Back to an old identity:

$$\sum_{i+j=n} \binom{2i}{i}\binom{2j}{j} = 4^n$$

Rui Duarte António Guedes de Oliveira

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Some references:

[1] Marta Sved: “Counting and recounting: the aftermath”, *Math. Intelligencer* **6** (1984) no. 2, 44–45.

[2] Valerio De Angelis: “Pairings and Signed Permutations” *Amer. Math. Monthly* **113** (2006), 642–644.

[3] Guisong Chang and Chen Xu: “Generalization and Probabilistic Proof of a Combinatorial Identity”, *Amer. Math. Monthly* **118** (2011), 175–177.

The probabilistic Approach

Chang & Xu, 2011

$$= \sum_{\substack{i_1,\ldots,i_m \geq 0 \\ i_1+\cdots+i_m=n}} \frac{n!}{2^n} \binom{2i_1}{i_1} \cdots \binom{2i_m}{i_m}.$$

Since the probability density function of $X_1^2 + X_2^2 + \cdots + X_m^2$ is identic probability density function of $\chi^2(m)$, we have

$$\mathbf{E}(X_1^2 + X_2^2 + \cdots + X_m^2)^n = \frac{2^n \Gamma(\frac{m}{2} + n)}{\Gamma(\frac{m}{2})}.$$

which completes the proof.

$$\sum_{i_1+\cdots+i_t=n} \binom{2i_1}{i_1}\binom{2i_2}{i_2}\cdots\binom{2i_t}{i_t} = \frac{4^n}{n!} \frac{\Gamma(\frac{t}{2}+n)}{\Gamma(\frac{t}{2})}$$

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The probabilistic Approach

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$$\sum_{i_1+\cdots+i_{2t}=n} \binom{2i_1}{i_1}\binom{2i_2}{i_2}\cdots\binom{2i_{2t}}{i_{2t}} = 4^n\binom{n+t-1}{n}$$

From Stanley's "Enumerative Combinatorics I" 2nd Ed.

Claim: $$\sum_{i+j=n}\binom{2i}{i}\binom{2j}{j} = 4^n$$

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Find a simple expression for $\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}f(i)$. (See equation (1.13).)

8. (a) [2–] Show that

$$\frac{1}{\sqrt{1-4x}} = \sum_{n\geq 0}\binom{2n}{n}x^n.$$

(b) [2–] Find $\sum_{n\geq 0}\binom{2n-1}{n}x^n$.

9. Let $f(m,n)$ be the number of paths from $(0,0)$ to $(m,n) \in \mathbb{N}\times\mathbb{N}$, where each

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(b) Note that $\binom{2n-1}{n} = \frac{1}{2}\binom{2n}{n}$, $n>0$ (see Exercise 1.3(e)).

9. (b) While powerful methods exist for solving this type of problem (see Example 6.3.8),

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Conjecture

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Conjecture-meaning

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
1 12 66 220 495 792 924 792 495 220 66 12 1

$$\binom{0}{0}\binom{8}{4}+\binom{2}{1}\binom{6}{3}+\binom{4}{2}\binom{4}{2}+\binom{6}{3}\binom{2}{1}+\binom{8}{4}\binom{0}{0}=1\times70+2\times20+6\times6+20\times2+70\times1=256=4^4$$

Conjecture-meaning

$$
\begin{array}{ccccccccccccccccccccccccc}
 & & & & & & & & & & & & & & 1 & & -4 & & 10 & & -20 & & 35 \\
 & & & & & & & & & & & & & 1 & & -3 & & 6 & & -10 & & 15 \\
 & & & & & & & & & & & & 1 & & -2 & & 3 & & -4 & & 5 & & -6 \\
 & & & & & & & & & & & 1 & & -1 & & 1 & & -1 & & 1 & & -1 \\
 & & & & & & & & & & 1 \\
 & & & & & & & & & 1 & & 1 \\
 & & & & & & & & 1 & & 2 & & 1 \\
 & & & & & & & 1 & & 3 & & 3 & & 1 \\
 & & & & & & 1 & & 4 & & 6 & & 4 & & 1 \\
 & & & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & & & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\
 & & 1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1 \\
 & 1 & & 9 & & 36 & & 84 & & 126 & & 126 & & 84 & & 36 & & 9 & & 1 \\
1 & & 10 & & 45 & & 120 & & 210 & & 252 & & 210 & & 120 & & 45 & & 10 & & 1 \\
1 & 11 & & 55 & & 165 & & 330 & & 462 & & 462 & & 330 & & 165 & & 55 & & 11 & & 1 \\
1 & 12 & 66 & & 220 & & 495 & & 792 & & 924 & & 792 & & 495 & & 220 & & 66 & & 12 & & 1
\end{array}
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																1		−4		10		−20		35
															1		−3		6		−10		15	
														1		−2		3		−4		5		−6
													1		−1		1		−1		1		−1	
												1		0		0		0		0		0		0
											1		1		0		0		0		0		0	
										1		2		1		0		0		0		0		0
									1		3		3		1		0		0		0		0	
								1		4		6		4		1		0		0		0		0
							1		5		10		10		5		1		0		0		0	
						1		6		15		20		15		6		1		0		0		0
					1		7		21		35		35		21		7		1		0		0	
				1		8		28		56		70		56		28		8		1		0		0
			1		9		36		84		126		126		84		36		9		1		0	
		1		10		45		120		210		252		210		120		45		10		1		0
	1		11		55		165		330		462		462		330		165		55		11		1	
1		12		66		220		495		792		924		792		495		220		66		12		1

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The "natural" approach

Claim: $$\sum_{i+j=n}\binom{2i}{i}\binom{2j}{j} = 4^n$$

Let $[n]=\{1,2,\ldots,n\}$ for a positive integer n (we also write $[0]=\varnothing$) and suppose $0\le i,j\le n$ with $i+j=n$, $i,j\in\mathbb{Z}$. Let $A=[2i]$ and $B=[2j]$. We count the pairs of form (X,Y), where

$$\begin{cases} X\subseteq A \text{ and } |X|=\frac{1}{2}|A|=i \\ Y\subseteq B \text{ and } |Y|=\frac{1}{2}|B|=j \end{cases}$$

and i and j take all possible values.

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Example:
Take $n = 11$ and $i = 5$. Hence, $A = \{1, 2, \ldots, 10\}$ and $B = \{1, 2, \ldots, 12\}$.

Let $X = \{2, 3, 4, 8, 9\}$ and $Y = \{2, 3, 5, 7, 8, 11\}$

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We represent (X, Y) by the rectangle

$R =$

1	○	○	○	5	1	■	■	4	■	6
6	7	○	○	10	■	■	9	10	■	12

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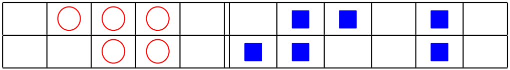

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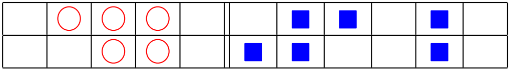
$R =$ 

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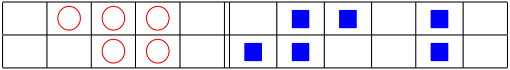
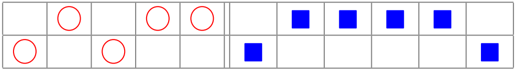


$R =$



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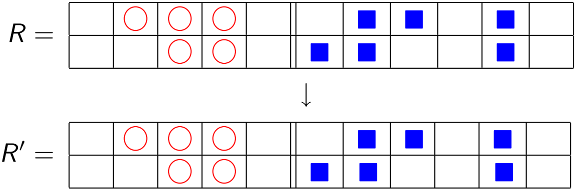

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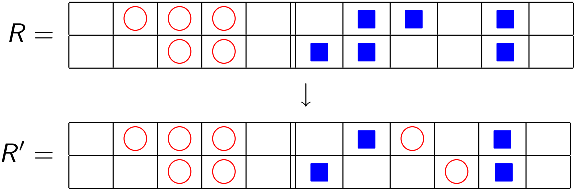

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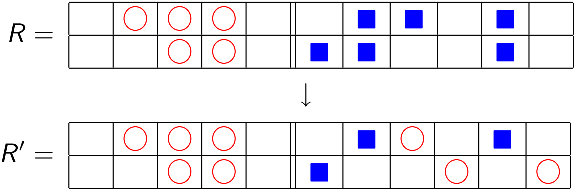

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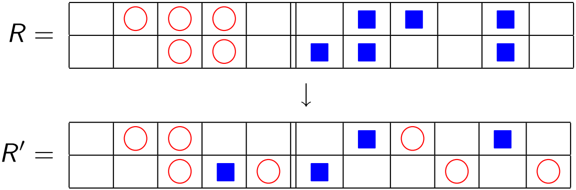

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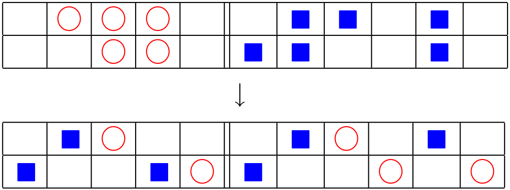
$R =$; $R' =$



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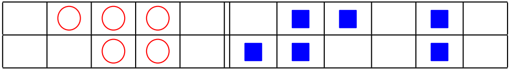
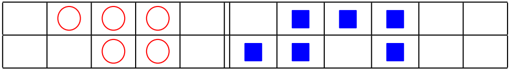


$S =$



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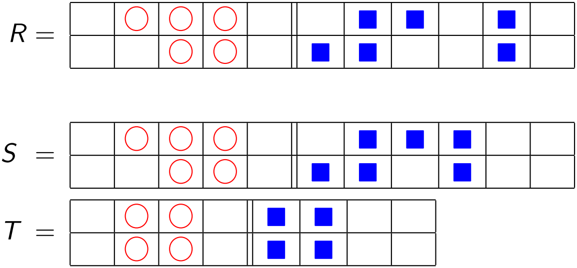

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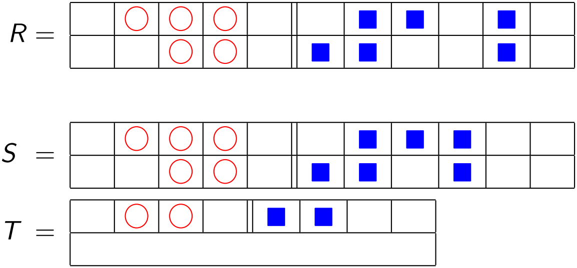

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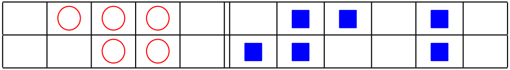
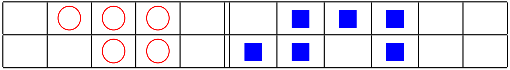
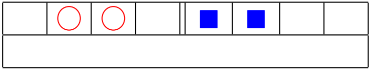


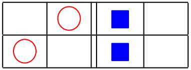
$S =$



$T =$



$U =$



Generalization (Combinatorial proof)

Claim: $$\sum_{i+j=n} \binom{2i-k}{i}\binom{2j+k}{j} = 4^n \; (k \in \mathbb{R})$$

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where $f_i(k) = (k+i-1)_i\,(k+2n)_{n-i}$

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$$= \frac{(-1)^n}{n!}\Delta^n f_\ell(k-2\ell)\big|_{\ell=0}$$

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$$\Delta^m f_\ell(k)_{|_{\ell=i}} = (-1)^m(n+1)\cdots(n+m)(k+i-1)_i\,(k+2n)_{n-i-m}$$

Generalization (Combinatorial proof)

Claim: $$\sum_{i+j=n}\binom{2i-k}{i}\binom{2j+k}{j} = 4^n \ (k \in \mathbb{R})$$

$$\iff \Delta^n f_\ell(k-2\ell)_{|\ell=0} \text{ is constant}$$

where $f_i(k) = (k+i-1)_i\,(k+2n)_{n-i}$

$$\Delta^m f_\ell(k)_{|\ell=i} = (-1)^m(n+1)\cdots(n+m)(k+i-1)_i\,(k+2n)_{n-i-m}$$

$$f_\ell(k) = p_n(\ell) + p_{n-1}(\ell)\,k + \cdots + p_0(\ell)\,k^n \ (\deg(p_i) \le i)$$

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$$f_\ell(k) = p_n(\ell) + p_{n-1}(\ell)\,k + \cdots + p_0(\ell)\,k^n \ (\deg(p_i) \le i)$$
$$f_\ell(k-2\ell) = p_n(\ell) + p_{n-1}(\ell)(k-2\ell) + \cdots + p_0(\ell)\,(k-2\ell)^n$$
$$f_\ell(k-2\ell) = q_n(\ell) + q_{n-1}(\ell)\,k + \cdots + q_0(\ell)\,k^n \ (\deg(q_i) \le i)$$

Generalization ("Analytic" approach)

Let $g_k(x) = \sum_{n\geq 0}\binom{2n+k}{n}x^n$. Then

$$g_0(x) = \frac{1}{\sqrt{1-4x}}$$

Let also

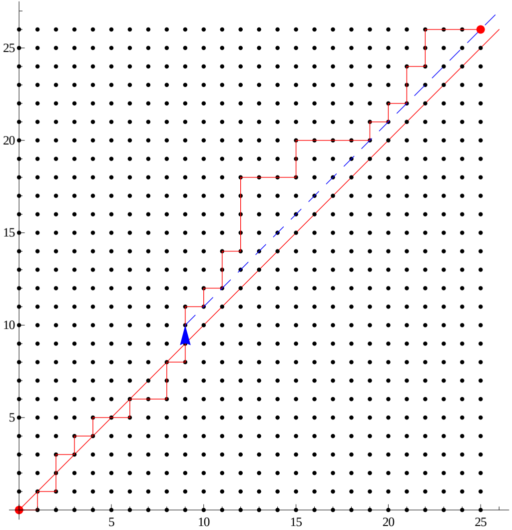
$$C(x) = \frac{2}{1+\sqrt{1-4x}}$$

($C(x)$ is the generating function of the Catalan numbers)
Then

$$g_k(x) = g_0(x)\, C^k(x).$$

Generalization ("Analytic" approach)

Example ($k = 1$):



Theorem. Let $k_1, \ldots, k_t$ be any integers such that $k_1 + \cdots + k_t = 0$. Then

$$\sum_{i_1+\cdots+i_t=n}\binom{2i_1+k_1}{i_1}\binom{2i_2+k_2}{i_2}\cdots\binom{2i_t+k_t}{i_t} = \sum_{i_1+\cdots+i_t=n}\binom{2i_1}{i_1}\binom{2i_2}{i_2}\cdots\binom{2i_t}{i_t}.$$