Enumeration of Shi regions with a fixed separating wall

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The extended Shi Arrangement

- $\{\varepsilon_1, \ldots, \varepsilon_{n+1}\}$ standard basis of $\mathbb{R}^{n+1}$
- $\langle \, | \, \rangle$ standard inner product
- $\Pi = \{\alpha_1, \ldots, \alpha_n\}$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $i = 1, \ldots, n$, is a basis of

$$V = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1 + x_2 + \cdots + x_{n+1} = 0\}.$$

- $\alpha_{ij} = \alpha_i + \cdots + \alpha_j = \varepsilon_i - \varepsilon_{j+1}$, with $\alpha_{ii} = \alpha_i$, and $\theta = \alpha_{1,n}$
- The elements of $\Delta = \{\varepsilon_i - \varepsilon_j : i \neq j\}$ are called roots
  A root α is positive ($\alpha > 0$) if $\alpha \in \Delta^+ = \{\varepsilon_i - \varepsilon_j : i < j\}$.

The extended Shi Arrangement

- For each $\alpha \in \Delta^+$ we define its reflecting hyperplane

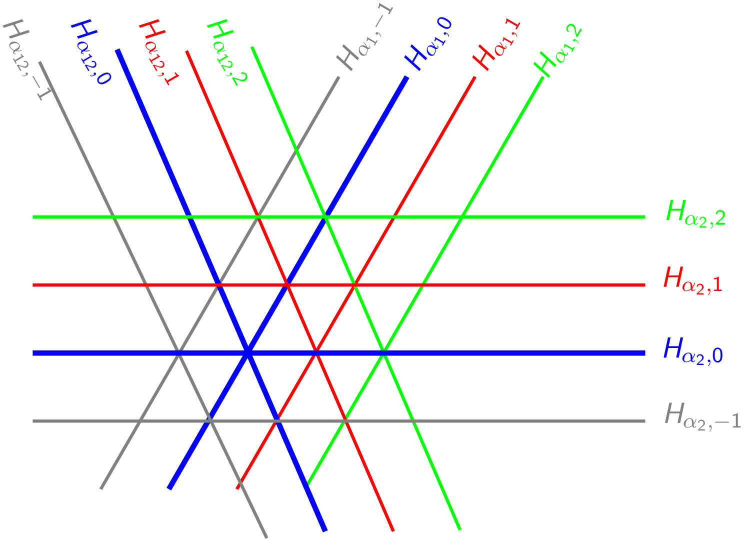
$$H_{\alpha,0} = \{v \in V : \langle v|\alpha\rangle = 0\},$$

  and for $k \in \mathbb{Z}$, the $H_{\alpha,0}$'s translate

$$H_{\alpha,k} = \{v \in V : \langle v|\alpha\rangle = k\}.$$

- The extended Shi arrangement, here called the m-Shi arrangement, is

$$\mathcal{H}_m = \{H_{\alpha,k} : \alpha \in \Delta^+, -m < k \leq m\}.$$

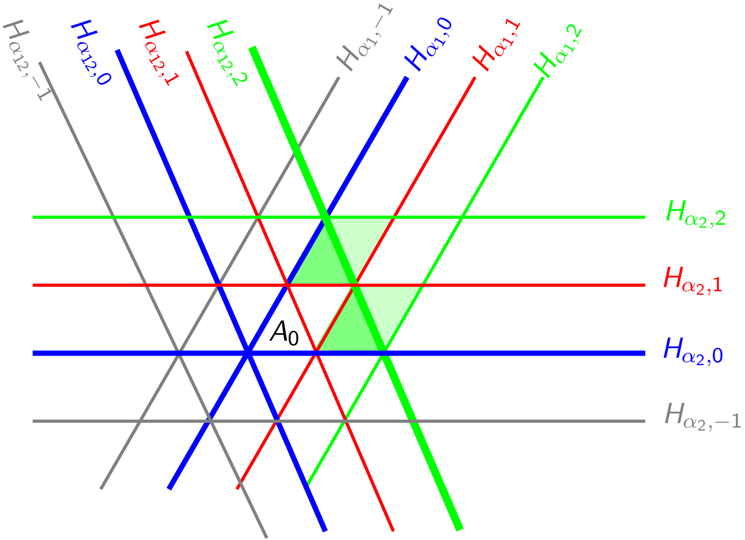


The 2-Shi arrangement $\mathcal{H}_2 = \{H_{\alpha,k} : \alpha \in \Delta^+, -2 < k \leq 2\}$.

- A region is a connected component of $V \setminus \bigcup_{H \in \mathcal{H}_m} H$
- For $\alpha \in \Delta^+$ and $k \in \mathbb{Z}$,

$$H^+_{\alpha,k} = \{v \in V : \langle v|\alpha\rangle \geq k\}.$$

- The dominant chamber of V is $V \cap \bigcap_{i=1}^n H^+_{\alpha_i,0}$.
- A dominant region is a region contained in the dominant chamber.
- A wall of a region R is an hyperplane which contains a facet of R.
- A separating wall for a region R is a wall of R which separates R from the region $A_0 := H^-_{\theta,1} \cap \bigcap_{i=1}^n H^+_{\alpha_i,0}$.



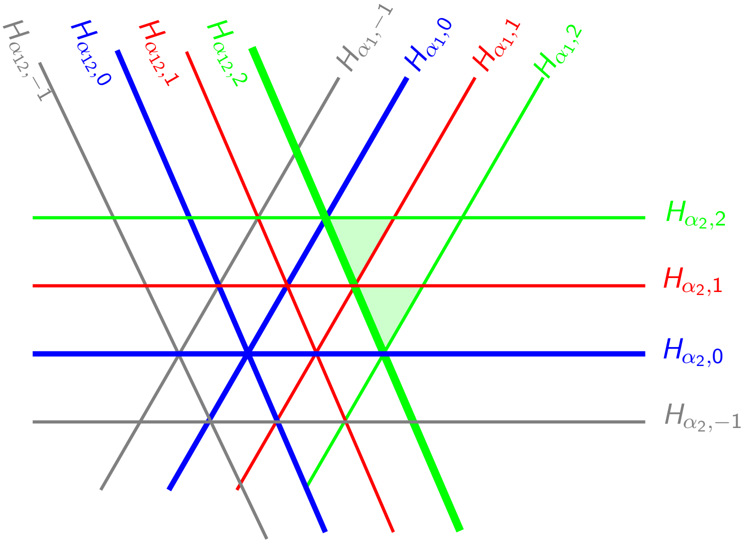
$H_{\alpha_{12},-1}$
$H_{\alpha_{12},0}$
$H_{\alpha_{12},1}$
$H_{\alpha_{12},2}$
$H_{\alpha_1,-1}$
$H_{\alpha_1,0}$
$H_{\alpha_1,1}$
$H_{\alpha_1,2}$
$H_{\alpha_2,2}$
$H_{\alpha_2,1}$
A_0
$H_{\alpha_2,0}$
$H_{\alpha_2,-1}$

Enumerative results - Motivation

- The number of dominant regions in the m-Shi arrangement is $\frac{1}{mn+1}\binom{(m+1)n}{n}$.
- The number of dominant regions having k separating walls of type $H_{\alpha_{ij},m}$ is equal to the kth m-Narayana number

$$\frac{1}{mn+1}\binom{n-1}{n-k-1}\binom{mn+1}{n-k}.$$

Problem: Fix $\alpha_{ij} \in \Delta^+$ and an integer $0 < k \leq m$. Find the number of dominant regions having separating wall $H_{\alpha_{ij},k}$.



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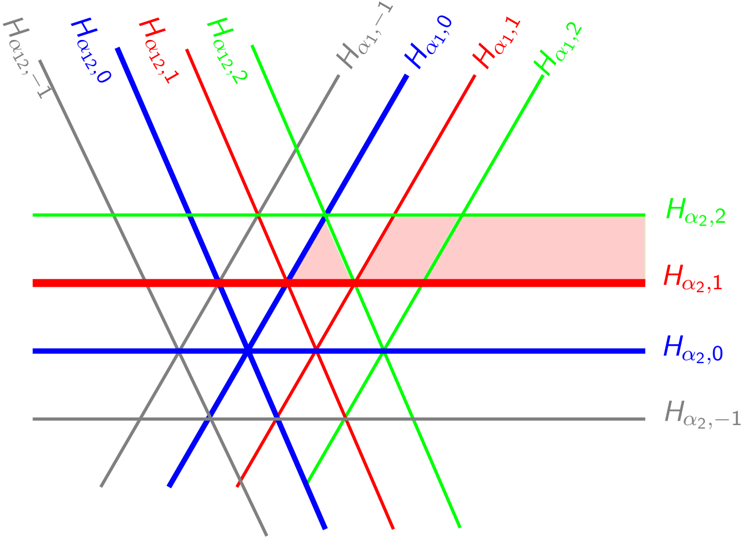
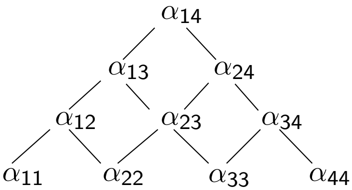


The root poset and m-Shi tableaux

- Root order on Δ^+: $\alpha \leq \beta$ if $\beta - \alpha$ is in the integer span of Π.



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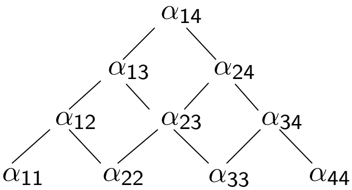


α_{14}	α_{13}	α_{12}	α_{11}
α_{24}	α_{23}	α_{22}	
α_{34}	α_{33}		
α_{44}			

- Arrange the roots α_{ij}, $1 \leq i < j \leq n$ in a n-staircase diagram such that α_{ij} is in box $(i, n - i + 1)$.

The root poset and m-Shi tableaux

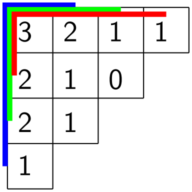
- To each root α_{ij} we associate an integer $0 \leq k_{ij} \leq m$

k_{14}	k_{13}	k_{12}	k_{11}
k_{24}	k_{23}	k_{22}	
k_{34}	k_{33}		
k_{44}			

- The filling of this n-staircase diagram is called an **m-Shi tableau** if the following two conditions hold:

$$k_{ij} = \begin{cases} k_{i\ell} + k_{\ell+1,j} \text{ or } k_{i\ell} + k_{\ell+1,j} + 1, & \text{if } k_{i\ell} + k_{\ell+1,j} \leq m-1 \\ & \quad \text{and } i \leq \ell < j \\ m, & \text{otherwise.} \end{cases}$$

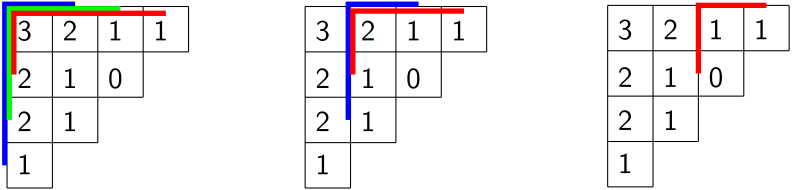
Example of a m-Shi tableau



- For each $k_{ij} \leq m-1$ we check if the sum of the values of the endpoints of each hook on k_{ij} of length $j-i+2$ sum up to k_{ij} or $k_{ij}-1$.
- For each $k_{ij} = m$ we check if the sum of the values of the endpoints of each hook on k_{ij} of length $j-i+2$ sum up to a value $\geq m$.

Example of a m-Shi tableau

Checking if the entries of the first row are valid:

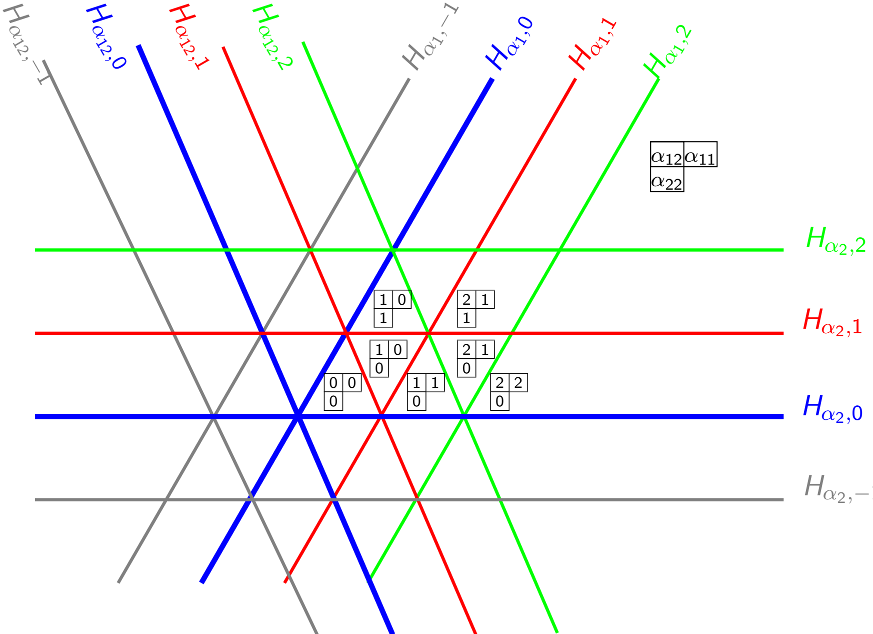


Theorem (Athanasiadis)

The m-Shi tableaux are in bijection with the dominant regions in the m-Shi arrangement.

The bijection:

- Given a dominant region R in the m-Shi arrangement, k_{ij} is the number of integer translates of $H_{\alpha_{ij},0}$ that separates R from the origin.
- Given an m-Shi tableau, the corresponding region R consists of those points x such that $\langle \alpha_{ij}, x \rangle \geq k_{ij}$, for all $1 \leq i \leq j \leq n$.



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$H_{\alpha_2,2}$
$H_{\alpha_2,1}$
$H_{\alpha_2,0}$
α_{12} α_{11}
α_{22}
1 0
1
2 1
1
1 0
0
2 1
0
0 0
0
1 1
0
2 2
0

m-Shi tableaux and separating walls

Theorem

Let T_R be the m-Shi tableau for the region R. The hyperplane $H_{\alpha_{ij},m}$ is a separating wall for the region R if and only if $k_{ij} = m$ and $k_{i\ell} + k_{\ell+1,j} = m - 1$ for all $i \leq \ell < j$.

Problem:

- Count the number of regions having $H_{\alpha_{ij},m}$ as separating wall.
- Count the number of m-Shi tableux with $k_{ij} = m$ and $k_{i\ell} + k_{\ell+1,j} = m - 1$ for all $i \leq \ell < j$.

Base Case

Theorem (Fishel,Tzanaki,Vazirani)
The number of regions in the m-Shi arrangement $\mathcal{H}_m$ with separating wall $H_{\theta,m}$ is equal to m^{n-1}.

- Let $w = w_1 \cdots w_{n-1}$ be a word over the alphabet $\{0, 1, \ldots, m-1\}$.
- Order w decreasingly and place it on the first row of T, placing also the correspondent entries on the first column.
- For all $i = 2, \ldots, n-1$ and all $j = i, \ldots, n-1$ we set

$$k_{i,j} = \begin{cases} k_{1,i-1} - k_{1,j} & \text{if } k_{1,i-1} \text{ is on the left of } k_{1,j} \text{ on } w \\ k_{1,i-1} - k_{1,j} - 1 & \text{otherwise} \end{cases}$$

- This filling produce a m-Shi tableaux and this operation is invertible.

Ex: $w = 61513 \in \{0, 1, 2, 3, 4, 5, 6\}^*$, the corresponding 7-Shi tableau is

7					

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7					

	7	6	5	3	1	1
	5					
	5				
	3			
	1		
$\rightarrow$	0	

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7					

$\rightarrow$

7	6	5	3	1	1
5					
5					
3					
1					
0					

$\rightarrow$

7	6	5	3	1	1
5	5	4	1	0	
5					
3					
1					
0					

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7					

$\rightarrow$

7	6	5	3	1	1
5					
5					
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1					
0					

$\rightarrow$

7	6	5	3	1	1
5	5	4	1	0	
5					
3					
1					
0					

7	6	5	3	1	1
5	5	4	1	0	
5	5	3	1		
3	3	2			
1	1				
0					

General case

- Let $h^n_{\alpha,k}$ be the set of dominant regions in the m-Shi arrangement having $H_{\alpha,k}$ as a separating wall.
- Given a fundamental region R, let

$$r(R) = \#\{(j,k) : R \in h^n_{\alpha_{1j}k} \text{ and } 1 \leq k \leq m\}$$

$$c(R) = \#\{(i,k) : R \in h^n_{\alpha_{i,n-1}k} \text{ and } 1 \leq k \leq m\}$$

- The generating function is

$$f^n_{\alpha_{ij}m}(p,q) = \sum_{R \in h^n_{\alpha_{ij}k}} p^{c(R)} q^{r(R)}.$$

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