Dual Families in Enveloping Algebras

Matthieu Deneufchâtel, G. H. E. Duchamp et H. N. Minh

Laboratoire d'Informatique de Paris Nord,
Université Paris 13

SLC 68, Ottrott

Plan

Context :

- Duality in Lie and enveloping algebras ;
- Numerical experimentations in the free algebra.

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- Discover the properties of the dual family of a basis from the properties of the basis ;
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Applications of Schützenberger's factorization :

- Polysystems and non linear differential equations (factorization of transport operators) ;
- Polyzetas and renormalization of divergent polyzetas.

Plan

- I a totally ordered set for $<$.
- $\mathfrak{A}$ an algebra with unit $1_{\mathfrak{A}}$.
- If $Y = (y_i)_{i \in I}$ is a totally ordered family in $\mathfrak{A}$ and $\alpha \in \mathbb{N}^{(I)}$,

$$Y^\alpha = y_{i_1}^{\alpha_{i_1}} y_{i_2}^{\alpha_{i_2}} \cdots y_{i_k}^{\alpha_{i_k}}$$

$\forall\, J = \{i_1, i_2 \cdots i_k\},\ i_1 > i_2 > \cdots > i_k$ such that $\mathrm{supp}(\alpha) \subset J$.

- If $(e_i)_{i \in I}$ denotes the canonical basis of $\mathbb{N}^{(I)}$ $(e_i(j) = \delta_{ij})$, one has : $Y^{e_i} = y_i$.
- Typically : $I \to \mathrm{Lyn}(X), < \to <_{\text{lex}}$.

Let $\mathfrak{g}$ a k-Lie algebra and $B = (b_i)_{i \in I}$ an ordered basis (for a total order $<$ on I) of $\mathfrak{g}$.

Poincaré-Birkhoff-Witt Basis

The elements

$$B^\alpha,\ \alpha = (\alpha_{i_1}, \ldots, \alpha_{i_p}) \text{ with } i_1 > \cdots > i_p,$$

form a basis of $\mathscr{U}(\mathfrak{g})$ (called PBW basis).

$(B_\alpha)_{\alpha \in \mathbb{N}^{(I)}}$ a basis, $\langle \cdot | \cdot \rangle$ a scalar product.

Dual family $S^{[\beta]}$: $\langle S^{[\beta]} | B_\alpha \rangle = \delta_{\alpha\beta}$ (does not necessarily exist).

Notation :

- $S^{[e_i]}$: atomic elements.
- S^α defined by $S^\alpha = \prod_{i \in \mathrm{supp}(\alpha)} S^{[e_i]}$.
- $S^{[e_i]} = S^{e_i}$.
- $\left(S^{[\beta]}\right)_{\beta \in \mathbb{N}^{(I)}}$ is of Poincaré-Birkhoff-Witt type iff

$$S^{[\beta]} = S^\beta, \ \forall \beta \in \mathbb{N}^{(I)}.$$

Plan

Theorem : [see Mélançon-Reutenauer]

Let

- k be a field of characteristic zero,
- $\mathfrak{g}$ a k-Lie algebra,
- $B = (b_i)_{i \in I}$ be an ordered basis of $\mathfrak{g}$,
- $(B^\alpha)_{\alpha \in \mathbb{N}^{(I)}}$ be the associated PBW basis.

Denoting $(S_\alpha)_{\alpha \in \mathbb{N}^{(I)}}$ the dual family of $(B^\alpha)_{\alpha \in \mathbb{N}^{(I)}}$ in $\mathscr{U}^*$, one gets the following relation :

$$\sum_{\alpha \in \mathbb{N}^{(I)}} S_\alpha \otimes B^\alpha = \prod_{i \in I}^{\rightarrow} \exp\left(S_{e_i} \otimes B^{e_i}\right).$$

N.B. : The product in the *r.h.s.* is $\star \otimes \mu_{\mathscr{U}}$.

- Partially commutative case - $k\langle X, \theta \rangle$:

$$\sum_{w \in M(X,\theta)} w \otimes w = \prod_{\ell \in \mathrm{Lyn}(X,\theta)}^{\searrow} \exp^{S_\ell \otimes P_\ell} .$$

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- Stuffle algebra : $Y=\{y_i\}_{i\geq 1}$, $|y_s|=s$.

$$u \uplus 1 = 1 \uplus u = u \ ;$$
$$y_i u \uplus y_j v = y_i(u \uplus y_j v) + y_j(y_i u \uplus v) + y_{i+j}(u \uplus v).$$

Let :

- $(B,<)$ be any totally ordered (homogeneous) basis of $\mathrm{Prim}(k\langle Y\rangle)$;
- $(S_\alpha)_{\alpha\in\mathbb{N}^{(B)}}$ the dual basis of $(B^\alpha)_{\alpha\in\mathbb{N}^{(B)}}$.

$$\prod_{b\in B}^{\searrow} \exp(S_b\otimes b) = \sum_{w\in Y^*} w\otimes w \ .$$

Property of the dual family $(S_\alpha)_{\alpha\in\mathbb{N}^{(I)}}$:

$$\begin{aligned} S_\alpha \star S_\beta &= \sum_{\gamma\in\mathbb{N}^{(I)}} \langle S_\alpha \star S_\beta | B^\gamma \rangle S^\gamma \\ &= \sum_{\gamma\in\mathbb{N}^{(I)}} \langle S_\alpha \otimes S_\beta | \Delta(B^\gamma) \rangle^{\otimes 2} S^\gamma \\ &= \sum_{\gamma\in\mathbb{N}^{(I)}} \langle S_\alpha \otimes S_\beta | \sum_{\gamma_1+\gamma_2=\gamma} \frac{\gamma!}{\gamma_1!\,\gamma_2!} B^{\gamma_1} \otimes B^{\gamma_2} \rangle^{\otimes 2} S^\gamma \\ &= \frac{(\alpha+\beta)!}{\alpha!\,\beta!} S_{\alpha+\beta}. \end{aligned}$$

Hence the elements $T_\alpha = \alpha! S_\alpha$ form a multiplicative family.

Let us consider the converse problem :

Let $(T_\alpha)_{\alpha\in\mathbb{N}^{(I)}}$ be a multiplicative family in $\mathscr{U}(\mathfrak{g})^*$:

$$T_\alpha \star T_\beta = T_{\alpha+\beta},\ \alpha,\beta \in \mathbb{N}^{(I)}$$

for the convolution $\star$.

Assume that there exist a dual family $(B^{[\alpha]})_{\alpha\in\mathbb{N}^{(I)}}$: $\langle S_\alpha | B^{[\beta]}\rangle = \delta_{\alpha\beta}$.

Theorem

The atomic elements $B^{[e_i]}, i \in \mathbb{N}$, form a basis of $\mathfrak{g}$.

Question : $B^\alpha = \prod_{i \in \mathrm{supp}(\alpha)} B^{[e_i]} \stackrel{?}{=} B^{[\alpha]}$

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$$\prod_{i\in I}^{\rightarrow} \exp\left(S_{e_i} \otimes B^{e_i}\right) = \sum_{k\geq 0} \sum_{\substack{i_1\geq\cdots\geq i_k \\ \alpha_1,\dots,\alpha_k}} \frac{(S_{e_{i_1}} \otimes B^{e_{i_1}})^{\alpha_1} \dots (S_{e_{i_k}} \otimes B^{e_{i_k}})^{\alpha_k}}{\alpha_1! \dots \alpha_k!}$$

$$= \sum_{k\geq 0} \sum_{\substack{i_1\geq\cdots\geq i_k \\ \alpha_1,\dots,\alpha_k}} \frac{S_{e_{i_1}}^{\alpha_1} \star \cdots \star S_{e_{i_k}}^{\alpha_k} \otimes (B^{e_{i_1}})^{\alpha_1} \dots (B^{e_{i_k}})^{\alpha_k}}{\alpha_1! \dots \alpha_k!}$$

$$= \sum_{k\geq 0} \sum_{\substack{i_1\geq\cdots\geq i_k \\ \alpha_1,\dots,\alpha_k}} \frac{S_{e_{i_1}}^{\alpha_1} \star \cdots \star S_{e_{i_k}}^{\alpha_k}}{\alpha_1! \dots \alpha_k!} \otimes B^\alpha$$

Summary :

Multiplicative family $(T_\alpha)_{\alpha\in\mathbb{N}^{(I)}} \to (S_\alpha)_{\alpha\in\mathbb{N}^{(I)}} \overset{\text{dualization}}{\to} \left(B^{[\alpha]}\right)_{\alpha\in\mathbb{N}^{(I)}}$.

$$Id_{\mathscr{U}} = \sum_{\alpha\in\mathbb{N}^{(I)}} S_\alpha \otimes B^{[\alpha]} \overset{?}{=} \sum_{\alpha\in\mathbb{N}^{(I)}} S_\alpha \otimes B^{\alpha} = \prod_{i\in I}^{\rightarrow} \exp\left(S_{e_i} \otimes B^{e_i}\right)$$

- Product for the *r.h.s.* : $\star \otimes$ conc ;
- One can try to write this factorization as soon as one has two bases in duality ;
- Problem : $\displaystyle\prod_{i\in\mathrm{supp}(\alpha)} (B^{e_i})^{\alpha_i} = B^{[\alpha]}$?
- $\phi : V^* \otimes V \to \mathrm{End}^{\text{finite}}(V) : \phi(f \otimes v) : b \mapsto f(b) \cdot v$.

Plan

Some convergence remarks for physicists

Let $I, \leq$ be an ordered set.

I is a directed set if for all pairs $\{i_1, i_2\}$

$$\text{there exists } i_3 \in I \quad s.t. \quad i_1 \leq i_3,\ i_2 \leq i_3.$$

If X is a topological space, a map $i : I \to X$ is called a generalized sequence or a net.

Notation : $(x_i)_{i \in I}$.

Let $(y_i)_{i\in I}$ be a net in X and $x \in X$.

$(y_i)_{i\in I}$ converges towards x in X iff

$$\forall U \text{ neighborhood of } x,\ \exists N \in I,\ \forall i > N,\ y_i \in U.$$

Application :

- V be a vector space (discrete topology) ;
- $(f_i)_{i\in I}$ a net in $\mathrm{End}(V)$ (pointwise convergence) ;
- $g \in \mathrm{End}(V)$.

Then $(f_i)_{i\in I}$ converges towards g iff

$$\forall v \in V,\ \exists N \in I,\ \forall i \geq N,\ f_i(v) = g(v).$$

Context : $\mathfrak{g}$: a Lie algebra $\to \mathscr{U}(\mathfrak{g})$;
$\mathrm{End}_k(\mathscr{U}(\mathfrak{g}))$ with pointwise convergence

Summable family : Directed set : $(\mathfrak{P}^{\mathrm{finite}}(\mathbb{N}^{(I)}), \subset)$.

Let $f = (f_i)_{i \in I}$, $f_i \in \mathrm{End}_k(\mathscr{U}(\mathfrak{g}))$ and $g \in \mathrm{End}_k(\mathscr{U}(\mathfrak{g}))$.

The family f is summable iff

$$\left(S_F = \sum_{j \in F} f_j \right)_{F \subset_{\mathrm{finite}} I} \quad \text{converges towards } g \in \mathrm{End}_k(\mathscr{U}(\mathfrak{g}))$$

Context : $\mathfrak{g}$: a Lie algebra $\rightarrow \mathscr{U}(\mathfrak{g})$;
$\mathrm{End}_k(\mathscr{U}(\mathfrak{g}))$ with pointwise convergence

Multipliable family : We assume that I is totally ordered.

Directed set : $(\mathfrak{P}^{\mathrm{finite}}(I), \subset)$.

Let $f = (f_i)_{i \in I}$, $f_i \in \mathrm{End}_k(\mathscr{U}(\mathfrak{g}))$ and $g \in \mathrm{End}_k(\mathscr{U}(\mathfrak{g}))$.

The family f is called multipliable iff

$$\left(M_F = \prod_{j \in F}^{\rightarrow} f_j \right)_{F \subset_{\mathrm{finite}} I} \quad \text{converges towards } g \in \mathrm{End}_k(\mathscr{U}(\mathfrak{g})).$$

Plan

Let $X = \{x_1, \ldots, x_n\}$.

$(k\langle\langle X\rangle\rangle)^* \sim k\langle X\rangle$ for the scalar product :

$$\langle S|P\rangle = \sum_{w\in X^*} \langle S|w\rangle\langle P|w\rangle.$$

$k\langle X\rangle = \mathscr{U}\,(\mathrm{Lie}_k(X))$.

A basis of $\mathrm{Lie}_k(X)$ is given by standard bracketing : $\forall \ell \in \mathrm{Lyn}(X)$

$$P_\ell = [P_{\ell_1}, P_{\ell_2}] \text{ if } \sigma(\ell) = (\ell_1, \ell_2).$$

PBW Basis of $k\langle X\rangle$:

$$P_w = P_{\ell_{i_1}}^{\alpha_1} \dots P_{\ell_{i_k}}^{\alpha_k} \text{ if } w = \ell_{i_1}^{\alpha_1} \dots \ell_{i_k}^{\alpha_k},\ \ell_{i_1} > \dots > \ell_{i_k}.$$

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S Basis :

$$S_w = \begin{cases} w & \text{if } |w| = 1; \\ xS_u & \text{if } w = xu \text{ and } w \in \mathrm{Lyn}(X); \\ \dfrac{S_{\ell_{i_1}}^{ш\alpha_1} ш \dots ш S_{\ell_{i_k}}^{ш\alpha_k}}{\alpha_1! \dots \alpha_k!} & \text{if } w = \ell_{i_1}^{\alpha_1} \dots \ell_{i_k}^{\alpha_k},\ \ell_{i_1} > \dots > \ell_{i_k}. \end{cases}$$

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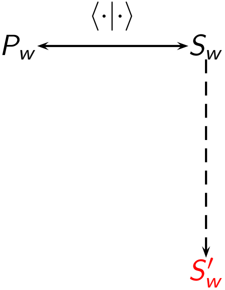
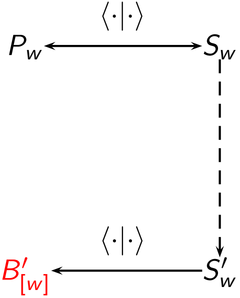
One can show that $(S_w)_w$ and $(P_w)_w$ are in duality :

$$\langle S_u | P_v \rangle = \delta_{uv}.$$

$$P_w \xleftrightarrow{\langle \cdot | \cdot \rangle} S_w$$



$$S'_w = \begin{cases} \ell & \text{if } \ell \in \mathrm{Lyn}(X); \\ \dfrac{{S'}_{\ell_{i_1}}^{ш\alpha_1} ш \ldots ш {S'}_{\ell_{i_k}}^{ш\alpha_k}}{\alpha_1! \ldots \alpha_k!} & \text{if } w = \ell_{i_1}^{\alpha_1} \ldots \ell_{i_k}^{\alpha_k} \ . \end{cases}$$



$$S'_w = \begin{cases} \ell & \text{if } \ell \in \mathrm{Lyn}(X); \\ \dfrac{S'^{\,ш\alpha_1}_{\ell_{i_1}} ш \ldots ш S'^{\,ш\alpha_k}_{\ell_{i_k}}}{\alpha_1! \ldots \alpha_k!} & \text{if } w = \ell_{i_1}^{\alpha_1} \ldots \ell_{i_k}^{\alpha_k}\ . \end{cases}$$

Properties of the family $\left(B'_{[w]}\right)_{w \in X^*}$?

- Triangularity ;
- PBW type ($B'_{[w]} \overset{?}{=} B'_w = B'^{\alpha_1}_{[\ell_1]} \ldots B'^{\alpha_n}_{[\ell_n]}$)...

Theorem 1

Let P belong to $k\langle X\rangle$ and $\ell \in \mathrm{Lyn}(X)$. Then

$$P = B'_\ell \iff \begin{cases} P = \ell + \sum_{\ell < u} \langle P|u\rangle u; \\ P \text{ is primitive}; \\ \forall \ell_1 \in \mathrm{Lyn}(X), \langle P|\ell_1\rangle = \delta_{\ell\ell_1}. \end{cases}$$

Reformulation

Let P belong to $k\langle X\rangle$ and $\ell \in \mathrm{Lyn}(X)$. Then

$$P = B'_\ell \Longleftrightarrow \begin{cases} P \text{ is primitive}\,; \\ |\mathrm{supp}(P) \cap \mathrm{Lyn}(X)| = 1; \\ \langle P|\ell\rangle = 1. \end{cases}$$

If $w \in X^*$ is a word of Lyndon factorization $w = \ell_1^{\alpha_1} \dots \ell_n^{\alpha_n}$, $\ell_1 > \dots > \ell_n$, N denotes the sum $\sum_{i=1}^{n} \alpha_i$.

If we had that, for every w s.t. $N \geq 2$,

$$\mathrm{supp}(B'_w) \cap \mathrm{Lyn}(X) = \emptyset,$$

then

$$B'_w = B'_{[w]}.$$

We would have the two bases in duality, namely :

$$\langle B'_w | S'_u \rangle = \delta_{wu}.$$

Further properties :

- Construction of the basis :

$$\begin{aligned}
B'_{\ell_m} &= P_{\ell_m} \\
B'_{\ell_{m-1}} &= P_{\ell_{m-1}} - \langle P_{\ell_{m-1}} | \ell_m \rangle B'_{\ell_m} \\
B'_{\ell_{m-2}} &= P_{\ell_{m-2}} - \langle P_{\ell_{m-2}} | \ell_{m-1} \rangle B'_{\ell_{m-1}} - \langle P_{\ell_{m-2}} | \ell_m \rangle B'_{\ell_m} \\
&\vdots \\
B'_{\ell_{m-k}} &= P_{\ell_{m-k}} - \sum_{j=0}^{k-1} \langle P_{\ell_{m-k}} | \ell_{m-j} \rangle B'_{\ell_{m-j}}.
\end{aligned}$$

Further properties :

- Construction of the basis :

$$\begin{aligned}
B'_{\ell_m} &= P_{\ell_m} \\
B'_{\ell_{m-1}} &= P_{\ell_{m-1}} - \langle P_{\ell_{m-1}} | \ell_m \rangle B'_{\ell_m} \\
B'_{\ell_{m-2}} &= P_{\ell_{m-2}} - \langle P_{\ell_{m-2}} | \ell_{m-1} \rangle B'_{\ell_{m-1}} - \langle P_{\ell_{m-2}} | \ell_m \rangle B'_{\ell_m} \\
&\vdots \\
B'_{\ell_{m-k}} &= P_{\ell_{m-k}} - \sum_{j=0}^{k-1} \langle P_{\ell_{m-k}} | \ell_{m-j} \rangle B'_{\ell_{m-j}}.
\end{aligned}$$

- Not a PBW basis : $B'_{aabbaabb} \neq B'_{aabb} B'_{aabb}$ (and not of PBW basis for any total order on $\mathrm{Lyn}(X)$).

"PBW Criterion"

Let $(S_\alpha)_{\alpha\in\mathbb{N}^{(I)}}$ be a basis of $\mathscr{U}(\mathscr{G})$ in duality with a basis $\left(B^{[\alpha]}\right)_{\alpha\in\mathbb{N}^{(I)}}$:

$$\langle S_\alpha | B^{[\beta]}\rangle = \delta_{\alpha\beta}.$$

Then $B^\beta = B^{[\beta]}$ (B is of Poincaré-Birkhoff-Witt type) if and only if

$$\forall i \in I,\ \forall\beta \in \mathbb{N}^{(I)},\ |\beta| \geq 2,\ \langle S_{e_i} | B^{[\beta]}\rangle = 0.$$

2 Sage worksheets :

- Shuffle algebra : http://sagenb.org/home/pub/4504/
- Stuffle algebra : http://sagenb.org/home/pub/4519/

Implement basic objects, the duality bracket, some bases, the bialgebra structures...

Perspective : Dual bases in stuffle algebra.

Merci !