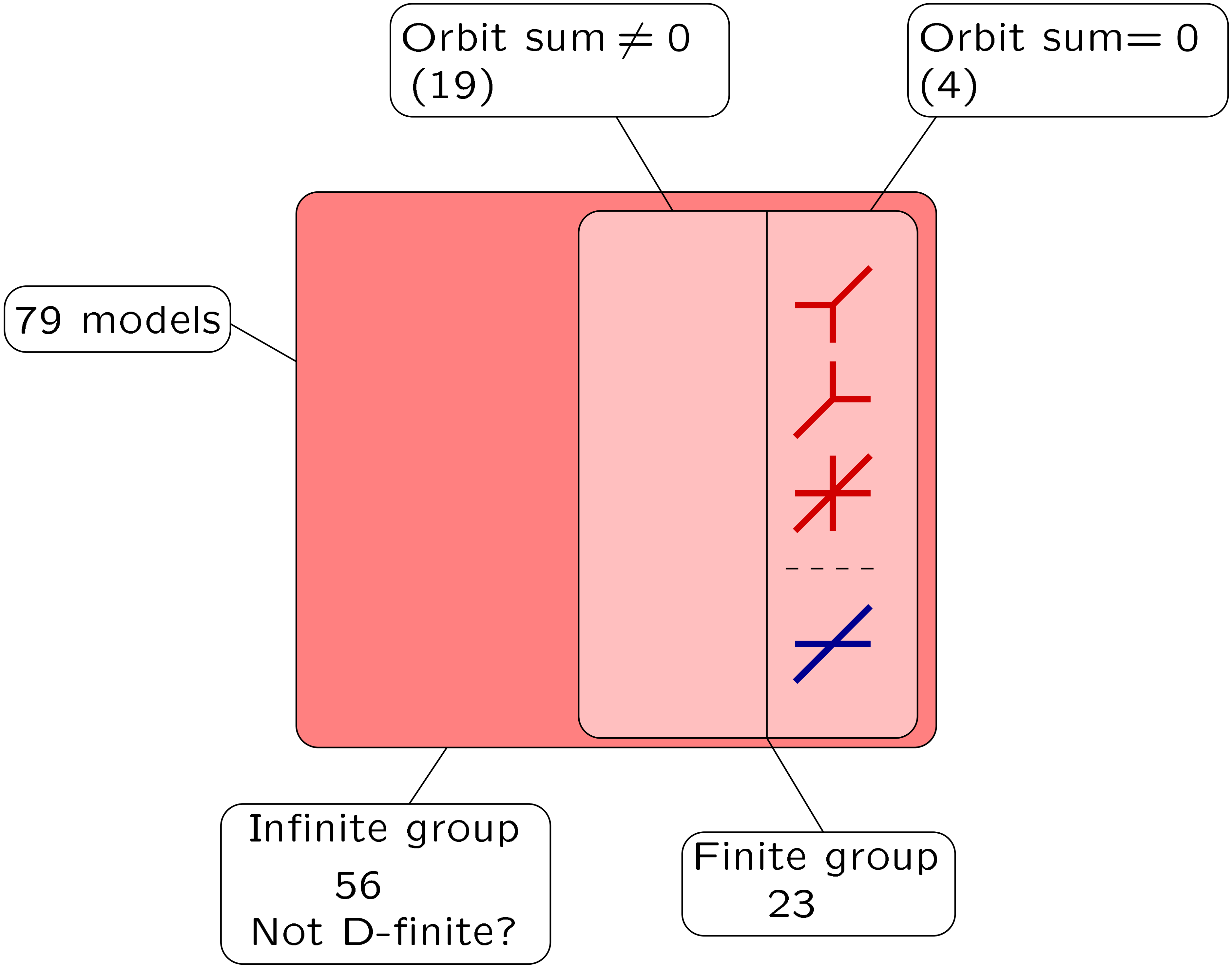


Walks with small steps in the quarter plane

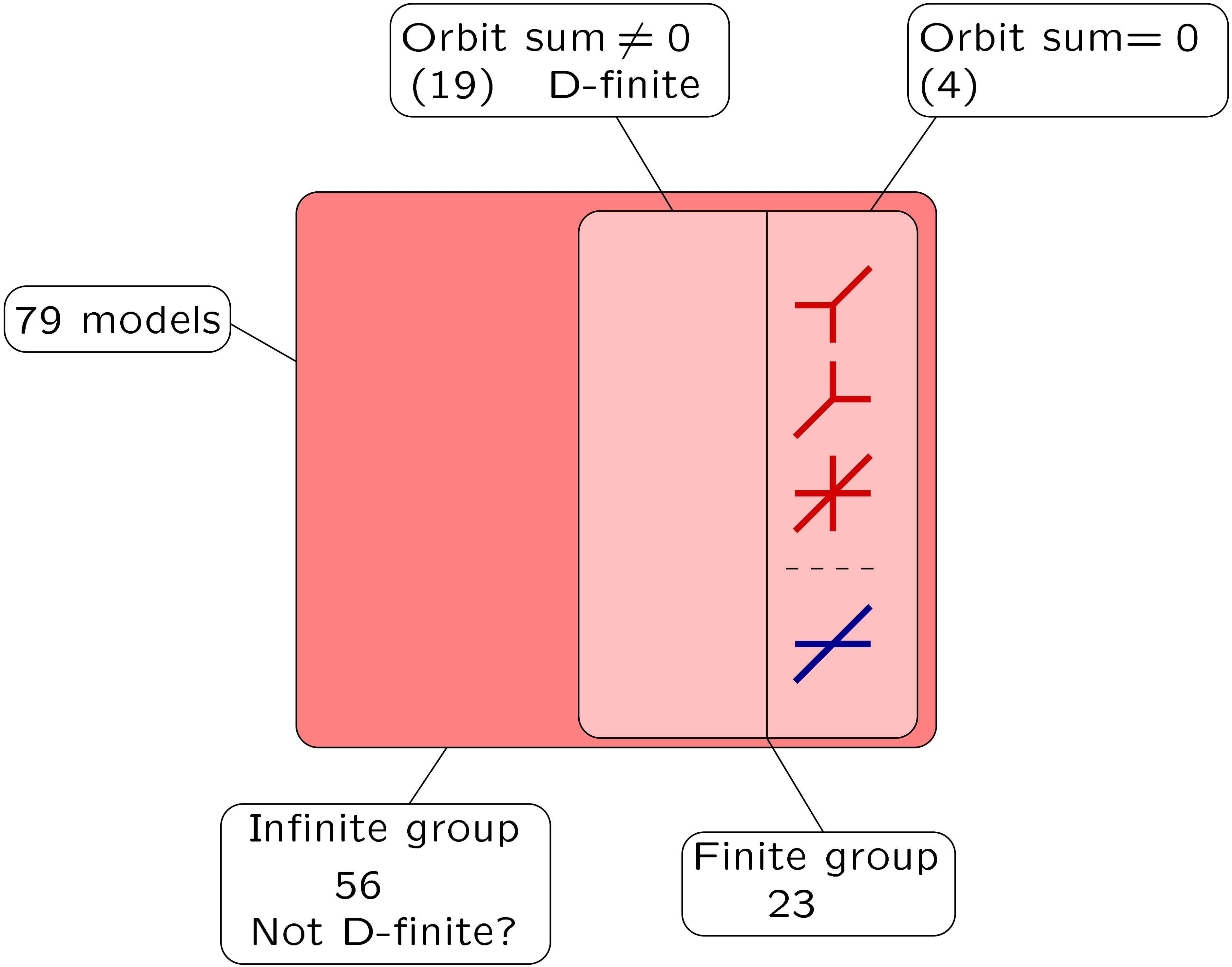


Walks with small steps in the quarter plane





Orbit sum $\neq 0$
(19)
Orbit sum$= 0$
(4)
79 models
Infinite group
56
Not D-finite?
Finite group
23



Orbit sum ≠ 0
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79 models
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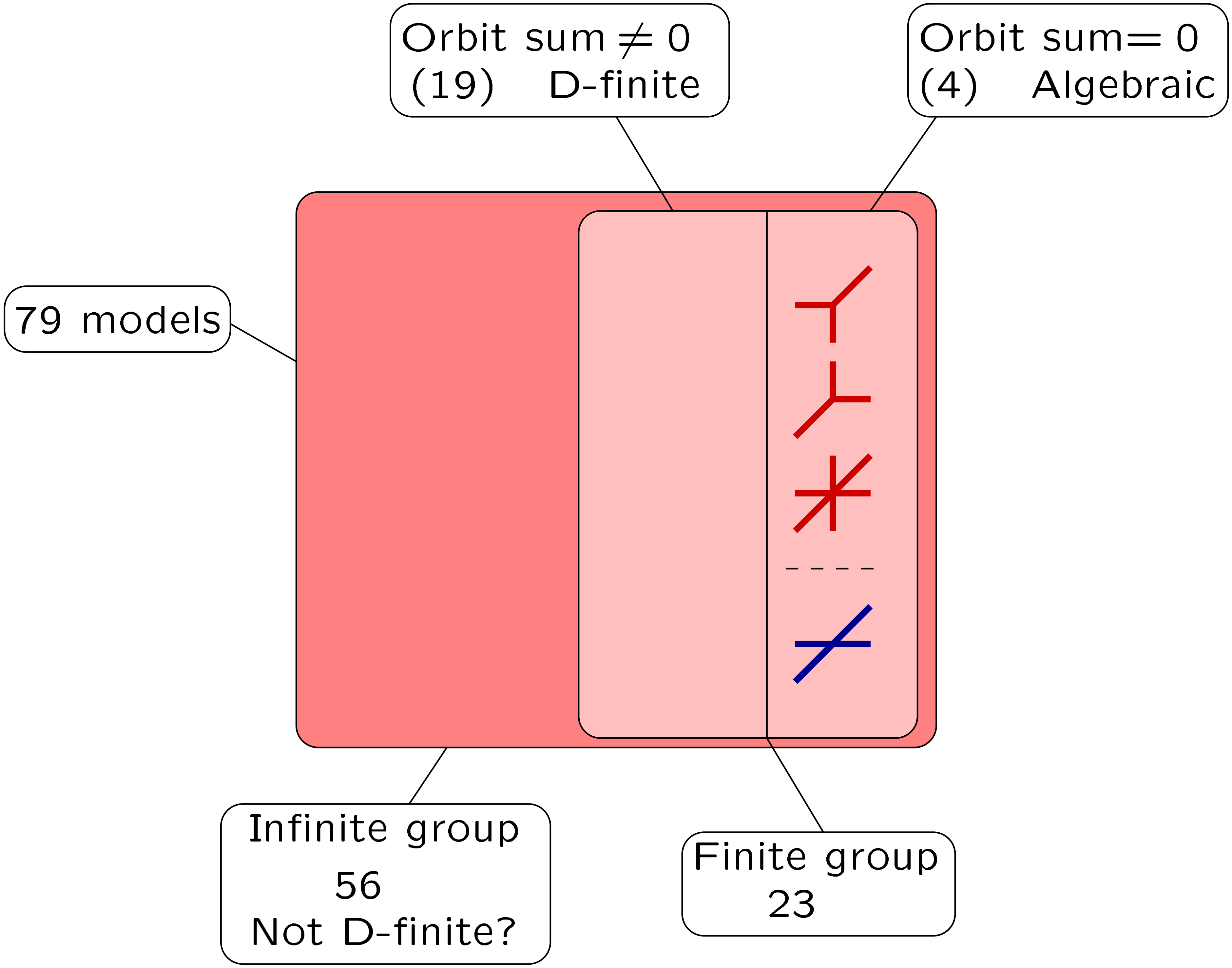
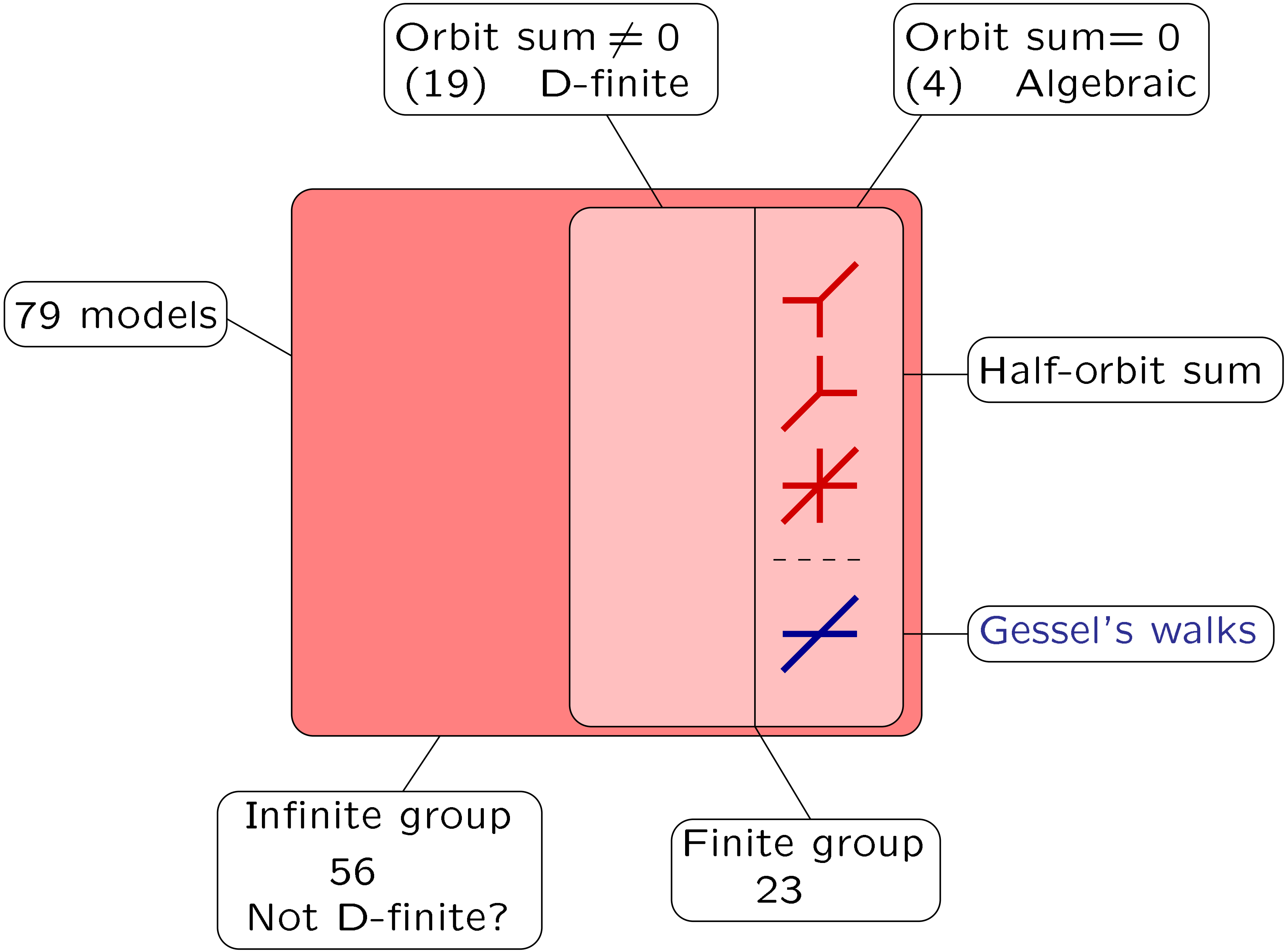


Orbit sum $\neq 0$
(19) D-finite
Orbit sum$= 0$
(4) Algebraic
79 models
Infinite group
56
Not D-finite?
Finite group
23



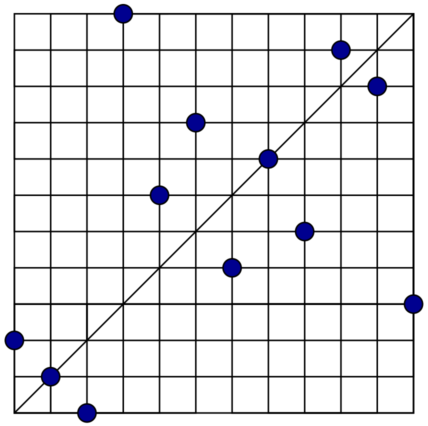
Orbit sum $\neq 0$
(19) D-finite
Orbit sum$= 0$
(4) Algebraic
79 models
Half-orbit sum
Gessel's walks
Infinite group
56
Not D-finite?
Finite group
23

Involutions avoiding $(m+1)m\cdots 21$ (Case $m=2\ell+1$)

Recursive construction: insert a cycle containing the largest value

$m=5$

$\ell=2$



This involution avoids 654321.

Involutions avoiding $(m+1)m\cdots 21$ (Case $m = 2\ell+1$)

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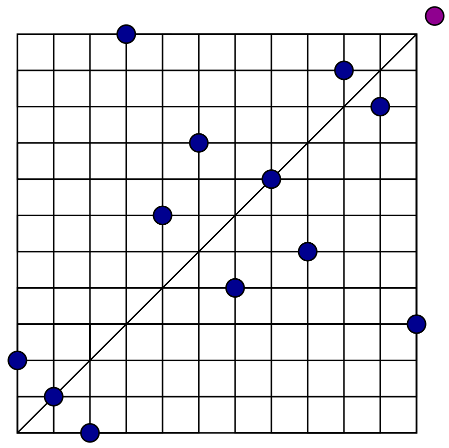

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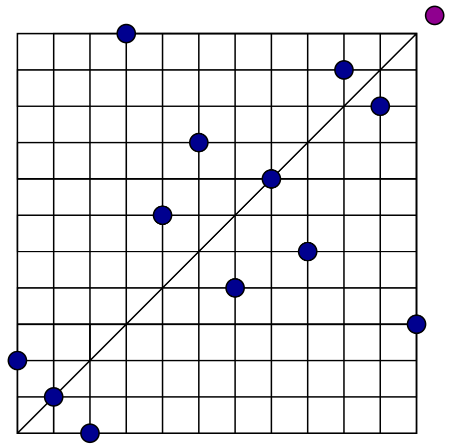

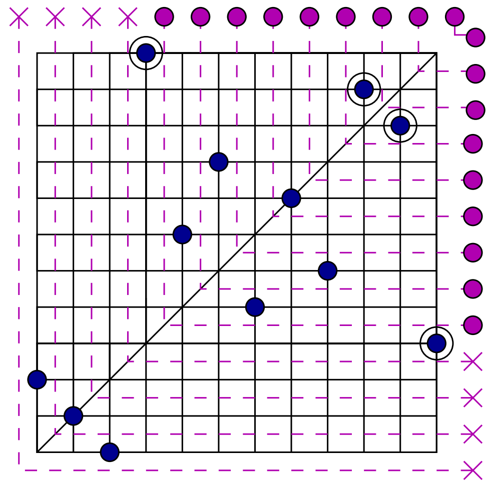

This involution avoids 654321. There are 9 admissible ways to insert a 2-cycle.

Involutions avoiding $(m+1)m\cdots 21$ (Case $m = 2\ell + 1$)

Recursive construction: insert a cycle containing the largest value

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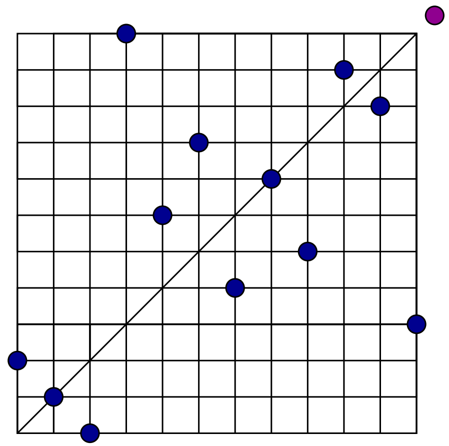

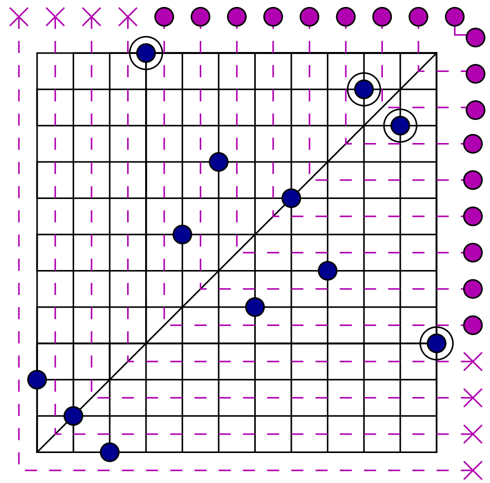

This involution avoids 654321. There are 9 admissible ways to insert a 2-cycle.

For $1 \leq j \leq \ell$, keep track of the size of the smallest NE square containing $(2j)\cdots 21$ ($\Rightarrow \ell$ catalytic variables $u_1, \ldots, u_\ell$) [Jaggard-Marincel 07]

Involutions avoiding $(m+1)m\cdots 21$ (Case $m = 2\ell+1$)

Recursive construction: insert a cycle containing the largest value

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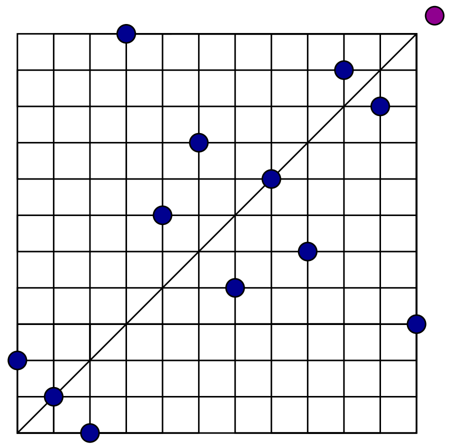
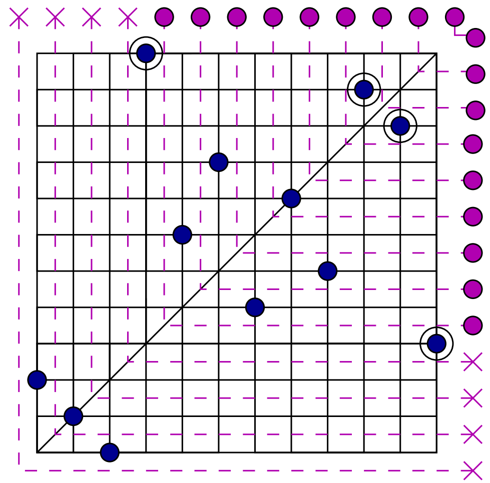
$\ell = 2$





This involution avoids 654321. There are 9 admissible ways to insert a 2-cycle.

For $1 \le j \le \ell$, keep track of the size of the smallest NE square containing $(2j)\cdots 21$ ($\Rightarrow \ell$ catalytic variables $u_1, \ldots, u_\ell$) [Jaggard-Marincel 07]

$$A(u) \equiv A(t; u_1, \ldots, u_\ell) = u_{1,\ell} + t u_{1,\ell} A(u) + t^2 u_{1,\ell} \sum_{k=1}^{\ell} u_{k,\ell} \frac{A(u) - A(u_1, \ldots, u_{k-1}u_k, 1, u_{k+1}, \ldots, u_\ell)}{u_k - 1}$$

with $u_{i,j} = u_i u_{i+1} \cdots u_j$.

Involutions avoiding $(m+1)\cdots 21$ (Case $m=2\ell+1$)

• Set $u_i = v_i/v_{i+1}$.
Then $B(t;v_1,\ldots,v_\ell) := A(t;u_1,\ldots,u_\ell)$ is a series in t with coefficients in $\mathbb{Q}[v_i]$ and

$$B(v) = v_1 + tv_1B(v) + t^2v_1\sum_{k=1}^{\ell} v_kv_{k+1}\frac{B(v) - B(v_1,\ldots,v_{k-1},v_{k+1},v_{k+1},\ldots,v_\ell)}{v_k - v_{k+1}}$$

with $v_{\ell+1} = 1$.

• The kernel reads

$$K(v) = 1 - tv_1 - t^2v_1\sum_{k=1}^{\ell}\frac{v_kv_{k+1}}{v_k - v_{k+1}}$$

One multiplied by $\prod_k(v_k - v_{k+1})$, it is quadratic in each v_k

• Look for the transformation of v_k that leaves the kernel unchanged
(Note that $B(v_1,\ldots,v_{k-1},v_{k+1},v_{k+1},\ldots,v_\ell)$ is independent from v_k)

A change of variables

- The equation:

$$B(v) = v_1 + tv_1 B(v) + t^2 v_1 \sum_{k=1}^{\ell} v_k v_{k+1} \frac{B(v) - B(v_1, \ldots, v_{k-1}, v_{k+1}, v_{k+1}, \ldots, v_\ell)}{v_k - v_{k+1}}$$

- The interesting transformations are best visible when setting

$$v_i = 1 - t(x_i + \cdots + x_\ell)$$

and $B(t; v_1, \ldots, v_\ell) = C(t; x_1, \ldots, x_\ell)$. Then

$$(1 - t - t \sum x_i - t \sum \bar{x}_i) C(x) = 1 - t \sum_{k=1}^{\ell} \frac{C(x_1, \ldots, x_{k-1} + x_k, 0, x_{k+1}, \ldots, x_\ell)}{x_k}$$

with $\bar{x}_i = 1/x_i$.

The kernel is invariant by all signed permutations of the x_i, i.e., by the hyperoctahedral group $\mathcal{B}_\ell$.

The orbit sum

- The equation:

$$(1-t-t\sum x_i - t\sum \bar{x}_i)C(x) = 1 - t\sum_{k=1}^{\ell} \frac{C(x_1,\ldots,x_{k-1}+x_k,0,x_{k+1},\ldots,x_\ell)}{x_k}$$

with $\bar{x}_i = 1/x_i$.

The kernel is invariant by all signed permutations of the x_i, i.e., by the hyper-octahedral group $\mathcal{B}_\ell$.

- Multiply by $\prod_i x_i^i$ and form the alternating sum over $\mathcal{B}_\ell$. This eliminates all $C(\cdot)$ occurring on the r.h.s and gives:

$$\sum_{\sigma\in\mathcal{B}_\ell} \varepsilon(\sigma)\sigma\left(x_1^1\cdots x_\ell^\ell C(x_1,\ldots,x_\ell)\right) = \sum_{\sigma\in\mathcal{B}_\ell} \varepsilon(\sigma)\sigma\left(x_1^1\cdots x_\ell^\ell\right) = \frac{\det\left((x_i^j - \bar{x}_i^j)\right)}{1-t-t\sum x_i - t\sum \bar{x}_i}$$

The orbit sum

Coefficient extraction

- We have obtained:

$$\sum_{\sigma \in \mathcal{B}_\ell} \varepsilon(\sigma)\sigma\left(x_1^1 \cdots x_\ell^\ell C(x_1, \ldots, x_\ell)\right) = \frac{\det\left((x_i^j - \bar{x}_i^j)\right)}{1 - t - t\sum x_i - t\sum \bar{x}_i}$$

- Extract the coefficient of $x_1^1 \cdots x_\ell^\ell$: this gives the length generating function of $(m+1)\cdots 21$ avoiding involutions as:

$$C(0, \ldots, 0) = [x_1^1 \cdots x_\ell^\ell]\ \frac{\det\left((x_i^j - \bar{x}_i^j)\right)}{1 - t - t\sum x_i - t\sum \bar{x}_i}$$

Coefficient extraction

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or, if we take the exponential generating function:

$$\tilde{C}(0,\ldots,0) = [x_1^1 \cdots x_\ell^\ell] \; \det\left((x_i^j - \bar{x}_i^j)\right) \exp(t + t\sum x_i + t\sum \bar{x}_i)$$

Coefficient extraction

• Extract the coefficient of $x_1^1 \cdots x_\ell^\ell$: this gives the length generating function of $(m+1)\cdots 21$ avoiding involutions as:

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or, if we take the exponential generating function:

$$\tilde{C}(0,\ldots,0) = [x_1^1 \cdots x_\ell^\ell]\ \det\left((x_i^j - \bar{x}_i^j)\right)\exp(t + t\sum x_i + t\sum \bar{x}_i)$$

• This decouples the variables x_i, and yields

$$\tilde{C}(0,\ldots,0) = \exp(t)\det\left(J_{|j-i|} - J_{i+j}\right)$$

where

$$J_i(t) = \sum_{n\geq 0} \frac{t^{2n+i}}{n!(n+i)!}$$

[Gordon 71], [Gessel 90]

Some references

- Walks in a quadrant

  ◦ Walks with small steps in the quarter plane, MBM & Mishna, Contemp. Math. 520 (2010)

  ◦ The complete generating function for Gessel's walks is algebraic, Bostan & Kauers, Proc. Amer. Math. Soc. (2010)

  ◦ Two non-holonomic lattice walks in the quarter plane, Mishna & Rechnitzer, Theoret. Comput. Sci. 410 (2009)

- Permutations with no long decreasing subsequence

  ◦ Counting permutations with no long monotone subsequence via generating trees and the kernel method, MBM, J. Alg. Combin. 33 (2011)

- More permutations

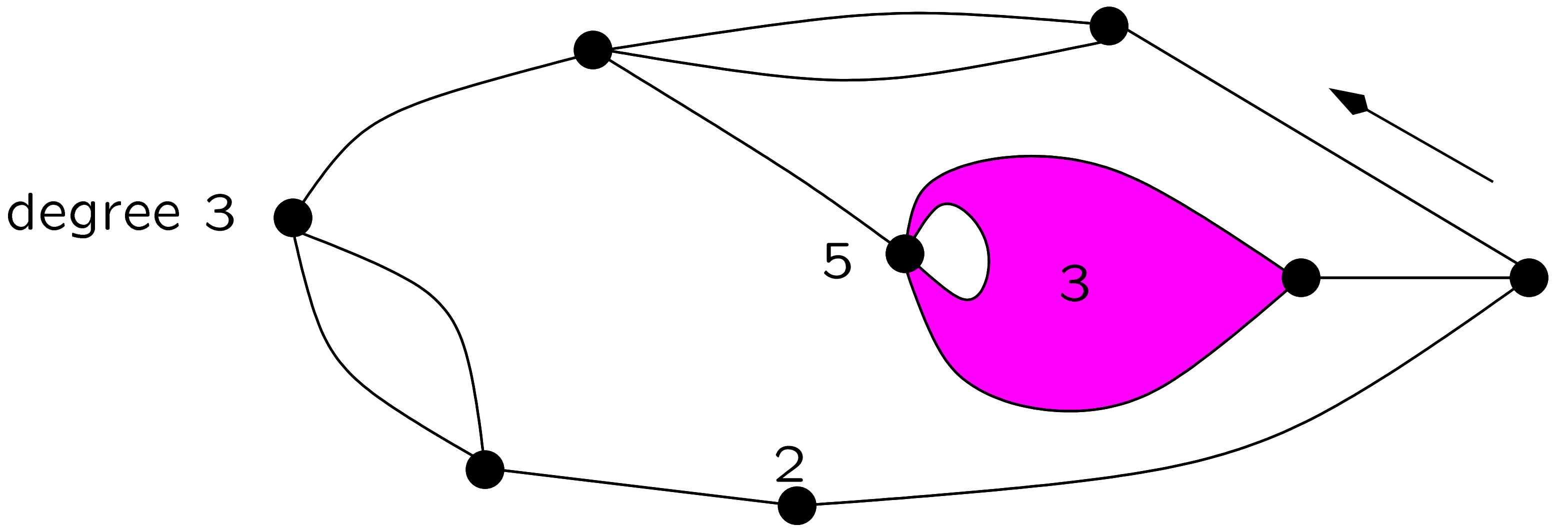
  ◦ Four classes of pattern-avoiding permutations under one roof: generating trees with two labels, MBM, Electronic J. Combinatorics 9 (2003)

IV. Polynomial equations with two (or more) catalytic variables

$$F(x,y) = xq(q-1) + \frac{xyt}{q} F(1,y) F(x,y) + xt\, \frac{F(x,y) - F(x,0)}{y} - x^2 yt\, \frac{F(x,y) - F(1,y)}{x-1}$$

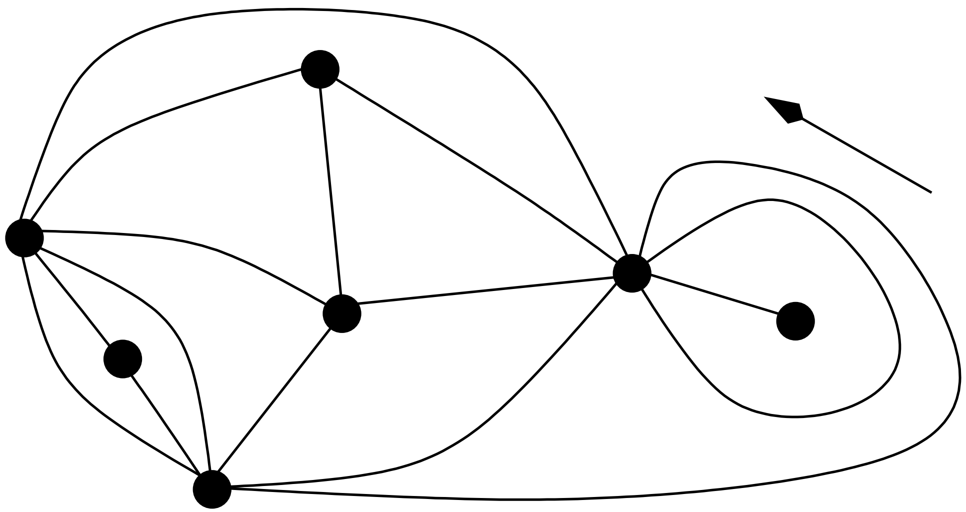
Examples only!

Rooted planar maps



- vertices
- edges
- and faces

Triangulations



Every face has degree 3.

Loops and multiple edges are allowed.

Enumeration of planar maps

- Equations with one catalytic variables

$$F(x) = 1 + tx^2 F(x)^2 + t\,\frac{xF(x) - F(1)}{x-1}$$

- Algebraic series

Arquès Bauer Bédard Bender Bernardi Bessis Bodirsky Bousquet-Mélou Boulatov Bouttier Brézin Brown Canfield Chauve Cori Di Francesco Duplantier Eynard Fusy Gao Goupil Goulden Guitter t'Hooft Itzykson Jackson Jacquard Kazakov Kostov Krikun Labelle Lehman Leroux Liskovets Liu Machì Mehta Mullin Parisi Poulalhon Richmond Robinson Schaeffer Schellenberg Strehl **Tutte** Vainshtein Vauquelin Visentin Walsh Wanless Wormald Zinn-Justin Zuber Zvonkine...

Maps equipped with an additional structure

In combinatorics, but mostly in statistical physics

How many maps equipped with...

– a spanning tree?
[Mullin 67]

– a spanning forest?
[Bouttier et al., Sportiello et al.]

– a self-avoiding walk?
[Duplantier-Kostov 88]

– a proper q-colouring?
[Tutte 74, Bouttier et al. 02]

What is the expected partition function of...

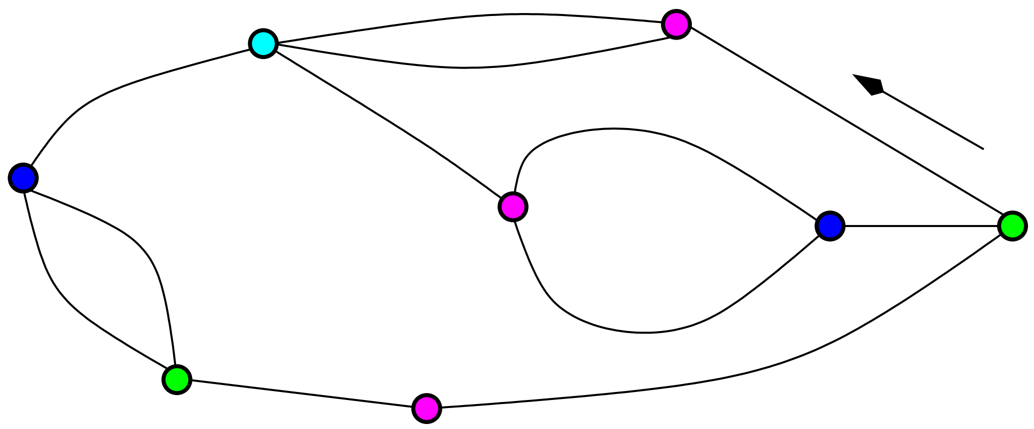
– the Ising model?
[Boulatov, Kazakov, MBM, Schaeffer, Bouttier et al.]

– the hard-particle model?
[MBM, Schaeffer, Jehanne, Bouttier et al. 02, 07]

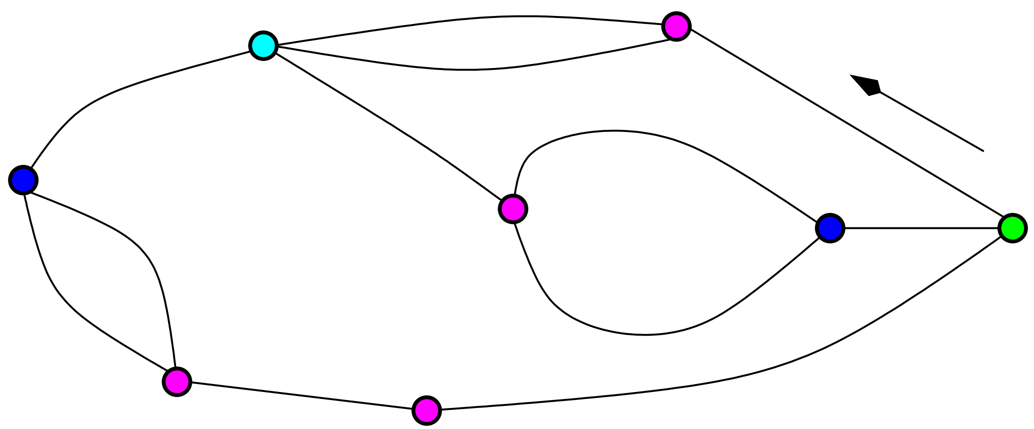
– the Potts model?
[Eynard-Bonnet 99, Baxter 01, MBM-Bernardi 09, Guionnet et al. 10

Colourings

Proper



Non-proper (general)



Monochromatic edge

The Potts model on planar maps

• Count all q-colourings of some family $\mathcal{M}$ of planar maps, keeping track of the number $m(M)$ of monochromatic edges:

$$M(q,\nu,t) := \sum_{M\ q-\text{coloured}} t^{\mathsf{e}(M)}\, \nu^{m(M)}$$

The Potts generating function of maps.

• In other words,

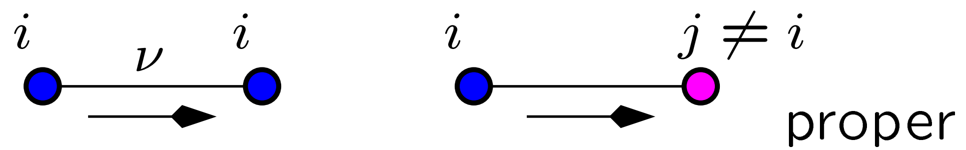
$$M(q,\nu,t) = \sum_{M} \mathsf{Z}_M(q,\nu) t^{\mathsf{e}(M)}$$

where

$$\mathsf{Z}_M(q,\nu) = \sum_{c:V(M)\rightarrow\{1,2,\ldots,q\}} \nu^{m(c)}$$

is the Potts partition function of M.

Example: When M has one edge and two vertices, $\mathsf{Z}_M(q,\nu) = q\nu + q(q-1)$



The Potts model on planar maps

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$$M(q,\nu,t) := \sum_{M \ q-\text{coloured}} t^{\mathsf{e}(M)}\,\nu^{m(M)}$$

The Potts generating function of maps.

• In particular,

$$M(q,0,t) := \sum_{M \ q-\text{prop. coloured}} t^{\mathsf{e}(M)} = \sum_{M} \chi_M(q) t^{\mathsf{e}(M)}$$

counts properly coloured maps.

The Potts model on planar maps

• Count all q-colourings of some family $\mathcal{M}$ of planar maps, keeping track of the number $m(M)$ of monochromatic edges:

$$M(q,\nu,t) := \sum_{M \ q-\text{coloured}} t^{\mathsf{e}(M)} \nu^{m(M)}$$

The Potts generating function of maps.

• Equivalently, find

$$\sum_{M\in\mathcal{M}} T_M(x,y)\ t^{\mathsf{e}(M)} = \cdots$$

where $T_M(x,y)$ is the Tutte polynomial of M. Connection:

$$(x-1)(y-1)^{\mathsf{v}(M)} T_M(x,y) = \sum_{q-\text{colourings of } M} \nu^{m(M)}$$

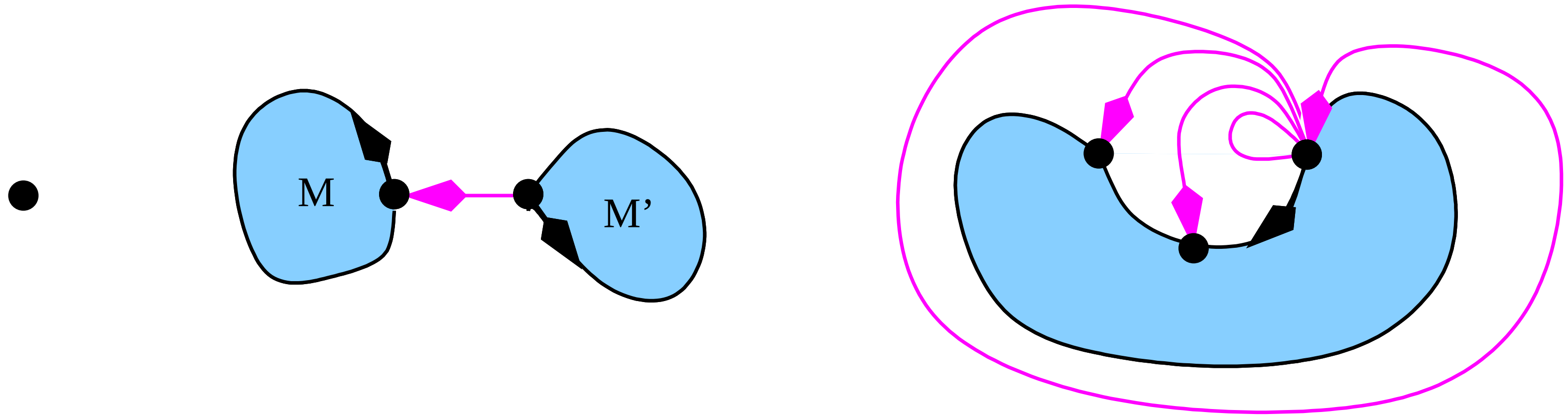
with $q=(x-1)(y-1)$ and $\nu = y-1$.

Recursive description of planar maps: deleting the root-edge

Let

$$F(t;x) \equiv F(x) = \sum_{M} t^{\mathsf{e}(M)} x^{\mathsf{df}(M)} = \sum_{d\geq 0} F_d(t) x^d$$

where $\mathsf{e}(M)$ is the number of edges and $\mathsf{df}(M)$ the degree of the outer face.



$$\begin{aligned} F(x) &= 1 + tx^2F(x)^2 + t\sum_{d\geq 0} F_d(t)\left(x^{d+1} + x^d + \cdots + x\right) \\ &= 1 + tx^2F(x)^2 + tx\frac{xF(x) - F(1)}{x-1} \end{aligned}$$

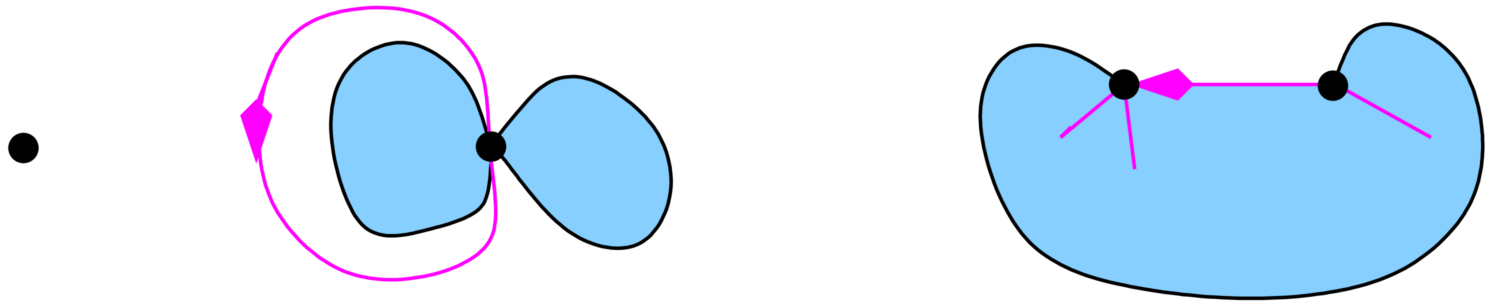
[Tutte 68] A quadratic equation with one catalytic variable, x

Recursive description of planar maps: contracting the root-edge

Let

$$F(t;y) \equiv F(y) = \sum_{M} t^{\mathsf{e}(M)} y^{\mathsf{dv}(M)} = \sum_{d \geq 0} F_d(t) y^d$$

where $\mathsf{e}(M)$ is the number of edges and $\mathsf{dv}(M)$ the degree of the root vertex.



$$\begin{aligned} F(y) &= 1 + ty^2 F(y)^2 + t \sum_{d \geq 0} F_d(t) \left(y^{d+1} + y^d + \cdots + y \right) \\ &= 1 + ty^2 F(y)^2 + ty \frac{yF(y) - F(1)}{y-1} \end{aligned}$$

The same equation... (duality)

Coloured planar maps: Forget algebraicity!

Theorem [Tutte 73]: For planar triangulations,

$$\sum_T \chi'_T(1)\, t^{\mathsf{v}(T)} = \sum_n (-1)^n b(n) t^{n+2}$$

where

$$b(n) = \frac{2\,(3n)!}{n!(n+1)!(n+2)!} \sim 27^n n^{-4},$$

and this asymptotic behaviour prevents the series $B(t) := \sum b_n t^n$ from being algebraic.

However, it satisfies a linear differential equation.

Catalytic variables

The Potts generating function of planar maps, being transcendental, cannot be described with one catalytic variable

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HOWEVER

it can be described with two catalytic variables

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it can be described with two catalytic variables

WHY IS THAT SO?

• The recursive description of the Potts partition function

$$\mathrm{Z}_G(q,\nu) = \mathrm{Z}_{G\backslash e}(q,\nu) + (\nu-1)\,\mathrm{Z}_{G/e}(q,\nu)$$

calls for a recursive description of maps by contraction and deletion of edges.

• This is possible if one keeps track of the degree of the outer face, and the degree of the root-vertex.

Equations with two catalytic variables

- Let

$$M(x,y) \equiv M(q,\nu,t;x,y) = \frac{1}{q}\sum_{M} \mathsf{Z}_M(q,\nu) t^{\mathsf{e}(M)} x^{\mathsf{dv}(M)} y^{\mathsf{df}(M)},$$

where $\mathsf{dv}(M)$ (resp. $\mathsf{df}(M)$) is the degree of the root-vertex (resp. root-face).

- The Potts generating function of planar maps satisfies:

$$\begin{aligned} M(x,y) &= 1 + xyt\left((\nu-1)(y-1)+qy\right) M(x,y) M(1,y) \\ &\quad + xyzt(x\nu-1) M(x,y) M(x,1) \\ &\quad + xyt(\nu-1)\frac{xM(x,y)-M(1,y)}{x-1} + xyzt\frac{yM(x,y)-M(x,1)}{y-1}. \end{aligned}$$

[Tutte 68]

This equation has been sleeping for 40 years

In the footsteps of W. Tutte

- For the GF $T(q,t;x,y) \equiv T(x,y)$ of properly q-coloured triangulations:

$$T(x,y) = xy^2q(q-1) + \frac{xt}{yq}T(1,y)T(x,y) + xt\frac{T(x,y) - y^2T_2(x)}{y} - x^2yt\frac{T(x,y) - T(1,y)}{x-1}$$

where $T_2(x)$ is the coefficient of y^2 in $T(x,y)$.

[Tutte 73] Chromatic sums for rooted planar triangulations: the cases $\lambda = 1$ and $\lambda = 2$
[Tutte 73] Chromatic sums for rooted planar triangulations, II : the case $\lambda = \tau + 1$
[Tutte 73] Chromatic sums for rooted planar triangulations, III : the case $\lambda = 3$
[Tutte 73] Chromatic sums for rooted planar triangulations, IV : the case $\lambda = \infty$
[Tutte 74] Chromatic sums for rooted planar triangulations, V : special equations
[Tutte 78] On a pair of functional equations of combinatorial interest
[Tutte 82] Chromatic solutions
[Tutte 82] Chromatic solutions II
[Tutte 84] Map-colourings and differential equations

[Tutte 95]: Chromatic sums revisited

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Theorem [Tutte]

- For $q = 2+2\cos\frac{2\pi}{m}$, $q \neq 4$, the series $T(1,y) \equiv T(t;1,y)$ satisfies a polynomial equation with one catalytic variable y.

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Theorem [Tutte]

• For $q = 2 + 2\cos\frac{2\pi}{m}$, $q \neq 4$, the series $T(1,y) \equiv T(t;1,y)$ satisfies a polynomial equation with one catalytic variable y.

• When q is generic, the generating function of properly q-coloured planar triangulations is differentially algebraic:

$$2q^2(1-q)t + (qt + 10H - 6tH')H'' + q(4-q)(20H - 18tH' + 9t^2H'') = 0$$

with $H(t) = t^2T_2(q, \sqrt{t}; 1)/q$.

Adapt this to other equations!

[Tutte 73] Chromatic sums for rooted planar triangulations: the cases $\lambda = 1$ and $\lambda = 2$
[Tutte 73] Chromatic sums for rooted planar triangulations, II : the case $\lambda = \tau + 1$
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[Tutte 95]: Chromatic sums revisited

Our results

- Let $M(q, \nu, t; x, y)$ be the Potts generating function of planar maps:

$$M(x,y) \equiv M(q,\nu,t;x,y) = \frac{1}{q} \sum_M \mathsf{Z}_M(q,\nu) t^{\mathsf{e}(M)} x^{\mathsf{dv}(M)} y^{\mathsf{df}(M)},$$

where dv(M) (resp. df(M)) is the degree of the root-vertex (resp. root-face).

Theorem

- For $q = 2 + 2\cos\frac{j\pi}{m}$, $q \neq 0, 4$, the series $M(q, \nu, t; 1, y) \equiv M(1, y)$ satisfies a polynomial equation with one catalytic variable y, and the complete Potts generating function $M(q, \nu, t; x, y)$ is algebraic.

- When q is generic, $M(q, \nu, t; 1, 1)$ is differentially algebraic:

(an explicit system of differential equations)

[mbm-Bernardi 09] Counting colored planar maps: algebraicity results. Arxiv:0909:1695

[mbm-Bernardi 11] Counting colored planar maps: differential equations

Example: The Ising model on planar maps ($q = 2$)

Let A be the series in t, with polynomial coefficients in ν, defined by

$$A = t \, \frac{\left(1 + 3\nu A - 3\nu A^2 - \nu^2 A^3\right)^2}{1 - 2A + 2\nu^2 A^3 - \nu^2 A^4}.$$

Then the Ising generating function of planar maps is

$$M(2, \nu, t; 1, 1) = \frac{1 + 3\nu A - 3\nu A^2 - \nu^2 A^3}{\left(1 - 2A + 2\nu^2 A^3 - \nu^2 A^4\right)^2} \, P(\nu, A)$$

where

$$\begin{aligned} P(\nu, A) = \nu^3 A^6 + 2\nu^2 (1 - \nu) A^5 + \nu (1 - 6\nu) A^4 \\ - \nu (1 - 5\nu) A^3 + (1 + 2\nu) A^2 - (3 + \nu) A + 1. \end{aligned}$$

$\rightsquigarrow$ Asymptotics: Phase transition at $\nu_c = \frac{3+\sqrt{5}}{2}$, critical exponents...

Example: properly 3-coloured planar maps ($q = 3, \nu = 0$)

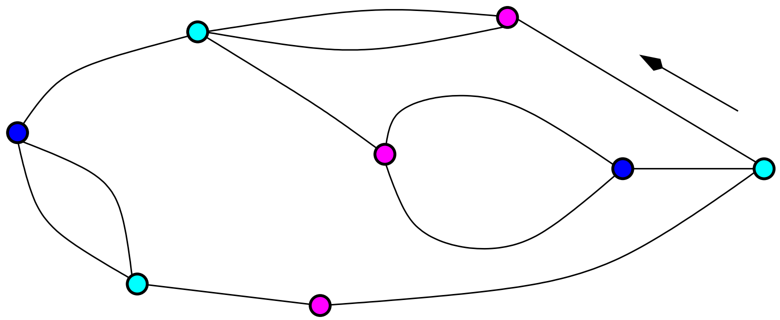
Let A be the quartic series in t defined by

$$A = t\,\frac{(1+2A)^3}{(1-2A^3)}.$$

Then the generating function of properly 3-coloured planar maps is

$$M(3,0,t;1,1) = \frac{(1+2A)(1-2A^2-4A^3-4A^4)}{(1-2A^3)^2}$$

$\rightsquigarrow$ Asymptotics: A random loopless planar map with n edges has approximately $(1.42...)^n$ proper 3-colourings



Our results: when q is generic

- Let $M(q,\nu,t;x,y)$ be the Potts generating function of planar maps:

$$M(x,y) \equiv M(q,\nu,t;x,y) = \frac{1}{q}\sum_M \mathsf{Z}_M(q,\nu) t^{\mathsf{e}(M)} x^{\mathsf{dv}(M)} y^{\mathsf{df}(M)},$$

where dv(M) (resp. df(M)) is the degree of the root-vertex (resp. root-face).

Theorem

- For $q = 2 + 2\cos\frac{j\pi}{m}$, $q \neq 4$, the series $M(q,\nu,t;1,y) \equiv M(1,y)$ satisfies a polynomial equation with one catalytic variable y, and the complete Potts generating function $M(q,\nu,t;x,y)$ is algebraic.

- When q is generic, $M(q,\nu,t;1,1)$ is differentially algebraic:

(an explicit system of differential equations)

An explicit system of differential equations

Let $D(t,v) = q\nu + (\nu-1)^2 - q(\nu+1)v + \Big(q + t(\nu-1)(q-4)(q+\nu-1)\Big)v^2.$

- There exists a unique 8-tuple $(P_1(t), \ldots, P_4(t), Q_1(t), Q_2(t), R_1(t), R_2(t))$ of series in t with polynomial coefficients in q and ν such that

$$\frac{1}{v^2R}\,\frac{\partial}{\partial v}\left(\frac{v^4R^2}{PD^2}\right) = \frac{1}{Q}\,\frac{\partial}{\partial t}\left(\frac{Q^2}{PD^2}\right),$$

where

$$\begin{aligned}
P(t,v) &= P_4(t)v^4 + P_3(t)v^3 + P_2(t)v^2 + P_1(t)v + 1,\\
Q(t,v) &= Q_2(t)v^2 + Q_1(t)v + 1,\\
R(t,v) &= R_2(t)v^2 + R_1(t)v + q + \nu - 3,
\end{aligned}$$

with the initial conditions (at $t=0$):

$$P(0,v) = (1-v)^2 \quad \text{and} \quad Q(0,v) = 1 - v.$$

An explicit system of differential equations (cont'd)

• The Potts generating function of planar maps, $M(1,1) \equiv M(q,\nu,t;1,1)$, satisfies

$$12\,t^2\left(q\nu+(\nu-1)^2\right)M(q,\nu,t;1,1) =$$
$$8\,t(q+\nu-3)Q_1(t)-Q_1(t)^2+P_2(t)-2\,Q_2(t)-4\,t\,(2-3\,\nu-q)-12\,t^2\,(q+\nu-3)^2\,.$$

Questions

1. Use the structure of

$$\frac{1}{v^2R}\frac{\partial}{\partial v}\left(\frac{v^4R^2}{PD^2}\right)=\frac{1}{Q}\frac{\partial}{\partial t}\left(\frac{Q^2}{PD^2}\right),$$

to obtain a single differential equation (or an expression?) for $M(q,\nu,t;1,1)$.

2. Relate this to elliptic functions, and to the papers of [Bonnet & Eynard 99], and [Guionnet, Jones, Shlyakhtenko & Zinn-Justin 10]

An analogous system for triangulations

Let $D(t,v) = q\nu^2 + (\nu-1)\,(4(\nu-1)+q)\,v + \Big(q\nu(\nu-1)(q-4)t + (\nu-1)^2\Big)\,v^2.$

• There exists a unique 7-tuple $(P_1(t),\ldots,P_3(t),Q_1(t),Q_2(t),R_0(t),R_1(t))$ of series in t with polynomial coefficients in q and ν such that

$$\frac{1}{v^2R}\,\frac{\partial}{\partial v}\left(\frac{v^5R^2}{PD^2}\right) = \frac{1}{Q}\,\frac{\partial}{\partial t}\left(\frac{Q^2}{PD^2}\right),$$

where

$$\begin{aligned} P(t,v) &= P_3(t)v^3 + P_2(t)v^2 + P_1(t)v + 1, \\ Q(t,v) &= Q_2(t)v^2 + Q_1(t)v + 2\nu, \\ R(t,v) &= R_1(t)v + R_0(t), \end{aligned}$$

with the initial conditions (at $t=0$):

$$P(0,v) = 1 + v/4 \quad \text{and} \quad Q(0,v) = 2\nu + v.$$

• Expression of the Potts GF of triangulations in terms of the P_i and Q_i

... and for properly q-coloured triangulations ($\nu = 0$)

Let $D(v) = v + 4 - q$.

• There exists a unique 4-tuple (P_1, P_2, P_3, Q_1) of zeries in t with polynomial coefficients in q such that

$$-\frac{4t}{v}\,\frac{\partial}{\partial v}\left(\frac{v^3}{P}\right) = \frac{1}{Q}\,\frac{\partial}{\partial t}\left(\frac{Q^2}{PD}\right).$$

where

$$\begin{aligned} P(t,v) &= P_3(t)v^3 + P_2(t) + P_1(t)v + 1, \\ Q(t,v) &= Q_1(t)v + 1, \end{aligned}$$

with the initial conditions (at $t = 0$):

$$P(0,v) = 1 + v/4 \quad \text{and} \quad Q(0,v) = 1.$$

• From the system, one can derive Tutte's differential equation,

$$2q^2(1-q)t + (qt + 10H - 6tH')H'' + q(4-q)(20H - 18tH' + 9t^2H'') = 0$$

with $H(t) = t^2 T_2(q, \sqrt{t}; 1)/q$.

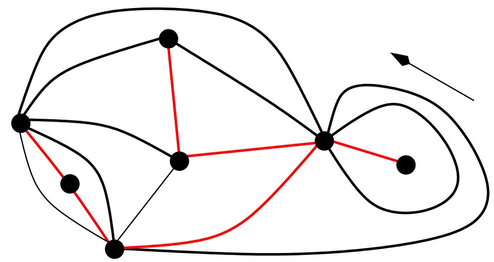
Coloured enumeration:

bijections?

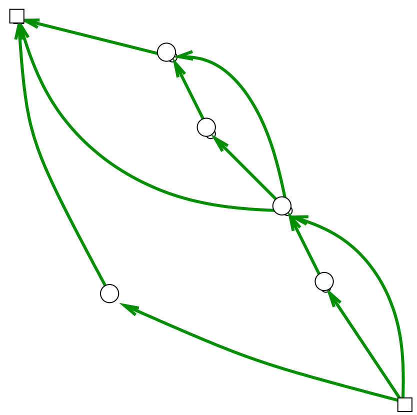
Some bijections exist in special cases... but most remain to be found

Some existing bijections

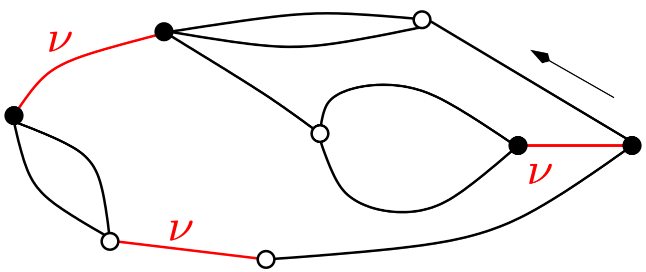
- Maps equipped with a spanning tree ($\mathsf{T}_M(1,1)$)
[Mullin 67], [Bernardi 07]



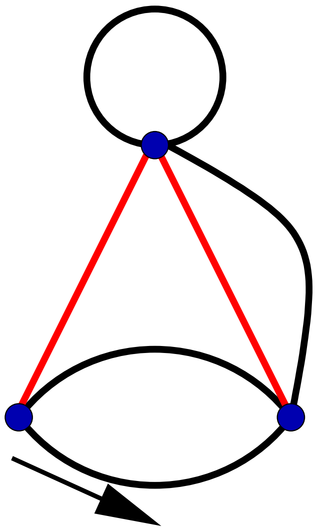
- Maps equipped with a bipolar orientation ($(-1)^{\mathsf{v}(M)}\chi'_M(1)$)
[Felsner-Fusy-Noy-Orden 08],
[Fusy-Poulalhon-Schaeffer 08],
[Bonichon-mbm-Fusy 08]



- The Ising model on planar maps (case $q=2$)
[MBM-Schaeffer 02], [Bouttier et al. 07]



Bijective counting of maps equipped with a spanning tree



n edges, $k+1$ vertices ($\Rightarrow k$ edges in the tree)

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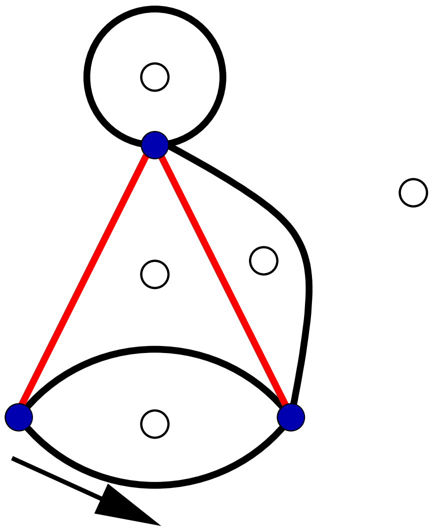

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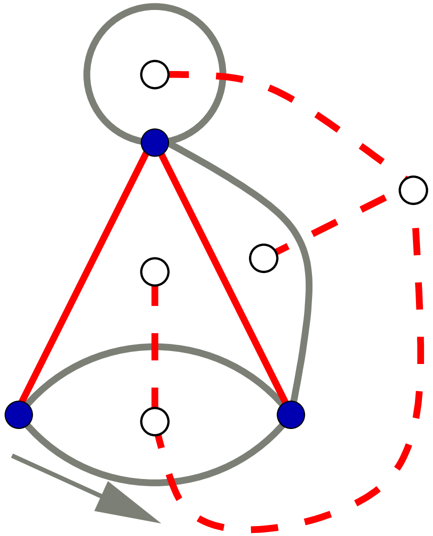

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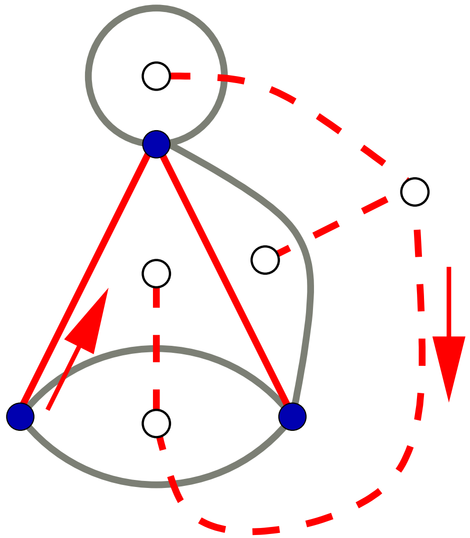

n edges, $k+1$ vertices ($\Rightarrow k$ edges in the tree)

A shuffle of two plane trees

$$\binom{2n}{2k} C_k C_{n-k}$$

with $C_k = \binom{2k}{k}/(k+1)$ counts rooted trees with k edges.

Some references

• Counting planar maps, coloured or uncoloured, MBM, Survey paper for the 23rd Bristish Combinatorial Conference, Exeter, July 2011. London Math. Soc. Lecture Note Ser. 392 (2011)

• Counting colored planar maps: algebraicity results, MBM & Bernardi, J. Combin. Theory ser. B 101 (2011)

• Dichromatic sums revisited, Tutte, J. Combin. Theory Ser. B, 66, (1996)

Perspectives

A. More combinatorics

- Understand algebraic series, e.g., for 3-coloured planar maps:

$$M(3,0,t;1,1) = \frac{(1+2A)(1-2A^2-4A^3-4A^4)}{(1-2A^3)^2} \quad \text{with} \quad A = t\,\frac{(1+2A)^3}{(1-2A^3)}$$

- Understand differential equations, e.g., for properly q-coloured triangulations:

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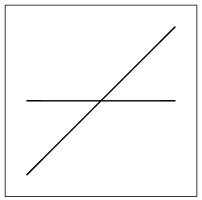
B. Equations with several catalytic variables

- Prove in a constructive manner the algebraicity of Gessel's walks in a quadrant



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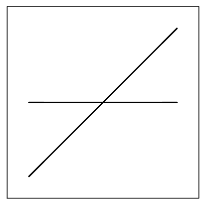

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C. Asymptotics

• Work out asymptotics and singularities directly from equations with catalytic variables?