Quadruply-Refined Enumeration of Alternating Sign Matrices and Doubly-Refined Enumeration of Descending Plane Partitions

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References

- RB, P Di Francesco and P Zinn-Justin *On the weighted enumeration of alternating sign matrices and descending plane partitions*
  J. Combin. Theory Ser. A **119** (2012) 331–363 **arXiv: 1103.1176**

- RB, P Di Francesco and P Zinn-Justin *The doubly-refined enumeration of alternating sign matrices and descending plane partitions*
  arXiv: 1202.1520

- RB *The quadruply-refined enumeration of alternating sign matrices*
  arXiv: 1203.3187

Plan

- Define alternating sign matrices (ASMs) & descending plane partitions (DPPs)
- Define certain bulk & boundary statistics for ASMs & DPPs
- Show that unrefined, singly-refined & doubly-refined enumerations (where order of refinement refers to number of boundary statistics) are the same for ASMs & DPPs
- Show that generating functions associated with these enumerations are given by explicit determinant formulae
- Show that quadruply-refined ASM generating function satisfies Desnanot–Jacobi-type identity & can be computed explicitly in terms of determinants

Alternating Sign Matrices (ASMs)

$$\mathrm{ASM}(n) := \left\{ n\times n \text{ matrices} \;\middle|\; \begin{array}{l} \bullet \text{ each entry 0, 1 or } -1 \\ \bullet \text{ along each row \& column, nonzero entries} \\ \quad \text{alternate in sign \& add up to 1} \end{array} \right\}$$

- Arose during study of Dodgson condensation algorithm for determinant evaluation *(Mills, Robbins, Rumsey 1982; Robbins, Rumsey 1986)*
- Many subsequent appearances in combinatorics, algebra, mathematical physics, . . .
- Any permutation matrix is an ASM
- Any ASM contains a single 1 & no -1's in first & last row & column
- e.g. ASM(3) =

$$\left\{ \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}, \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, \begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix} \right\}$$

- e.g. $\begin{pmatrix}0&0&0&1&0&0\\0&1&0&-1&1&0\\1&-1&1&0&0&0\\0&0&0&1&0&0\\0&1&0&-1&0&1\\0&0&0&1&0&0\end{pmatrix} \in \mathrm{ASM}(6)$

Descending Plane Partitions (DPPs)

DPP(n) :=

$$\left\{ \begin{array}{l} \text{arrays} \; \begin{array}{cccccc} D_{11} & D_{12} & D_{13} & \cdots\cdots\cdots & D_{1,\lambda_1} \\ & D_{22} & D_{23} & \cdots\cdots & D_{2,\lambda_2+1} \\ & & D_{33} & \cdots & D_{3,\lambda_3+2} \\ & & \ddots & & ⋰ \\ & & & D_{tt} \cdots\cdots & D_{t,\lambda_t+t-1} \end{array} \end{array} \middle| \begin{array}{l} \bullet \text{ each part (entry) a positive integer} \\ \bullet \text{ parts decrease weakly along rows} \\ \bullet \text{ parts decrease strictly down columns} \\ \bullet \; n \geq D_{11} > \lambda_1 \geq D_{22} > \ldots \geq D_{tt} > \lambda_t \end{array} \right\}$$

- Arose during study of cyclically symmetric plane partitions *(Andrews 1979)*

- e.g. $\text{DPP}(3) = \left\{ \emptyset, \begin{array}{cc} 3 & 3 \\ & 2 \end{array}, 2, 3\,3, 3, 3\,2, 3\,1 \right\}$

- e.g. $\begin{array}{ccccc} 6 & 6 & 6 & 5 & 2 \\ & 4 & 4 & 1 & \\ & & 3 & & \end{array} \in \text{DPP}(6)$

ASM Statistics

For $A \in \mathsf{ASM}(n)$

Bulk statistics:

- $\nu(A) := \sum_{\substack{1\le i<i'\le n \\ 1\le j'\le j\le n}} A_{ij}\, A_{i'j'} \;=\; \#$ of 'inversions' in A
- $\mu(A) := \#$ of -1's in A

Boundary statistics:

- $\rho_{\mathsf{T}}(A) := \#$ of 0's left of the 1 in top row of A
- $\rho_{\mathsf{R}}(A) := \#$ of 0's below the 1 in right-most column of A
- $\rho_{\mathsf{B}}(A) := \#$ of 0's right of the 1 in bottom row of A
- $\rho_{\mathsf{L}}(A) := \#$ of 0's above the 1 in left-most column of A

- e.g. $A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$

$\Rightarrow\;\; \nu(A) = 5,\;\; \mu(A) = 3,\;\; \rho_{\mathsf{T}}(A) = 3,\;\; \rho_{\mathsf{R}}(A) = 1,\;\; \rho_{\mathsf{B}}(A) = 2,\;\; \rho_{\mathsf{L}}(A) = 2$

DPP Statistics

For $D \in \mathrm{DPP}(n)$

Bulk statistics:

- $\nu(D) :=$ # of parts of D for which $D_{ij} > j - i$
  $=$ # of 'nonspecial' parts in D
- $\mu(D) :=$ # of parts of D for which $D_{ij} \leq j - i$
  $=$ # of 'special' parts in D

Boundary statistics:

- $\rho_{\mathsf{T}}(D) :=$ # of n's in D
- $\rho_{\mathsf{B}}(D) :=$ (# of $(n-1)$'s in D) $+$
  (# of rows of D of length $n-1$)

- Only row 1 can contain n's
- Only rows 1 & 2 can contain $(n-1)$'s
- Only row 1 can have length $n-1$

- e.g. $D = \begin{array}{ccccc} 6 & 6 & 6 & 5 & 2 \\ & 4 & 4 & 1 & \\ & & 3 & & \end{array} \in \mathrm{DPP}(6)$ (special parts: 2 & 1)

  $\Rightarrow \ \nu(D) = 7, \ \ \mu(D) = 2, \ \ \rho_{\mathsf{T}}(D) = 3, \ \ \rho_{\mathsf{B}}(D) = 2$

ASM & DPP Unrefined Generating Functions

- $Z_n^{\mathsf{ASM}}(x,y) := \sum_{A \in \mathsf{ASM}(n)} x^{\nu(A)}\, y^{\mu(A)}$
- $Z_n^{\mathsf{DPP}}(x,y) := \sum_{D \in \mathsf{DPP}(n)} x^{\nu(D)}\, y^{\mu(D)}$

- e.g. ASM(3) =

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}$$

$$\& \quad \mathsf{DPP}(3) = \left\{ \emptyset,\ \begin{matrix} 3\,3 \\ \;\;2 \end{matrix},\ 2,\ 3\,3,\ 3,\ 3\,2,\ 3\,1 \right\}$$

$$\Rightarrow \quad Z_3^{\mathsf{ASM}}(x,y) = Z_3^{\mathsf{DPP}}(x,y) = 1 + 2x + 2x^2 + x^3 + xy$$

Unrefined Enumeration (2 Bulk, 0 Boundary Statistics)

Theorem $Z_n^{\mathsf{ASM}}(x,y) = Z_n^{\mathsf{DPP}}(x,y)$

Equivalently (# of $n\times n$ ASMs with p 'inversions' & m -1's) $=$
(# of order n DPPs with p nonspecial parts & m special parts)

- Conjectured *Mills, Robbins, Rumsey 1983*
- Previously-known special case: $x=y=1$, $\left|\mathsf{ASM}(n)\right| = \left|\mathsf{DPP}(n)\right| \left(= \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}\right)$

Proof structure

1. (a) Use bijection between ASM(n) & {configurations of statistical mechanical six-vertex model with domain-wall boundary conditions on $n\times n$ grid}
   (b) Apply Izergin–Korepin determinant formula for partition function of model

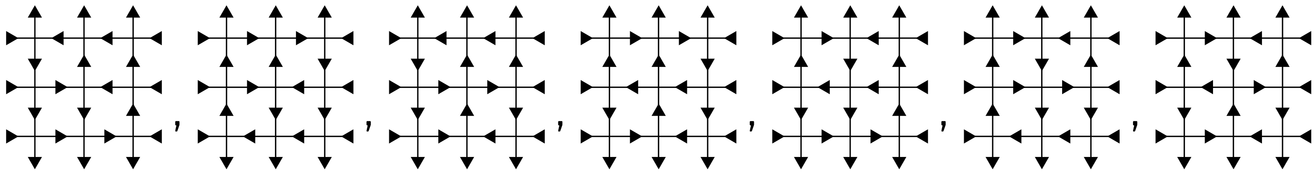
2. (a) Use bijection between DPP(n) & {certain nonintersecting lattice paths on $n\times n$ grid}
   (b) Apply Lindström–Gessel–Viennot theorem for weighted enumeration of nonintersecting paths with fixed endpoints in terms of determinant

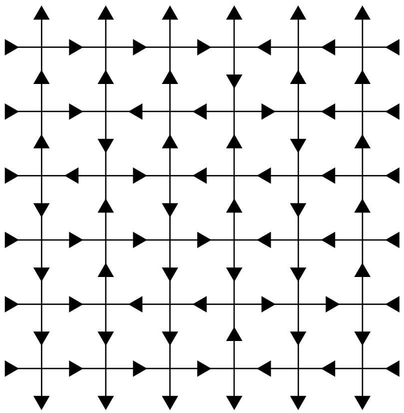
3. Determinant transformations give $Z_n^{\mathsf{ASM}}(x,y) = Z_n^{\mathsf{DPP}}(x,y) = \det_{0\le i,j\le n-1} K_n(x,y)_{ij}$

   where $K_n(x,y)_{ij} = -\delta_{i,j+1} + \sum_{k=0}^{\min(i,j+1)} \binom{i-1}{i-k}\binom{j+1}{k} x^k y^{i-k}$

Configurations of Six-Vertex (Square Ice) Model with Domain-Wall Boundary Conditions (DWBC)

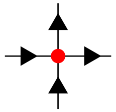
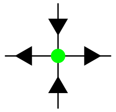
$6\text{VDW}(n) := \left\{ \begin{array}{l|l} \text{edge orientations} & \bullet \text{ 2 inward \& 2 outward arrows at each internal} \\ \text{of } n \times n \text{ grid} & \quad \text{vertex } (\Rightarrow \text{ 6 possible vertex configurations}) \\ & \bullet \text{ upper \& lower boundary arrows all outward,} \\ & \quad \text{left \& right boundary arrows all inward} \end{array} \right\}$

- e.g. 6VDW(3) = {  }

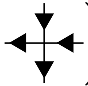
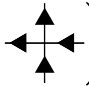
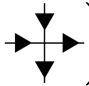
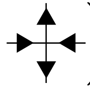
- e.g.  $\in$ 6VDW(6)

Six-Vertex Model with DWBC Statistics

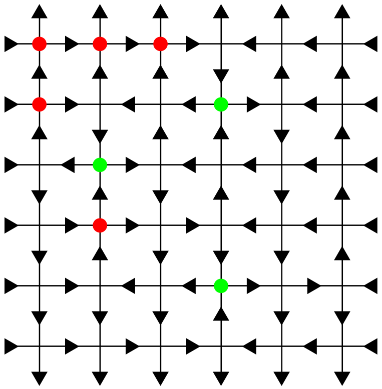
For $C \in 6\text{VDW}(n)$

- $\nu(C) :=$ # of  vertex configurations in C
- $\mu(C) :=$ # of  vertex configurations in C

- numbers of other 4 vertex configuration types in C satisfy

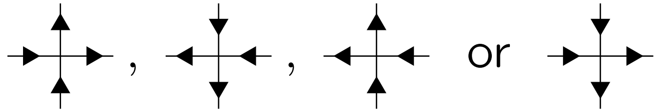
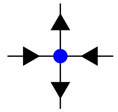
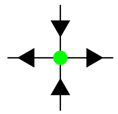
(# ) $= \nu(C)$, (# ) $=$ (# ) $= \frac{n(n-1)}{2} - \nu(C) - \mu(C)$, (# ) $= \mu(C) + n$

- e.g.



$\nu(C) = 5, \quad \mu(C) = 3$

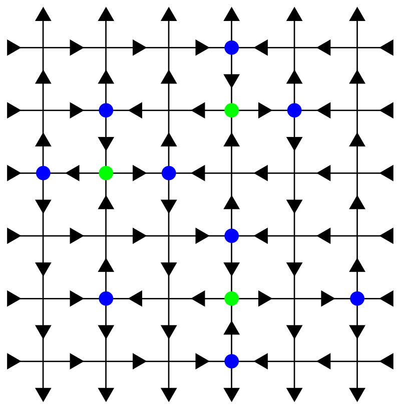
ASM(n) − 6VDW(n) **Bijection**

ASM		six-vertex model with DWBC
0	$\longleftrightarrow$	
1	$\longleftrightarrow$	
-1	$\longleftrightarrow$	

- Gives bijection between $\{A \in \text{ASM}(n) \mid \nu(A) = p,\ \mu(A) = m\}$ & $\{C \in \text{6VDW}(n) \mid \nu(C) = p,\ \mu(C) = m\}$

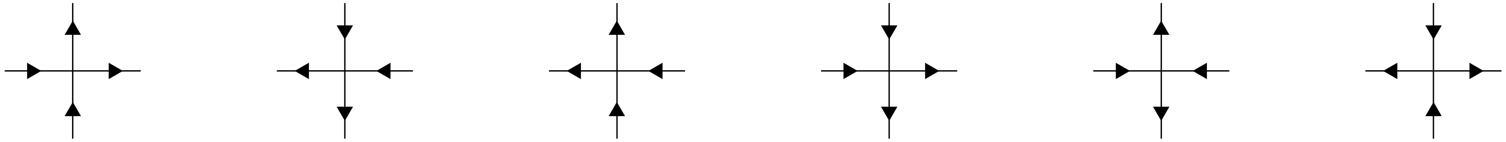
- e.g.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \longleftrightarrow$$



Izergin–Korepin Formula

- Integrable vertex weights:



$a(u,v) \qquad a(u,v) \qquad b(u,v) \qquad b(u,v) \qquad c(u,v) \qquad c(u,v)$

$$a(u,v) := uq - \tfrac{v}{q} \qquad b(u,v) := \tfrac{u}{q} - vq \qquad c(u,v) := (q^2 - \tfrac{1}{q^2})\sqrt{uv}$$

u: row 'spectral parameter', v: column 'spectral' parameter, q: 'crossing' parameter

- Yang-Baxter equation satisfied

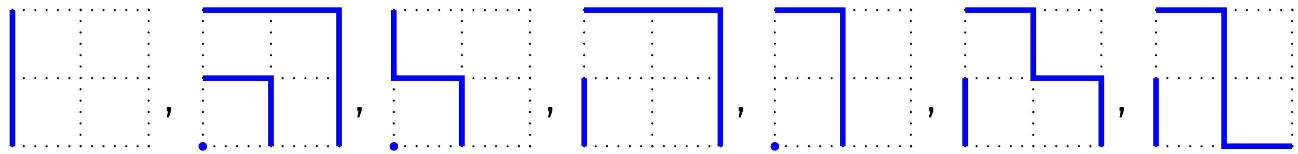
- Izergin–Korepin formula for partition function of six-vertex model with DWBC:

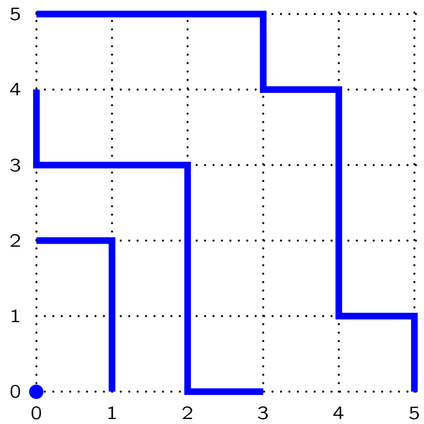
$$Z_n^{\mathsf{6V}}(u_1,\dots,u_n;v_1,\dots,v_n) := \sum_{C \in \mathsf{6VDW}(n)} \prod_{i,j=1}^{n} \begin{pmatrix} \text{weight at vertex } (i,j) \text{ with} \\ \text{parameters } u_i,\ v_j \text{ for config'n } C \end{pmatrix}$$

$$= \frac{\prod_{i,j=1}^{n} a(u_i,v_j)\, b(u_i,v_j)}{\prod_{1\le i<j\le n}(u_i-u_j)(v_j-v_i)} \det_{1\le i,j\le n}\left(\frac{c(u_i,v_i)}{a(u_i,v_j)\, b(u_i,v_j)}\right) \qquad \textit{(Izergin 1987)}$$

Nonintersecting Lattice Paths

$\mathrm{NILP}(n) := \Big\{$ sets P of nonintersecting paths on $n \times n$ grid $\Big|$ P consists of paths from $(0, \lambda_{i-1} - 1)$ to $(\lambda_i, 0)$ for each $i = 1, \ldots, t+1$, with each step rightward or downward, for some $0 \leq t \leq n-1$ & $n = \lambda_0 > \lambda_1 > \ldots > \lambda_t > \lambda_{t+1} = 0$ $\Big\}$

- e.g. $\mathrm{NILP}(3) = \Big\{$  $\Big\}$

- e.g. 

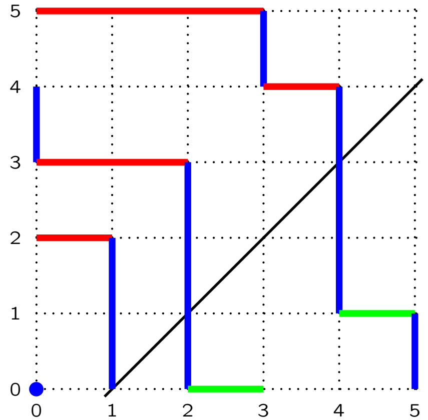
$\in \mathrm{NILP}(6)$

Nonintersecting Lattice Path Statistics

For $P \in \mathrm{NILP}(n)$

- $\nu(P) := \#$ of rightward steps in P above subdiagonal line
- $\mu(P) := \#$ of rightward steps in P below subdiagonal line

- e.g.



$$\nu(P) = 7, \quad \mu(P) = 2$$

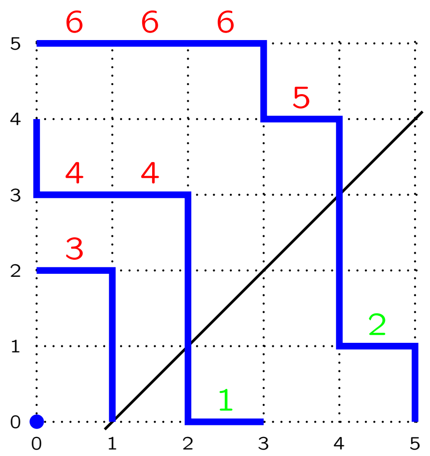
DPP(n) − NILP(n) **Bijection**

DPP		nonintersecting path set
$D_{ij} - 1$	$\longleftrightarrow$	height of $(j - i + 1)$th rightward step of ith path from top

- Gives bijection between $\{D \in \text{DPP}(n) \mid \nu(D) = p,\ \mu(D) = m\}$ & $\{P \in \text{NILP}(n) \mid \nu(P) = p,\ \mu(P) = m\}$

- e.g.

$$\begin{array}{ccccc} 6 & 6 & 6 & 5 & 2 \\ & 4 & 4 & 1 & \\ & & 3 & & \end{array} \quad \longleftrightarrow$$



Lindström–Gessel–Viennot Theorem

- Consider an acyclic directed graph G
- Assign weight $w(e)$ to each edge e of G
- Let $W(p) := \prod_{\text{edges } e \text{ of } p} w(e)$ for any path p on G
- Let $\mathcal{P}_{u,v} := \{\text{paths on } G \text{ from vertex } u \text{ to vertex } v\}$
- Let $\mathcal{N}_G(u_1,\dots,u_m;v_1,\dots,v_m) :=$

$$\left\{ \text{sets } P \text{ of paths on } G \;\middle|\; \begin{array}{l} \bullet\ P \text{ consists of path of } \mathcal{P}_{u_i,v_i} \text{ for each } i=1,\dots,m \\ \bullet\ \text{different paths of } P \text{ do not intersect} \end{array} \right\}$$

- Assume $\mathcal{N}_G(u_{\sigma_1},\dots,u_{\sigma_m};v_1,\dots,v_m)=\emptyset$ for each nonidentity permutation σ of $\{1,\dots,m\}$
- Then $$\sum_{P\in\mathcal{N}_G(u_1,\dots,u_m;v_1,\dots,v_m)} \prod_{p\in P} W(p) = \det_{1\le i,j\le m}\left(\sum_{p\in\mathcal{P}_{u_i,v_j}} W(p)\right)$$

(Lindström 1973; Gessel, Viennot 1989)

Unrefined Enumeration Result: Further Details

Theorem $Z_n^{\mathsf{ASM}}(x,y) = Z_n^{\mathsf{DPP}}(x,y) = \det_{0\le i,j\le n-1} K_n(x,y)_{ij}$

where $K_n(x,y)_{ij} = -\delta_{i,j+1} + \sum_{k=0}^{\min(i,j+1)} \binom{i-1}{i-k}\binom{j+1}{k} x^k y^{i-k}$

Proof outline

1. (a) Use bijection between ASM(n) & 6VDW(n)

   (b) Transform Izergin–Korepin determinant formula & then use row spectral parameters $u_1 = \ldots = u_n = r$, column spectral parameters $v_1 = \ldots = v_n = \frac{1}{r}$

   (c) Parameterize x, y in terms of q, r as $x = \left(\frac{a(r,\frac{1}{r})}{b(r,\frac{1}{r})}\right)^2$, $y = \left(\frac{c(r,\frac{1}{r})}{b(r,\frac{1}{r})}\right)^2$

2. (a) Use bijection between DPP(n) & NILP(n)

   (b) Apply Lindström–Gessel–Viennot theorem to $n \times n$ directed grid with horizontal edge weights x above subdiagonal line, horizontal edge weights y below subdiagonal line, vertical edge weights 1 & certain fixed endpoints of paths

   (c) Sum over possible endpoints of paths to give $Z_n^{\mathsf{DPP}}(x,y) = \det_{0\le i,j\le n-1} K_n(x,y)_{ij}$

3. Apply determinant transformations to give $Z_n^{\mathsf{ASM}}(x,y) = Z_n^{\mathsf{DPP}}(x,y)$

Singly-Refined Enumeration (1 Boundary Statistic)

- Define singly-refined generating functions $Z_n^{\mathsf{ASM}}(x,y;z) := \sum_{A\in\mathsf{ASM}(n)} x^{\nu(A)}\, y^{\mu(A)}\, z^{\rho_{\mathsf{T}}(A)}$

  & $Z_n^{\mathsf{DPP}}(x,y;z) := \sum_{D\in\mathsf{DPP}(n)} x^{\nu(D)}\, y^{\mu(D)}\, z^{\rho_{\mathsf{T}}(D)}$

Theorem $Z_n^{\mathsf{ASM}}(x,y;z) = Z_n^{\mathsf{DPP}}(x,y;z) = \det_{0\le i,j\le n-1} K_n(x,y;z)_{ij}$

where $K_n(x,y;z)_{ij} = -\delta_{i,j+1} + \begin{cases} \sum_{k=0}^{\min(i,j+1)} \binom{i-1}{i-k}\binom{j+1}{k} x^k y^{i-k}, & j \le n-2 \\ \sum_{k=0}^{i}\sum_{l=0}^{k} \binom{i-1}{i-k}\binom{n-l-1}{k-l} x^k y^{i-k} z^l, & j = n-1 \end{cases}$

First equality means (# of $n\times n$ ASMs with p 'inversions', m -1's & k 0's left of the 1 in top row) $=$ (# of order n DPPs with p nonspecial parts, m special parts & k n's)

- Conjectured *Mills, Robbins, Rumsey 1983*
- Several special cases previously-known, e.g. $x = y = 1$

Proof outline

- Similar to proof of unrefined case, but now use transformed Izergin–Korepin formula with row spectral parameters $u_1 = t$, $u_2 = \ldots = u_n = r$ & Lindström–Gessel–Viennot theorem with horizontal edge weights xz in top row of grid

Doubly-Refined Enumeration (2 Boundary Statistics)

- Define doubly-refined generating functions

$$Z_n^{\mathsf{ASM}}(x,y;z_1,z_2) \;:=\; \sum_{A\in\mathsf{ASM}(n)} x^{\nu(A)}\, y^{\mu(A)}\, z_1^{\rho_{\mathsf{T}}(A)}\, z_2^{\rho_{\mathsf{B}}(A)}$$

$$\&\quad Z_n^{\mathsf{DPP}}(x,y;z_1,z_2) \;:=\; \sum_{D\in\mathsf{DPP}(n)} x^{\nu(D)}\, y^{\mu(D)}\, z_1^{\rho_{\mathsf{T}}(D)}\, z_2^{\rho_{\mathsf{B}}(D)}$$

Theorem $Z_n^{\mathsf{ASM}}(x,y;z_1,z_2) = Z_n^{\mathsf{DPP}}(x,y;z_1,z_2) = \det\limits_{0\le i,j\le n-1} K_n(x,y;z_1,z_2)_{ij}$

where $K_n(x,y;z_1,z_2)_{ij} = -\delta_{i,j+1} + \begin{cases} \sum_{k=0}^{\min(i,j+1)} \binom{i-1}{i-k}\binom{j+1}{k} x^k y^{i-k}, & j\le n-3 \\ \sum_{k=0}^{i}\sum_{l=0}^{k} \binom{i-1}{i-k}\binom{n-l-2}{k-l} x^k y^{i-k} z_2^{l+1}, & j=n-2 \\ \sum_{k=0}^{i}\sum_{l=0}^{k}\sum_{m=0}^{l} \binom{i-1}{i-k}\binom{n-l-2}{k-l} x^k y^{i-k} z_1^m z_2^{l-m}, & j=n-1 \end{cases}$

- Statistic ρ_{B} for DPPs not previously studied

Proof Outline for Doubly-Refined Case

- Apply ASM(n)– 6VDW(n) & DPP(n)–NILP(n) bijections as previously

- Now consider row spectral parameters $u_1 = t_1$, $u_2 = \ldots = u_{n-1} = r$, $u_n = t_2$ for six-vertex model with DWBC & consider horizontal edge weights xz_1 in top row, xz_2 in second-top row of grid for nonintersecting lattice paths

- A simple case of Bazin determinant identity is
$$\det N^{k_1,k_2} \det N^{k_3,k_4} - \det N^{k_1,k_3} \det N^{k_2,k_4} + \det N^{k_1,k_4} \det N^{k_2,k_3} = 0$$
for any $(n+2) \times n$ matrix N, where $N^{k,k'}$ denotes N with rows k and k' omitted

- Use Bazin identity with Izergin–Korepin determinant & summed Lindström–Gessel–Viennot determinant $\det\limits_{0 \le i,j \le n-1} K_n(x,y;z_1,z_2)_{ij}$ to show that $Z_n^{\mathsf{ASM}}(x,y;z_1,z_2)$ & $Z_n^{\mathsf{DPP}}(x,y;z_1,z_2)$ both satisfy
$$(z_1-z_2)\, Z_n(x,y;z_1,z_2)\, Z_{n-1}(x,y) = (z_1-1)z_2\, Z_n(x,y;z_1)\, Z_{n-1}(x,y;z_2) - z_1(z_2-1)\, Z_{n-1}(x,y;z_1)\, Z_n(x,y;z_2)$$

- $Z_n^{\mathsf{ASM}}(x,y;z_1,z_2) = Z_n^{\mathsf{DPP}}(x,y;z_1,z_2)$ then follows from $Z_n^{\mathsf{ASM}}(x,y;z) = Z_n^{\mathsf{DPP}}(x,y;z)$

Quadruply-Refined ASM Enumeration

- Define adjacent-boundary doubly-refined & quadruply-refined ASM generating functions

$$Z_n^{\mathrm{adj}}(x,y;z_1,z_2) := \sum_{A\in\mathrm{ASM}(n)} x^{\nu(A)}\, y^{\mu(A)}\, z_1^{\rho_{\mathrm{T}}(A)}\, z_2^{\rho_{\mathrm{L}}(A)}$$

$$\widetilde{Z}_n^{\mathrm{adj}}(x,y;z_1,z_2) := \sum_{A\in\mathrm{ASM}(n)} x^{\nu(A)}\, y^{\mu(A)}\, z_1^{\rho_{\mathrm{T}}(A)}\, z_2^{\rho_{\mathrm{R}}(A)} = x^{\frac{n(n-1)}{2}}\,(z_1z_2)^{n-1}\, Z_n^{\mathrm{adj}}(\tfrac{1}{x},\tfrac{y}{x};\tfrac{1}{z_1},\tfrac{1}{z_2})$$

& $$Z_n^{\mathrm{ASM}}(x,y;z_{\mathrm{T}},z_{\mathrm{R}},z_{\mathrm{B}},z_{\mathrm{L}}) := \sum_{A\in\mathrm{ASM}(n)} x^{\nu(A)}\, y^{\mu(A)}\, z_{\mathrm{T}}^{\rho_{\mathrm{T}}(A)}\, z_{\mathrm{R}}^{\rho_{\mathrm{R}}(A)}\, z_{\mathrm{B}}^{\rho_{\mathrm{B}}(A)}\, z_{\mathrm{L}}^{\rho_{\mathrm{L}}(A)}$$

Theorem $y(z_{\mathrm{T}}-z_{\mathrm{B}})(z_{\mathrm{L}}-z_{\mathrm{R}})\, Z_n^{\mathrm{ASM}}(x,y;z_{\mathrm{T}},z_{\mathrm{R}},z_{\mathrm{B}},z_{\mathrm{L}})\, Z_{n-2}^{\mathrm{ASM}}(x,y) =$

$$((z_{\mathrm{T}}-1)(z_{\mathrm{R}}-1)-yz_{\mathrm{T}}z_{\mathrm{R}})((z_{\mathrm{B}}-1)(z_{\mathrm{L}}-1)-yz_{\mathrm{B}}z_{\mathrm{L}})\, Z_{n-1}^{\mathrm{adj}}(x,y;z_{\mathrm{T}},z_{\mathrm{L}})\, Z_{n-1}^{\mathrm{adj}}(x,y;z_{\mathrm{B}},z_{\mathrm{R}}) -$$
$$z_{\mathrm{T}}\, z_{\mathrm{R}}\, z_{\mathrm{B}}\, z_{\mathrm{L}}(x(z_{\mathrm{T}}-1)(z_{\mathrm{L}}-1)-y)(x(z_{\mathrm{B}}-1)(z_{\mathrm{R}}-1)-y)\, \widetilde{Z}_{n-1}^{\mathrm{adj}}(x,y;z_{\mathrm{T}},z_{\mathrm{R}})\, \widetilde{Z}_{n-1}^{\mathrm{adj}}(x,y;z_{\mathrm{B}},z_{\mathrm{L}}) -$$
$$(z_{\mathrm{B}}-1)(z_{\mathrm{R}}-1)((z_{\mathrm{T}}-1)(z_{\mathrm{L}}-1)-yz_{\mathrm{T}}z_{\mathrm{L}})\, Z_{n-1}^{\mathrm{adj}}(x,y;z_{\mathrm{T}},z_{\mathrm{L}})\, Z_{n-2}^{\mathrm{ASM}}(x,y) -$$
$$(z_{\mathrm{T}}-1)(z_{\mathrm{L}}-1)((z_{\mathrm{B}}-1)(z_{\mathrm{R}}-1)-yz_{\mathrm{B}}z_{\mathrm{R}})\, Z_{n-1}^{\mathrm{adj}}(x,y;z_{\mathrm{B}},z_{\mathrm{R}})\, Z_{n-2}^{\mathrm{ASM}}(x,y) +$$
$$z_{\mathrm{T}}\, z_{\mathrm{R}}\, (xz_{\mathrm{B}}z_{\mathrm{L}})^{n-1}(z_{\mathrm{B}}-1)(z_{\mathrm{L}}-1)(x(z_{\mathrm{T}}-1)(z_{\mathrm{R}}-1)-y)\, \widetilde{Z}_{n-1}^{\mathrm{adj}}(x,y;z_{\mathrm{T}},z_{\mathrm{R}})\, Z_{n-2}^{\mathrm{ASM}}(x,y) +$$
$$z_{\mathrm{B}}\, z_{\mathrm{L}}\, (xz_{\mathrm{T}}z_{\mathrm{R}})^{n-1}(z_{\mathrm{T}}-1)(z_{\mathrm{R}}-1)(x(z_{\mathrm{B}}-1)(z_{\mathrm{L}}-1)-y)\, \widetilde{Z}_{n-1}^{\mathrm{adj}}(x,y;z_{\mathrm{B}},z_{\mathrm{L}})\, Z_{n-2}^{\mathrm{ASM}}(x,y) -$$
$$(z_{\mathrm{T}}-1)(z_{\mathrm{R}}-1)(z_{\mathrm{B}}-1)(z_{\mathrm{L}}-1)((x^2z_{\mathrm{T}}z_{\mathrm{R}}z_{\mathrm{B}}z_{\mathrm{L}})^{n-1}-1)\, Z_{n-2}^{\mathrm{ASM}}(x,y)^2$$

- No DPP statistics currently known which, together with ρ_{T} & ρ_{B} for DPPs, have the same joint distribution as ρ_{T}, ρ_{R}, ρ_{B} & ρ_{L} for ASMs

Proof Outline for Quadruply-Refined ASM Relation

- Apply ASM(n)$-$ 6VDW(n) bijection as previously
- Now consider row spectral parameters $u_1 = t_{\mathsf{T}}$, $u_2 = \ldots = u_{n-1} = r$, $u_n = t_{\mathsf{B}}$ & column spectral parameters $v_1 = t_{\mathsf{L}}$, $v_2 = \ldots = v_{n-1} = s$, $v_n = t_{\mathsf{R}}$ for six-vertex model with DWBC
- Desnanot–Jacobi determinant identity is

$$M_{\mathsf{C}} \det M = M_{\mathsf{TL}}\, M_{\mathsf{BR}} - M_{\mathsf{TR}}\, M_{\mathsf{BL}}$$

  for any $n \times n$ matrix M, where M_{TL}, M_{TR}, M_{BR} & M_{BL} are the $(n-1)\times(n-1)$ connected minors in top-left, top-right, bottom-right & bottom-left corners, & M_{C} is the central $(n-2)\times(n-2)$ minor
- Using Desnanot–Jacobi identity with Izergin–Korepin determinant, applying properties of ASMs with 1's in corners & parameterizing x, y, z_{T}, z_{R}, z_{B}, z_{L} in terms of q, r, s, t_{T}, t_{R}, t_{B}, t_{L} gives quadruply-refined ASM relation

Explicit Quadruply-Refined ASM Enumeration

- Setting $z_{\mathsf{R}} = z_{\mathsf{B}} = 1$ in quadruply-refined ASM identity & solving a recursion relation gives

$$Z_n^{\mathsf{adj}}(x, y; z_1, z_2) = Z_{n-1}^{\mathsf{ASM}}(x, y) \Biggl(1 + \sum_{k=1}^{n-1} \Bigl(\tfrac{y\, z_1\, z_2}{(z_1-1)(z_2-1)}\Bigr)^{n-k} \Bigl(1 + \tfrac{(x(z_1-1)(z_2-1)-y)\, Z_k^{\mathsf{ASM}}(x,y;z_1)\, Z_k^{\mathsf{ASM}}(x,y;z_2)}{y\, Z_{k-1}^{\mathsf{ASM}}(x,y)\, Z_k^{\mathsf{ASM}}(x,y)}\Bigr)\Biggr)$$

- Therefore, adjacent-boundary doubly-refined ASM generating function can be expressed in terms of singly-refined and unrefined ASM generating functions

- So, using quadruply-refined ASM identity & previous determinant formulae, quadruply-refined ASM generating function can be computed explicitly

- Very recently, a different formula for quadruply-refined ASM generating function with $x = y = 1$ was obtained independently by *Ayyer & Romik*, *arXiv: 1202.3651*

- Quadruply-refined & other ASM enumeration results can also be expressed in terms of boundary correlation functions for six-vertex model with DWBC

Possible Further Work

- Obtain some similar results involving vertically symmetric ASMs & DPPs invariant under a certain known symmetry operation

- Find DPP properties corresponding to certain natural ASM properties
  e.g. DPP statistics corresponding to ASM statistics ρ_{L} & ρ_{R}?
  DPP operations corresponding to $\pi/2$-rotation or transposition of ASM?

- Find ASM properties corresponding to certain natural DPP properties
  e.g. ASM statistic corresponding to # of rows of DPP?
  ASM statistic corresponding to sum of parts of DPP?

- Study multiply-refined ASM & DPP enumeration with same bulk statistics as here, but slightly different boundary statistics, e.g. associated with several adjacent rows near ASM boundary

- Investigate connection between quadruply-refined ASM identity & result of Ayyer–Romik