Skew polynomial rings, Gröbner bases and the letterplace embedding of the free associative algebra

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$$
\begin{tikzcd}
F \arrow[rr, "\iota"] \arrow[dr, "\iota'"'] & & S \\
& P \arrow[ur, "\xi"'] &
\end{tikzcd}
$$

- $X = \{x_0, x_1, \ldots\}$ a finite or countable set;
- $K\langle X\rangle$ the free associative algebra generated by X;
- $K[X]$ the polynomial ring with commuting variables in X.

Trivial

- To relate commutative structures (ideals, modules, etc) to non-commutative ones is easy.
- $K\langle X\rangle \to K[X]$ surjective homomorphism of algebras. To work modulo commutators $[x_i, x_j]$.

Problem

- Is it possible to relate non-commutative structures to commutative ones?
- Apparently impossible: no algebra homomorphism $K[X] \to K\langle X\rangle$.
- But . . . there is one way.

Kickoff

- $\mathbb{N}^* = \{1, 2, \ldots\}$ set of positive integers;
- $P = K[X \times \mathbb{N}^*]$ the polynomial ring in the variables $x_i(j) = (x_i, j)$.
- Doubilet, Rota & Stein (1974): $\iota' : F = K\langle X \rangle \to P$ such that $x_{i_1} \cdots x_{i_d} \mapsto x_{i_1}(1) \cdots x_{i_d}(d)$ is an injective K-**linear map** (infact $\mathbb{S}_d$-modules homomorphism for each degree d).
- Drawback: this is **not algebra homomorphism**.

Goal

- Extend P to a new algebra S and transform ι' into an **algebra embedding** $\iota : F \to S$.
- S is generated by all commutative variables $x_i(j)$ except for a single new variable s satisfying the identity $s x_i(j) = x_i(j+1) s$.

Σ-invariant ideals

- P a K-algebra, $\Sigma \subset \mathrm{End}_K(P)$ a submonoid.
- An ideal $I \subset P$ is **Σ-invariant** if $\varphi(I) \subset I$ for all $\varphi \in \Sigma$.
- $G \subset I$ is a Σ-basis of I if $\Sigma(G)$ is a basis of I.
- I may be not finitely generated as an ideal (say P not Noetherian), but finitely generated as Σ-invariant ideal (Σ infinite monoid).

Applications

- **The ring of partial difference polynomials:**
- $\Sigma = \langle \sigma_1, \ldots, \sigma_r \rangle \approx \mathbb{N}^r$, P the polynomial ring in the variables $\sigma^\alpha(u_i)$ where $u_i = u_i(t_1, \ldots, t_r)$ are functions and $\sigma^\alpha = \prod_k \sigma_k^{\alpha_k}$ are shift operators $u_i \mapsto u_i(t_1 + \alpha_1 h, \ldots, t_r + \alpha_r h)$.
- **Algebraic statistic**, e.g. Gaussian two-factor model:
- $P = K[x_{ij} \mid i, j \in \mathbb{N}]$, $\mathrm{Inc}(\mathbb{N}) = \{f : \mathbb{N} \to \mathbb{N} \mid f \text{ strictly increasing}\}$, $\Sigma = \{x_{ij} \mapsto x_{f(i)f(j)} \mid f \in \mathrm{Inc}(\mathbb{N})\}$.
- **Representation/Invariant theory, PI-algebras and T-ideals, etc**

Skew monoid rings

- To encode the action of P and Σ over ideals as a single left module structure defined by an appropriate ring extending P.
- $S = P \star \Sigma$ the **skew monoid ring** that is the free P-module with (left) basis Σ and multiplication defined by the identity $\sigma f = \sigma(f)\sigma$ for all $\sigma \in \Sigma, f \in P$.
- I is a Σ-invariant ideal of P if and only if $I \subset P$ is a left S-submodule.
- Left Noetherianity of S implies I is finitely Σ-generated.

Skew letterplace algebra

- $P = K[X \times \mathbb{N}^*]$, $\sigma : P \to P$ such that $x_i(j) \mapsto x_i(j+1)$, $\Sigma = \langle \sigma \rangle \subset \mathrm{End}_K(P)$.
- P domain, σ injective implies $S = P \star \Sigma$ domain.
- σ of infinite order implies $S = P[s;\sigma]$ skew polynomial ring (Ore extension where σ-derivation is zero).
- We identify σ with variable s and denote $\sigma(f) = s \cdot f$.
- Element of S are $\sum_i f_i s^i$ with $f_i \in P$. Multiplication ruled by the identity $sf = (s \cdot f)s$.

Skew letterplace embedding

- $\iota : F \to S$ algebra homomorphism such that $x_i \mapsto x_i(1)s$.
- ι is embedding since words $w = x_{i_1} \cdots x_{i_d}$ maps to monomials $x_{i_1}(1) \cdots x_{i_d}(d)s^d = \iota'(w)s^d$.
- ι is homogeneous map w.r.t. grading $S = \bigoplus_i S_i$ where $S_i = Ps^i$.

Following Doubilet & al., we call P the **letterplace algebra** and hence S, ι the **skew letterplace algebra** and **skew letterplace embedding**.

Ideal (and bases) correspondences

- We introduce 3 correspondences between the ideals of the rings F, S and P able to **transfer Gröbner bases computations** from one ring to another.
- **F** $\rightarrow$ **S**: between (graded) ideals of F and S is an **extension-contraction** process due to the embedding ι.
- **P** $\rightarrow$ **S**: between (graded Σ-invariant) ideals of P and S is due to a **suitable grading** for P that maps into the natural grading of S ($S_i = Ps^i$) by a new embedding $\xi : P \rightarrow S$. This correspondence works in a more general context that the letterplace one.
- **F** $\rightarrow$ **P**: between ideals of F and P which is essentially the **composition of the above ones**. This mapping appeared already in – & Levandovskyy, J. Symbolic Comput., 44 (2009).

From F to S: ideals

- $F = K\langle X\rangle$ isomorphic to a graded subring $R \subset S$ implies that graded two-sided ideals $I \subset F \approx R$ can be **extended** to graded two-sided ideals of $J \subset S$.
- Such extension process behaves well that is $J \cap R = \iota(I)$. Hence, extension-contraction defines a **one-to-one correspondence** between all graded ideals of $I \subset F$ and some class of ideals $J \subset S$.
- We call J **skew letterplace analogue** of I.

Problem

- There exists some correspondence between skew letterplace analogues in S and some class of ideals of P?
- Note: $\pi : S \to P$ such that $s^i \mapsto 1$ is a left S-module epimorphism. Left ideals of S maps into Σ-invariant ideals of P.
- What about a right inverse $P \to S$?

General setting

- P polynomial ring with a countable set of variables, $\sigma : P \to P$ injective (infinite order) sending monomials into monomials.
- Denote $\hat{\mathbb{N}} = \{-\infty\} \cup \mathbb{N}$. Then $(\hat{\mathbb{N}}, \max, +)$ is a commutative idempotent semiring (or commutative dioid, max-plus algebra, etc)
- Assume P endowed also with a **weight function** that is $\mathrm{w} : \mathrm{Mon}(P) \to \hat{\mathbb{N}}$ such that for all monomials m, n:
- $\mathrm{w}(1) = -\infty$; $\mathrm{w}(mn) = \max(\mathrm{w}(m), \mathrm{w}(n))$;
- $\mathrm{w}(\sigma^i(m)) = i + \mathrm{w}(m)$.

Examples

- $P = K[X \times \mathbb{N}]$ with $\sigma : x_i(j) \mapsto x_i(j+1)$. One has the weight function $\mathrm{w}(x_i(j)) = j$.
- Note: P is both the letterplace algebra and the **ring of ordinary difference polynomials** where $x_i(0) = u_i = u_i(t)$ are univariate functions and $x_i(j) = \sigma^j(u_i)$.

w-grading of P

- Weight function implies a grading $P = \bigoplus_{i \in \hat{\mathbb{N}}} P_i$ of the algebra P defined by the monoid $(\hat{\mathbb{N}}, \max)$.
- Such grading is compatible with the action of $\Sigma = \langle \sigma \rangle$ that is $\sigma^i P_j \subset P_{i+j}$.

The right inverse of π

- Extend Σ with the endomorphism $\sigma^{-\infty} : x_i \mapsto 0$.
- $\hat{\Sigma} = \{\sigma^{-\infty}\} \cup \Sigma$, $\hat{S} = P \star \hat{\Sigma}$.
- Recall: $\pi : \hat{S} \to P, s^i \mapsto 1$ is a left $\hat{S}$-module epimorphism.
- Weight function implies existence of mapping $\xi : P \to \hat{S}$ such that $f \mapsto fs^i$ for all $f \in P_i$.
- ξ is an injective $\hat{\Sigma}$-equivariant map such that $\pi\xi = id$.

From P to S: ideals

- To avoid the use of extended ring $\hat{S}$ we restrict to ideals $J \neq P$.
- Let $J \neq P$ a w-graded Σ-invariant ideal of P.
- Denote J^S the graded two-sided ideal generated by $\xi(J) \subset S$.
- Then $\pi(J^S) = J$ that is there is a **one-to-one correspondence** between all w-graded Σ-invariant ideals $J \subsetneq P$ and some class of graded ideals $J^S \subset S$.
- We call J^S the **skew analogue** of J.

Note: the correspondence is given in a general setting, not only for the letterplace context.

From F to P: ideals

- $P = K[X \times \mathbb{N}^*]$, $\sigma : x_i(j) \mapsto x_i(j+1)$, $\mathrm{w}(x_i(j)) = j$.
- Recall: $\iota' = \pi\iota : F = K\langle X\rangle \to P$ s.t. $x_{i_1}\cdots x_{i_d} \mapsto x_{i_1}(1)\cdots x_{i_d}(d)$.
- Let $I \neq F$ be a graded two-sided ideal of F.
- Denote I^P the w-graded Σ-invariant ideal which is Σ-generated by $\iota'(I) \subset P$.
- Since $\iota = \xi\iota'$ (**THE DIAGRAM**) one has that $(I^P)^S$ coincides with the extension of $\iota(I) \subset R$ to S that is the skew letteplace analogue of I.
- Composing previous correspondences, one obtains a **one-to-one correspondence** between all graded two-sided ideals $I \subsetneq F$ and some class of w-graded Σ-invariant ideals $I^P \subsetneq P$.
- We call I^P the **letterplace analogue** of I.

New goals

- To develop of a Gröbner bases theory for (graded) two-sided ideals of $S = P \star \Sigma$.
- To develop of a Gröbner Σ-bases theory for (w-graded) Σ-invariant ideals of P.
- To understand how such generating sets maps into the ideal correspondences we found.

Applications

- **Unification** of commutative and non-commutative Gröbner bases theory (graded case) under a more general theory.
- Commutative CAS **gain new abilities** to compute with non-commutative expressions (Hecke algebras, enveloping algebras, etc) and also with (non-linear) difference equations (approximation of differential equations, combinatorics, etc).

General setting

- $P = K[x_0, x_1, \ldots]$ polynomial ring in a countable number of variables endowed with monomial ordering $\prec$.
- $\sigma : P \to P$ algebra endomorphism of infinite order such that:
- $\sigma(x_i)$ are monomials; $\gcd(\sigma(x_i), \sigma(x_j)) = 1$ for $i \neq j$;
- $m \prec n$ implies $\sigma(m) \prec \sigma(n)$ for all monomials m, n.
- It follows that $\mathrm{lm}(\sigma(f)) = \sigma(\mathrm{lm}(f))$ and $\mathrm{spoly}(\sigma(f), \sigma(g)) = \sigma(\mathrm{spoly}(f, g))$ for all $f, g \in P$.

Not only maps sending variables into variables, but also for instance the endomorphism $x_i \mapsto x_i^d$ (Weispfenning) fits this setting.
A similar setting appeared in Brouwer & Draisma, Math. Comp. 80, (2011)

Monomial orderings of S

- $\mathrm{Mon}(S) = \{ms^i \mid m \in \mathrm{Mon}(P), i \geq 0\}$.
- $\mathrm{Mon}(S)$ is closed under multiplication: $(ms^i)(ns^j) = m(s^i \cdot n)s^{i+j}$.
- As usual, monomial orderings are well-orderings of $\mathrm{Mon}(S)$ compatible with multiplication.
- A monomial ordering for S is for instance defined: $ms^i \prec ns^j$ if and only if $i < j$ or $i = j$ and $m \prec n$.
- Note: this is also a monomial ordering for S as **free P-module**.

Homogeneous Gröbner bases

- J a graded (two-sided) ideal of S, $G \subset J$ a set of homogeneous elements.
- G is a (Gröbner) basis of J if and only if $\Sigma G \Sigma$ is a (Gröbner) basis of J as **P-submodule of S**.
- Homogeneity simplifies two-sided generation: $fs^i(gs^j)hs^k = f(s^{i+j} \cdot h)\, s^i(gs^j)s^k$ for all $f, g, h \in P$.

Reduce computations by symmetry

- To compute a homogeneous Gröbner basis G in the ring S is equivalent to compute a Gröbner basis $G' = \Sigma G \Sigma$ in the free P-module S (module Buchberger algorithm) and then extract G from G'.
- Since $\mathrm{spoly}(sfs, sgs) = s\,\mathrm{spoly}(f, g)s$ for all $f, g \in S$, many S-polynomial computations are clearly **redundant**.
- Then, we give a criterion that reduces computations up to two-sided action defined by Σ.

Σ-criterion in S

- $G \subset S$ is a homogeneous Gröbner basis if and only if the S-polynomials $\mathrm{spoly}(f, s^i g s^j)$ and $\mathrm{spoly}(fs^i, s^j g)$ have a Gröbner representation with respect to $\Sigma G \Sigma$ for any $f, g \in G$ and $i, j \geq 0$.

Note: auto S-polynomials arise as $\mathrm{spoly}(gs^i, s^i g)$ for all i.

The polynomial ring P has an infinite number of variables and the free P-module S has infinite rank. No hope of termination?

Truncated termination in S

- Let $J \subset S$ be a graded ideal, H a homogeneous basis of J.
- **Assume** $H_d = \{f \in H \mid \deg_s(f) \leq d\}$ is a **finite set** for some d.
- Then, there exists a homogeneous Gröbner basis $G \subset J$ such that G_d is also finite that is d**-truncated Buchberger algorithm terminates**.
- In particular, **membership problem** has algorithmic solution for finitely generated graded ideals of the ring S.
- Proof: If $\Sigma_d = \{s^i\}_{i \leq d}$ then $H'_d = \Sigma_d H_d \Sigma_d$ is also a finite set. Define $P^{(d)}$ the polynomial ring in the finite number of variables occurring in elements of H'_d and put $S^{(d)} = \bigoplus_{i \leq d} P^{(d)} s^i$. The termination of d-truncated Buchberger algorithm for $H'_d \subset S^{(d)}$ is provided by Noetherianity of the ring $P^{(d)}$ and free $P^{(d)}$-module $S^{(d)}$ of finite rank.

In the same general setting, we have Gröbner basis theory for (w-graded) Σ-invariant ideals of P.

Gröbner Σ-bases

- Let $I \subset P$ be a Σ-invariant ideal.
- Define G a **Gröbner Σ-basis** of I if $\mathrm{lm}(G)$ is a Σ-basis of $\mathrm{LM}(I) = \langle \mathrm{lm}(I) \rangle$. In other words, $\Sigma \cdot G$ is a Gröbner basis of I.

Such bases are called “equivariant Gröbner bases” in Brouwer & Draisma. Owing to $\mathrm{spoly}(s \cdot f, s \cdot g) = s \cdot \mathrm{spoly}(f, g)$ for any $f, g \in P$, one has

Σ-criterion in P

- $G \subset P$ is a Gröbner basis if and only if the S-polynomial $\mathrm{spoly}(f, s^i \cdot g)$ has a Gröbner representation with respect to $\Sigma \cdot G$ for any $f, g \in G$ and $i \geq 0$.

Note: auto S-polynomials arise as $\mathrm{spoly}(g, s^i \cdot g)$ for all i.

Algorithm: SigmaGBasis

Input: H, a Σ-basis of a Σ-invariant ideal $I \subset P$.
Output: G, a Gröbner Σ-basis of I.

$G := H$;
$B := \{(f, g) \mid f, g \in G\}$;
while $B \neq \emptyset$ do
  choose $(f, g) \in B$;
  $B := B \setminus \{(f, g)\}$;
  for all $i \geq 0$
    $h := \text{Reduce}(\mathrm{spoly}(f, s^i \cdot g), \Sigma \cdot G)$;
    if $h \neq 0$ then
      $P := P \cup \{(h, g), (g, h), (h, h) \mid g \in G\}$;
      $G := G \cup \{h\}$;
return G.

No termination is provided in general. This is, for instance one, of main problems for computing with difference ideals. Nevertheless, weight functions provide some termination.

Truncated termination in P

- Let $I \subset P$ be a w-graded Σ-invariant ideal, H be a w-homogeneous Σ-basis of I.
- **Assume** $H_d = \{f \in H \mid \mathrm{w}(f) \leq d\}$ is a **finite set** for some d.
- Then, there exists a w-homogeneous Gröbner Σ-basis $G \subset I$ such that G_d is also finite that is d**-truncated Buchberger algorithm terminates**.
- In particular, **membership problem** has algorithmic solution for finitely generated w-graded Σ-invariant ideals of P.
- We argue in a similar way as for termination in S. Note that $\mathrm{w}(s^i \cdot f) = i + \mathrm{w}(f) \leq d$ implies that $i \leq d$.

In fact, truncated termination results for P and S are **equivalent** under skew ideal correspondence.

From P to S: GBs

- Let $I \subsetneq P$ be a w-graded Σ-invariant ideal and denote $J = I^S$ its skew analogue.
- Recall: $J \subset S$ is the graded two-sided ideal generated by $\xi(I)$, where $\xi : f \mapsto fs^i$ for all $f \in P_i$.
- Let $G \subset I$ be a subset of w-homogeneous elements. Then $\xi(G) \subset J$ is also a subset of homogeneous elements.
- G is a (Gröbner) Σ-basis of I if and only if $\xi(G)$ is a (Gröbner) basis of J.

Back to letterplace context

- $P = K[X \times \mathbb{N}^*], \sigma : x_i(j) \mapsto x_i(j+1), \mathrm{w}(x_i(j)) = j$, etc.
- For all words $w = x_{i_1} \cdots x_{i_d} \in \mathrm{Mon}(F)$ one has $\iota(w) = \iota'(w)s^d = x_{i_1}(1) \cdots x_{i_d}(d)s^d \in \mathrm{Mon}(S)$.
- The set $\mathrm{Mon}(R)$ of such elements is a submonoid of $\mathrm{Mon}(S)$.
- Then, monomial orderings of S can be restricted to $R \approx F$.
- The ring P (and hence S) is endowed with a multigrading.
- If $m = x_{i_1}(j_1) \cdots x_{i_d}(j_d) \in \mathrm{Mon}(P)$ we define $\partial(m) = (\mu_k)_{k \in \mathbb{N}^*}$ where μ_k is the number of times the integer k occurs in $j_1, \ldots, j_d$.
- A homogeneous element $fs^d \in S$ belongs to the subring R if and only if f is multi-homogeneous and $\partial(f) = 1^d = (1, \ldots, 1, 0, \ldots)$.

From F to S: GBs

- Let $I \subset R$ be a graded two-sided ideal and denote $J \subset S$ the extension of I.
- Let $G \subset J$ be a subset of multi-homogeneous elements. Then $G \cap R \subset I$ is also a subset of homogeneous elements.
- If G is a (Gröbner) basis of J then $G \cap R$ is a (Gröbner) basis of I.

Composing this result with the skew GB correspondence one obtains finally

From F to P: GBs

- Denote $V = \pi(R) \subset P$, left R-module isomorphic to R (inverse ξ).
- Let $I \subsetneq F$ be a graded two-sided ideal and denote $J \subsetneq P$ its letterplace analogue.
- Let $G \subset J$ be a set of w-homogeneous elements. Then $G \cap V \subset \iota'(I)$ is also a subset of homogeneous elements.
- If G is a (Gröbner) Σ-basis of J then $\iota'^{-1}(G \cap V)$ is a (Gröbner) basis of I.

This last result and corresponding algorithm appeared in $-$ & Levandovskyy, J. Symbolic Comput., 44 (2009).

Algorithm: FreeGBasis

Input: H, a homogeneous basis of a graded two-sided ideal $I \subsetneq F$.
Output: G, a homogeneous Gröbner basis of I.

$G := \iota'(H)$;
$B := \{(f, g) \mid f, g \in G\}$;
while $B \neq \emptyset$ do
    choose $(f, g) \in B$;
    $B := B \setminus \{(f, g)\}$;
    for all $i \geq 0$ **such that** $\mathrm{spoly}(f, s^i \cdot g) \in V$
        $h := \textsc{Reduce}(\mathrm{spoly}(f, s^i \cdot g), \Sigma \cdot G)$;
        if $h \neq 0$ then
            $P := P \cup \{(h, g), (g, h), (h, h) \mid g \in G\}$;
            $G := G \cup \{h\}$;
return $\iota'^{-1}(G)$.

As usual, termination is provided via truncation up to some degree.

Algorithmic unification

- FreeGBasis is a variation of the algorithm SigmaGBasis: we just added a **new criterion** $\mathrm{spoly}(f, s^i \cdot g) \in V$.
- In this **general algorithmic scheme** one obtains computation of Gröbner bases for (ordinary) difference ideals and for graded two-sided ideals of the free associative algebra.
- Such scheme is based on polynomials rings in **commutative variables** and therefore it can be implemented in any commutative computer algebra system.
- The essential difference with usual commutative Gröbner bases computations is the use of the **Σ-criterion** and **V-criterion** reducing the number of S-polynomials to be considered.
- Termination of truncated computations can be obtained via suitable gradings.

Examples of the encoding of difference and non-commutative ideals in $P = K[X \times \mathbb{N}]$:

Lorentz attractor

$$\begin{aligned} &D \cdot x_1(0) - A(x_2(0) - x_1(0)),\\ &D \cdot x_2(0) - x_1(0)(B - x_3(0)) + x_2(0),\\ &D \cdot x_3(0) - x_1(0)x_2(0) + Cx_3(0), \end{aligned}$$

where $D = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 + \ldots$ and $\Delta \cdot x_i(0) = x_i(1) - x_i(0)$, $\Delta^2 \cdot x_i(0) = x_i(2) - 2x_i(1) + x_i(0)$, etc.

Hecke algebra of the symmetric group $\mathbb{S}_n$

$$\begin{aligned} &x_i(1)x_{i+1}(2)x_i(3) - x_{i+1}(1)x_i(2)x_{i+1}(3),\\ &x_i(1)x_j(2) - x_j(1)x_i(2) \text{ for } |i - j| \geq 2,\\ &x_i(1)x_i(2) - (q - 1)x_i(1) - q. \end{aligned}$$

Implementation & experiments

- We developed an implementation of the algorithm FREEGBASIS in the computer algebra system **Singular**.
- See **www.singular.uni-kl.de/Manual/latest** and search for **letterplace**.
- Implementation is still in progress.
- Comparisons with the best implementations of non-commutative Gröbner bases (classic algorithm) are very encouraging.

Miscellanea

Example	BERG	GBNP	SING	#In	#Out
`nilp3-6`	**0:01**	0:07	**0:01**	192	110
`nilp3-10`	0:23	1:49	**0:03**	192	110
`nilp4-6`	1:22	1:12	**0:14**	2500	891
`nilp4-7`	**1:24**	7:32	1:40	2500	1238
`nilp4s-8`	**13:52**	1h:14:54	0:57†	1200	1415
`nilp4s-9`	**5h:50:26**	40h:23:19	1:32†	1200	1415
`metab5-10`	**0:20**	13:58††	0:22	360	76
`metab5-11`	27:23	14:42†	**1:11**	360	113
`metab5s-10`	**0:32**	1h:42:43††	0:34	45	76
`metab5s-11`	27:33	25:27†	**2:05**	45	113
`tri4-7`	0:48	18h†	**0:08**	12240	672
`tri4s-7`	0:40	3:37	**0:07**	3060	672
`ufn3-6`	0:31	1:43	**0:23**	125	1065
`ufn3-8`	**2:18**	9:33	2:20	125	1763
`ufn3-10`	**5:24**	20:37	3:25†	125	2446

Examples on Serre's relations

Example	BERG	GBNP	SING	#In	#Out
`ser-f4-15`	16:05	1h:25:48	**0:08**	9	43
`ser-e6-12`	0:49	5:39	**0:07**	20	76
`ser-e6-13`	2:36	14:52	**0:14**	20	79
`ser-ha11-10`	0:04	7:82	**0:01**	5	33
`ser-ha11-15`	1h:03:21	4h:06:00	**1:58**	5	112
`ser-eha112-12`	0:56	3:44	**0:37**	5	126
`ser-eha112-13`	1h:12:50	34:53	**4:08**	5	174

Future developments

- Extend theory and methods to finitely generated free commutative semigroups $\Sigma = \langle \sigma_1, \ldots, \sigma_r \rangle$ to cover **partial difference ideals** (in progress).
- Extend letterplace approach to **non-graded ideals** of $K\langle X \rangle$ via (de)homogeneization techniques (in progress).
- Consider endomorphism semigroups defined by $\mathrm{Inc}(\mathbb{N})$ for applications to algebraic statistic.
- Develop all algorithms related to Gröbner bases computations (free resolutions, Hilbert functions, ideal decomposition etc) in the general context of Σ-invariant ideals and modules.
- Implementing all of this.

$$
\begin{tikzcd}
F \arrow[rr, "\iota"] \arrow[dr, "\iota'"'] & & S \\
& P \arrow[ur, "\xi"'] &
\end{tikzcd}
$$

Thank you.