Arithmetic matroids and Tutte polynomial (joint work with Luca Moci)

Michele D'Adderio

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Matroid			
Definit	tion of Matroic	l -	
	use the word <i>list</i> fractional formula $\mathfrak{M} = \mathfrak{M}_{\mathbf{Y}}$		

function $rk : \mathbb{P}(X) \to \mathbb{N} \cup \{0\}$ such that:

- 1 if $A \subseteq X$, then $rk(A) \leq |A|$;
- **2** if $A, B \subseteq X$ and $A \subseteq B$, then $rk(A) \leq rk(B)$;

3 if $A, B \subseteq X$, then $rk(A \cup B) + rk(A \cap B) \leq rk(A) + rk(B)$.

In particular $rk(\emptyset) = 0$.

We say that a sublist A is independent \Leftrightarrow rk(A) = |A|.

An independent sublist of maximal rank rk(X) is called a *basis*.

rk(X) is called the *rank* of the matroid.

The independent sublists determine the matroid structure:

We use the word *list* for *multiset* (repetitions allowed).

A matroid $\mathfrak{M} = \mathfrak{M}_X = (X, rk)$ is a list of vectors X with a rank function $rk : \mathbb{P}(X) \to \mathbb{N} \cup \{0\}$ such that:

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- - independent = cycle-free (forests).

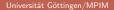
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 X is a finite list of vectors of a vector space (e.g. ℝⁿ); *rk*(A) = dim(span(A)); independent = linearly independent;
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Matroid			
	A		
Dual N	Matroid		

$$rk^*(A) := |A| - rk(X) + rk(X \setminus A).$$

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In particular the rank of \mathfrak{M}^* is |X| - rk(X).

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The *dual* of the matroid $\mathfrak{M} = (X, rk)$ is defined as the matroid with the same set X of vectors, and with bases the complements of the bases of \mathfrak{M} .

We will denote it by \mathfrak{M}^* . The rank function of \mathfrak{M}^* is given by

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$$T_X(x,y) := \sum_{A \subseteq X} (x-1)^{rk(X) - rk(A)} (y-1)^{|A| - rk(A)}.$$

From the definition it is clear that $T_X(1,1)$ is equal to the number of bases of the matroid.

The coefficients of the Tutte polynomial are positive, and they have a nice combinatorial interpretation in terms of *internal* and *external activity*.

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We fix a total order on X, and let B be a basis extracted from X. We say that $v \in X \setminus B$ is externally active on B if v is dependent on the list of elements of B following it. We say that $v \in B$ is internally active on B if v is externally active on the complement $B^c := X \setminus B$ in the dual matroid.

The number e(B) of externally active vectors is called the *external* activity of B, while the number $i(B) = e^*(B^c)$ of internally active vectors is called the *internal activity* of B.

Theorem (Crapo)

$$T_X(x,y) = \sum_{\substack{B \subseteq X \\ B \text{ basis}}} x^{e^*(B^c)} y^{e(B)}.$$

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An arithmetic matroid is a pair (\mathfrak{M}_X, m) , where \mathfrak{M}_X is a matroid on a list of vectors X, and m is a multiplicity function, i.e. $m : \mathbb{P}(X) \to \mathbb{N} \setminus \{0\}$ has the following properties:

- if $A \subseteq X$ and $v \in X$ is dependent on A, then $m(A \cup \{v\})$ divides m(A);
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- **13** if $A \subseteq B \subseteq X$ and B is a disjoint union $B = A \cup F \cup T$ such that for all $A \subseteq C \subseteq B$ we have $rk(C) = rk(A) + |C \cap F|$, then $m(A) \cdot m(B) = m(A \cup F) \cdot m(A \cup T)$.

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- If $A \subseteq B = X$, then we denote $\mu_X(A)$ simply by $\mu(A)$. Similarly for $\mu^*(A)$.
- Setting m(A) = 1 for all $A \subseteq X$ we get a *trivial* multiplicity function, which essentially does not add anything to the matroid structure.
- So any matroid is trivially an arithmetic matroid.
- In this sense the notion of an arithmetic matroid is a generalization of the one of a matroid.
- But of course there are more interesting examples.



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 $rk(A) := maximal rank of a free abelian subgroup of \langle A \rangle$;

 $m(A) := |G_A : \langle A \rangle|$, where G_A is the maximal subgroup of G such that $\langle A \rangle \leq G_A$ and $|G_A : \langle A \rangle| < \infty$.

If we set $\mathfrak{M}_X := (X, rk)$, then (\mathfrak{M}_X, m) is an arithmetic matroid.

Arithmetic matroids of this form are called representable.

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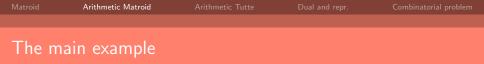
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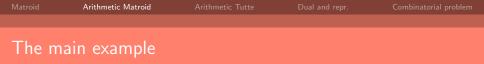
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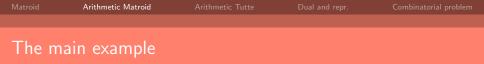
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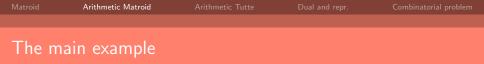
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Remark

The multiplicity of $A \subseteq X$ is the *GCD* of the minors of maximal rank in the submatrix corresponding to *A*.

So $m(\emptyset) = 1$, $m(\{v_2\}) = 2$, $m(\{v_1, v_2, v_3\}) = 3$, $m(\{v_2, v_3\}) = 1$, $m(\{v_1, v_2\}) = 6$, $m(\{v_1\}) = m(\{v_3\}) = 3$, $m(\{v_1, v_3\}) = 9$.

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The arithmetic Tutte polynomial of the arithmetic matroid (\mathfrak{M}_X, m) is defined as

$$M_X(x,y) := \sum_{A \subseteq X} m(A)(x-1)^{rk(X)-rk(A)}(y-1)^{|A|-rk(A)}.$$

From the definition it is clear that $M_X(1,1)$ is equal to the sum of the multiplicities of the bases of the matroid. For the trivial multiplicity function m(A) = 1 for all $A \subseteq X$ we get the Tutte polynomial $T_X(x, y)$ of \mathfrak{M}_X .

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	Arithmetic Tutte	
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An example

Let $X = \{v_1 := (3,0), v_2 := (2,-2), v_3 := (-3,3)\} \subseteq G := \mathbb{Z}^2$. Consider the matrix $\begin{pmatrix} 3 & 2 & -3 \\ 0 & -2 & 3 \end{pmatrix}$ whose columns are v_1, v_2, v_3 . Then $m(\emptyset) = m(\{v_2, v_3\}) = 1, m(\{v_1, v_2\}) = 6, m(\{v_2\}) = 2, m(\{v_1\}) = m(\{v_3\}) = m(\{v_1, v_2, v_3\}) = 3, m(\{v_1, v_3\}) = 9.$ $M_X(x, y) = \sum_{A \subseteq X} m(A)(x - 1)^{rk(X) - rk(A)}(y - 1)^{|A| - rk(A)} = (x - 1)^2 + (3 + 2 + 3)(x - 1) + (x - 1)(y - 1) + (6 + 9) + 3(y - 1) = x^2 + 5x + 6 + xy + 2y.$

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Positive coefficients!

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Given an arithmetic matroid (\mathfrak{M}_X, m) , its *dual* is (\mathfrak{M}_X^*, m^*) , where \mathfrak{M}_X^* is the dual matroid of \mathfrak{M}_X , and for all $A \subseteq X$ we set $m^*(A) := m(X \setminus A)$.

Lemma (D.-Moci)

The dual of an arithmetic matroid is an arithmetic matroid.

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The dual of a representable arithmetic matroid is representable.

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In fact we give an explicit construction.

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In fact we give an explicit construction.

Michele D'Adderio

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Let (\mathfrak{M}_X, m) be an arithmetic matroid, and $M_X(x, y)$ its arithmetic Tutte polynomial.

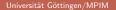
Question

Does $M_X(x, y)$ have positive coefficients for any arithmetic matroid?

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Is there a combinatorial interpretation of $M_X(x, y)$?

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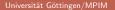
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Does $M_X(x, y)$ have positive coefficients for any arithmetic matroid? YES!

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Is there a combinatorial interpretation of $M_X(x, y)$? YES!

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Remember that $M_X(1,1)$ is the sum of the multiplicities of the bases extracted from X. $X_1 := \{v_1 := (3,0), v_2 := (2,-2)\} \subseteq G := \mathbb{Z}^2$. $m(\{v_1, v_2\}) = 6, m(\{v_1\}) = 3, m(\{v_2\}) = 2, m(\emptyset) = 1$. $M_{X_1}(x, v) = x^2 + 3x + 2$

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Same bases give different statistics!

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Consider an arithmetic matroid (\mathfrak{M}_X, m) . Let $S \subseteq X$ be of maximal rank, i.e. rk(S) = rk(X). Then $\mu(S) = \sum_{X \supseteq T \supseteq S} (-1)^{|T| - |S|} m(T) \ge 0$. We call L_X the list in which every maximal rank sublist S appears $\mu(S)$ many times. We construct dually L_X^* from (\mathfrak{M}_X^*, m^*) using $\mu^*(S)$. We define the lists $\mathcal{B} := \{(B, T) \mid B \text{ basis}, B \subseteq T, T \in L_X\}$ and its dual $\mathcal{B}^* := \{(B^c, \widetilde{T}) \mid B \text{ basis}, B^c \subseteq \widetilde{T}, \widetilde{T} \in L_X^*\}$. Each basis B appears m(B) times in \mathcal{B} (by inclusion-exclusion).

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the same order).

Are we done?

Not quite: we need to decide how to match the pairs from \mathcal{B} with the pairs from \mathcal{B}^* .

Clearly $(B, T) \in \mathcal{B}$ goes to some $(B^c, \tilde{T}) \in \mathcal{B}^*$, but how do we choose \tilde{T} ?

In fact it is even worst: from the computations of $M_X(x, y)$ we can see that sometimes the same copy of (B, T) needs to go to different (B^c, \tilde{T}) 's! We fix a total order on X. For every $(B, T) \in \mathcal{B}$ we define its *local* external activity e(B, T) to be the number of elements of $T \setminus B$ that are externally active on B. We define $e^*(B^c, \tilde{T})$ dually (using the same order).

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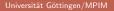
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We define a matching $\psi : \mathcal{B} \to \mathcal{B}^*$: given a basis $B \subseteq X$, we identify the pairs $(B, T) \in \mathcal{B}$ having the same elements in T active on B, ignoring the non-active elements. We do the same with the pairs $(B^c, \widetilde{T}) \in \mathcal{B}^*$. Then we equidistribute these pairs among each others.

Theorem (D.-Moci)

$$M_X(x,y) = \sum_{(B,T)\in\mathcal{B}} x^{e^*(\psi(B,T))} y^{e(B,T)}.$$

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An exa	ample		

$$\begin{aligned} X &= \{ \mathbf{v}_1 := (3,0) < \mathbf{v}_2 := (2,-2) < \mathbf{v}_3 := (-3,3) \} \subseteq G := \mathbb{Z}^2. \\ \text{Consider the matrix} \begin{pmatrix} 3 & 2 & -3 \\ 0 & -2 & 3 \end{pmatrix} \text{ whose columns are } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3. \\ \text{Then } m(\emptyset) &= m(\{\mathbf{v}_2, \mathbf{v}_3\}) = 1, \ m(\{\mathbf{v}_1, \mathbf{v}_2\}) = 6, \ m(\{\mathbf{v}_2\}) = 2, \\ m(\{\mathbf{v}_1\}) &= m(\{\mathbf{v}_3\}) = m(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}) = 3, \ m(\{\mathbf{v}_1, \mathbf{v}_3\}) = 9. \\ L_X &= (\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}^3, \{\mathbf{v}_1, \mathbf{v}_2\}^3, \{\mathbf{v}_1, \mathbf{v}_3\}^6) \\ L_X^* &= (\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}, \{\mathbf{v}_1, \mathbf{v}_2\}^2, \{\mathbf{v}_1, \mathbf{v}_3\}, \{\mathbf{v}_2, \mathbf{v}_3\}^2, \{\mathbf{v}_2\}^4, \{\mathbf{v}_3\}^2) \end{aligned}$$

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Then $m(\emptyset) = m(\{v_2, v_3\}) = 1, m(\{v_1, v_2\}) = 6, m(\{v_2\}) = 2, m(\{v_1\}) = m(\{v_3\}) = m(\{v_1, v_2, v_3\}) = 3, m(\{v_1, v_3\}) = 9.$
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Matroid			Combinatorial problem
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$$\begin{aligned} X &= \{ \mathbf{v}_1 := (3,0) < \mathbf{v}_2 := (2,-2) < \mathbf{v}_3 := (-3,3) \} \subseteq G := \mathbb{Z}^2. \\ \text{Consider the matrix} \begin{pmatrix} 3 & 2 & -3 \\ 0 & -2 & 3 \end{pmatrix} \text{ whose columns are } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3. \\ \text{Then } m(\emptyset) &= m(\{\mathbf{v}_2, \mathbf{v}_3\}) = 1, \ m(\{\mathbf{v}_1, \mathbf{v}_2\}) = 6, \ m(\{\mathbf{v}_2\}) = 2, \\ m(\{\mathbf{v}_1\}) &= m(\{\mathbf{v}_3\}) = m(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}) = 3, \ m(\{\mathbf{v}_1, \mathbf{v}_3\}) = 9. \\ L_X &= (\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}^3, \{\mathbf{v}_1, \mathbf{v}_2\}^3, \{\mathbf{v}_1, \mathbf{v}_3\}^6) \\ L_X^* &= (\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}, \{\mathbf{v}_1, \mathbf{v}_2\}^2, \{\mathbf{v}_1, \mathbf{v}_3\}, \{\mathbf{v}_2, \mathbf{v}_3\}^2, \{\mathbf{v}_2\}^4, \{\mathbf{v}_3\}^2) \end{aligned}$$

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 $x^2 + 3x + 2$

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Consider the basis $\{v_1, v_3\}$
 $x^2 + 3x + 2 + xy + 2y$

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Consider the basis $\{v_1, v_3\}$
 $x^2 + 3x + 2 + xy + 2y + 2x + 4$

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Consider the basis $\{v_1, v_3\}$
 $x^2 + 3x + 2 + xy + 2y + 2x + 4 = = x^2 + 3x + 2 + (y + 2)(x + 2)$

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 $x^2 + 3x + 2 + xy + 2y + 2x + 4 = = x^2 + 3x + 2 + (y + 2)(x + 2) = = x^2 + 5x + 6 + xy + 2y$

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			Combinatorial problem
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References

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2 C. De Concini, C. Procesi, *Topics in hyperplane arrangements, polytopes and box-splines*, Springer 2010.

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THANKS!

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