Physique Combinatoire.

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&
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Domaine St Jacques, le 27 mars 2012

Combinatorial Physics

When you're asked : "By the way, what is Combinatorial Physics ?" (and you have one minut to reply), what to say ?

Tentative definition

It is the art of solving physical problems with combinatorics and, conversely, to solve combinatorial problems using physical insights and methods.

Some topics (all relating to Physics, see also later JoCP)

- Combinatorics of data structures
  - graphs
  - words
  - discrete matchings
  - boards, diagrams, cells, polyominoes, etc ...
- Enumeration of data stuctures
  - constructible species
  - ranking
  - OGF, EGF, other denominators
  - asymptotic and exact properties

Some topics /2

- Representation theory
  - special functions and polynomials
  - symmetric polynomials
  - combinatorial modules
  - Fock spaces, normal forms
- Special functions (noncommutative)
  - noncommutative classical functions (trigonometry, etc ...)
  - noncommutative continued fractions
  - Frobenius characteristics

Some topics /3

- Combinatorial algebras, coalgebras and their actions
  - combinatorial and diagrammatic Hopf algebras
  - coherent states
  - (pro-)polynomial realizations
  - infinite dimensional Lie groups and differential equations
  - automata theory, representative functions, duality (dual laws, structure constants)
- Geometry of graphs, maps and matroids
  - constructions and operations on graphs
  - polynomials associated to graphs (Szymanczyk, Jones, Tutte polynomials)
  - braids and links
  - geometric and diagrammatic representation of tensors

Some topics /4

- Evolution groups
  - one-parameter groups of infinite matrices
  - one-parameter groups of differential operators
  - combinatorics of infinite-dimensional Lie algebras and their integration (local and global)
- Probabilities, distributions and measures
  - statistical mechanics
  - resolutions of unity for coherent states
  - moment problems, growth, unicity
  - orthogonal polynomials
  - parametric moment problems

Classical Fock space for bosons and q-ons

- Heisenberg-Weyl (two-dimensional) algebra is defined by two generators (a^+, a) which fulfill the relation

$$[a , a^+]=aa^+ -a^+a=1$$

- Known to have no (faithful) representation by bounded operators in a Banach space.

There are many « combinatorial » (faithful) representations by operators. The most famous one is the Bargmann-Fock representation

a ---> d/dx ; a+ ---> x

where a has degree -1 and a^+ has degree 1.

- These were bosons, there are also fermions. The relation for fermions is

$$aa^+ + a^+a = 1$$

- This provides a framework for the q-analogue which is defined by

$$[a\ ,\ a^+]_q = aa^+ - q a^+ a = 1$$

- For which Bargmann-Fock representation reads

$$a \text{ ---> } D_q \ ;\ a+ \text{ ---> } x$$

where a has degree -1 and a^+ has degree 1 and D_q is the (classical) q-derivative.

- For a faithful representation, one needs an infinite-dimensional space. The smallest, called Fock space, has a countable basis $(e_n)_{n\geq 0}$ (the actions are described below, each e_n is represented by a circled state « n »).

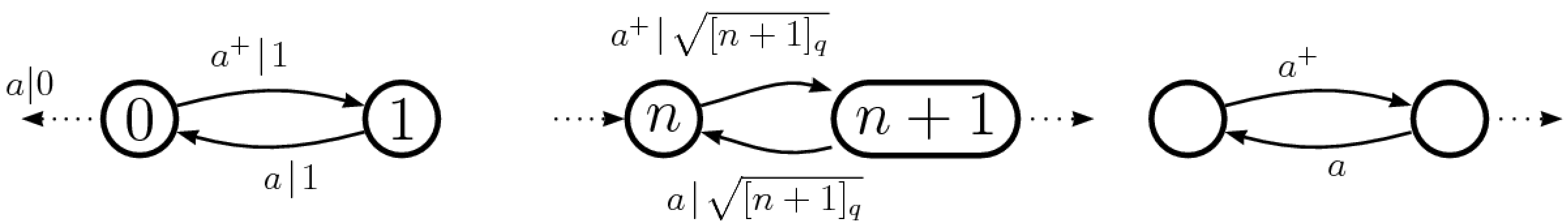
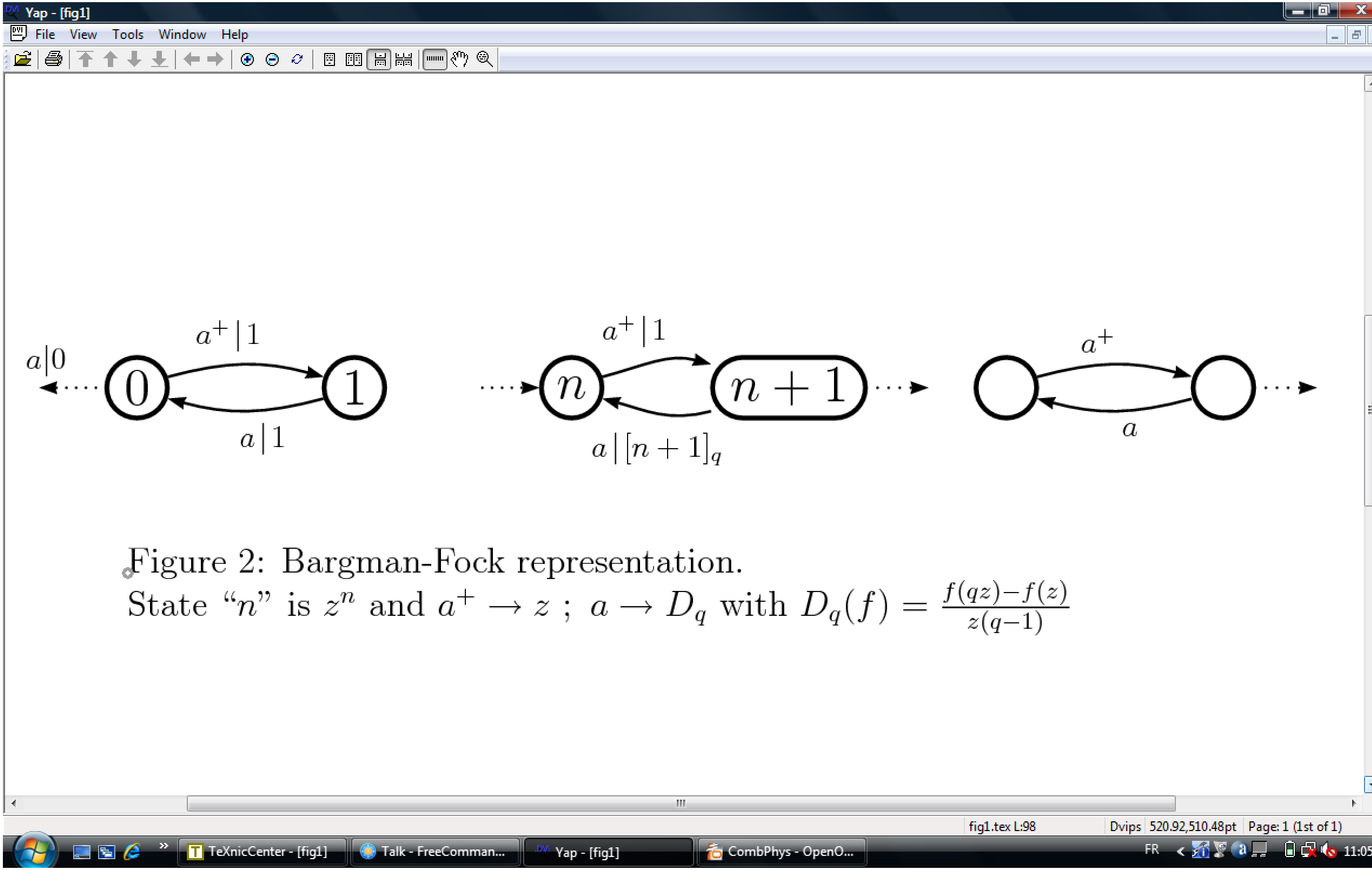


Figure 1: Classical Fock space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$a^+|1$
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0
1
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n
$n+1$
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a^+
a
Figure 2: Bargman-Fock representation.
State "n" is z^n and $a^+ \to z$; $a \to D_q$ with $D_q(f) = \frac{f(qz)-f(z)}{z(q-1)}$
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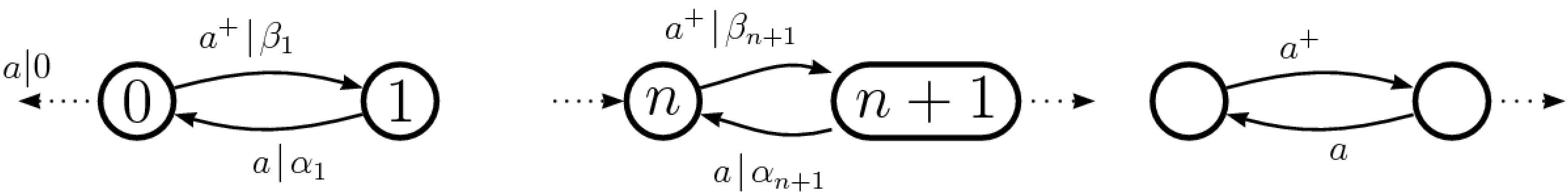


Figure 3: General setting: in order that the Fock space be bounded below, one must have $\alpha_0 = 0$.

• Physicists need to know the sum of all weights created when one passes from level « n » to level « m ». This problem has been called the « transfer packet problem » and is at once rephrased by combinatorists as the computation of a formal power series.

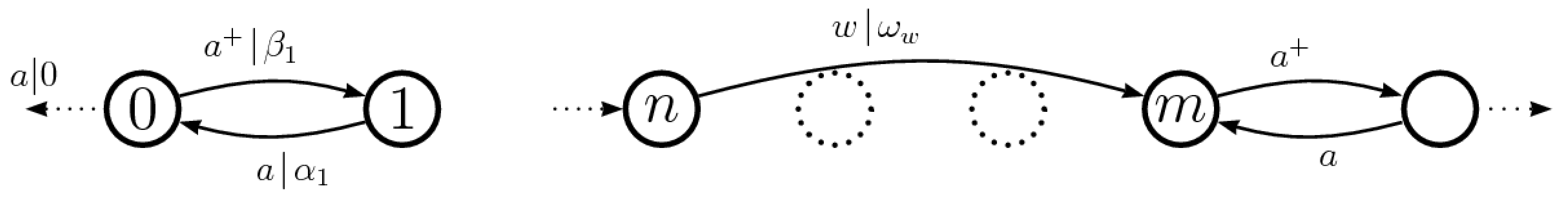
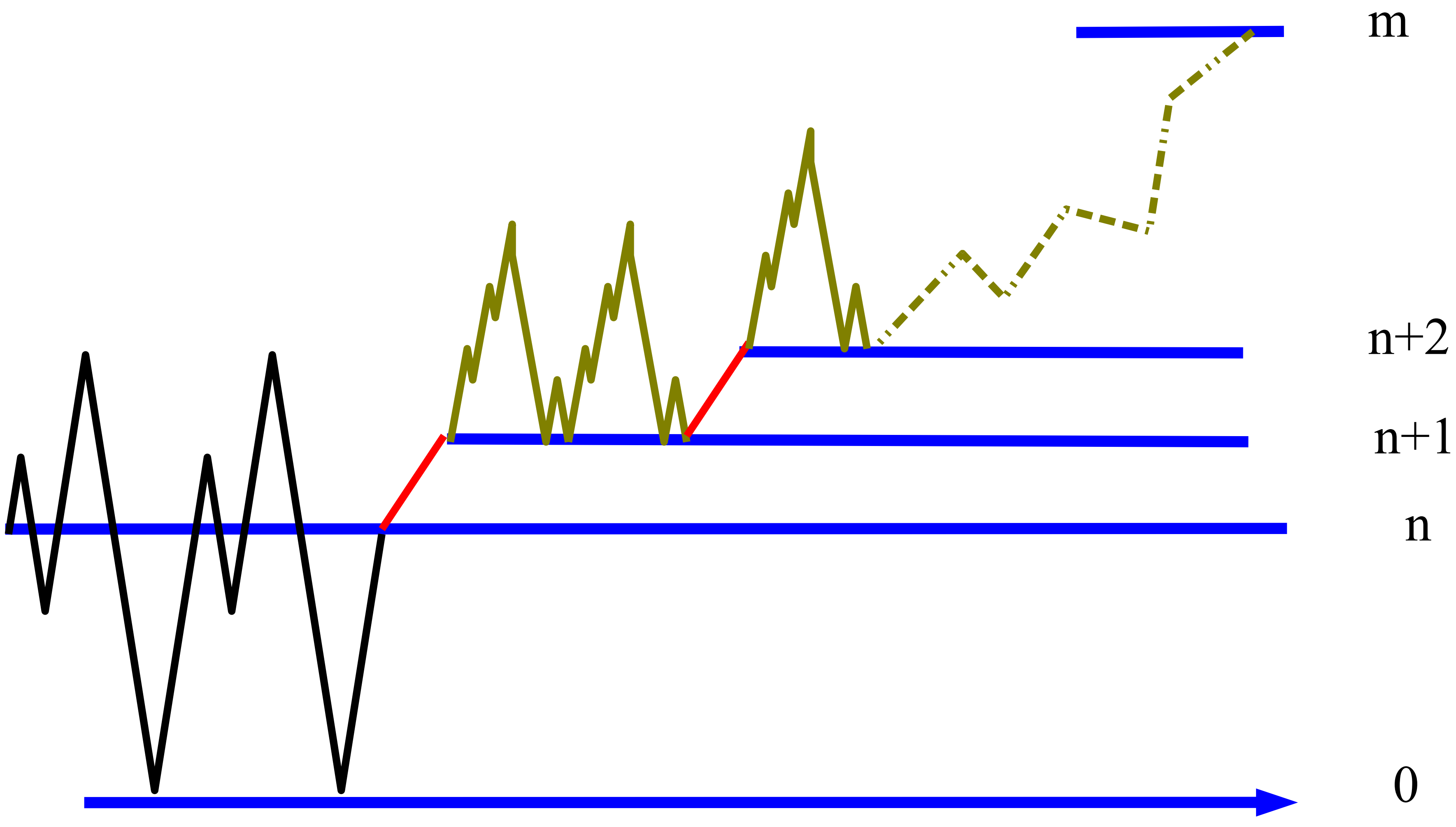


Figure 4: The transfer packet problem

Change of level



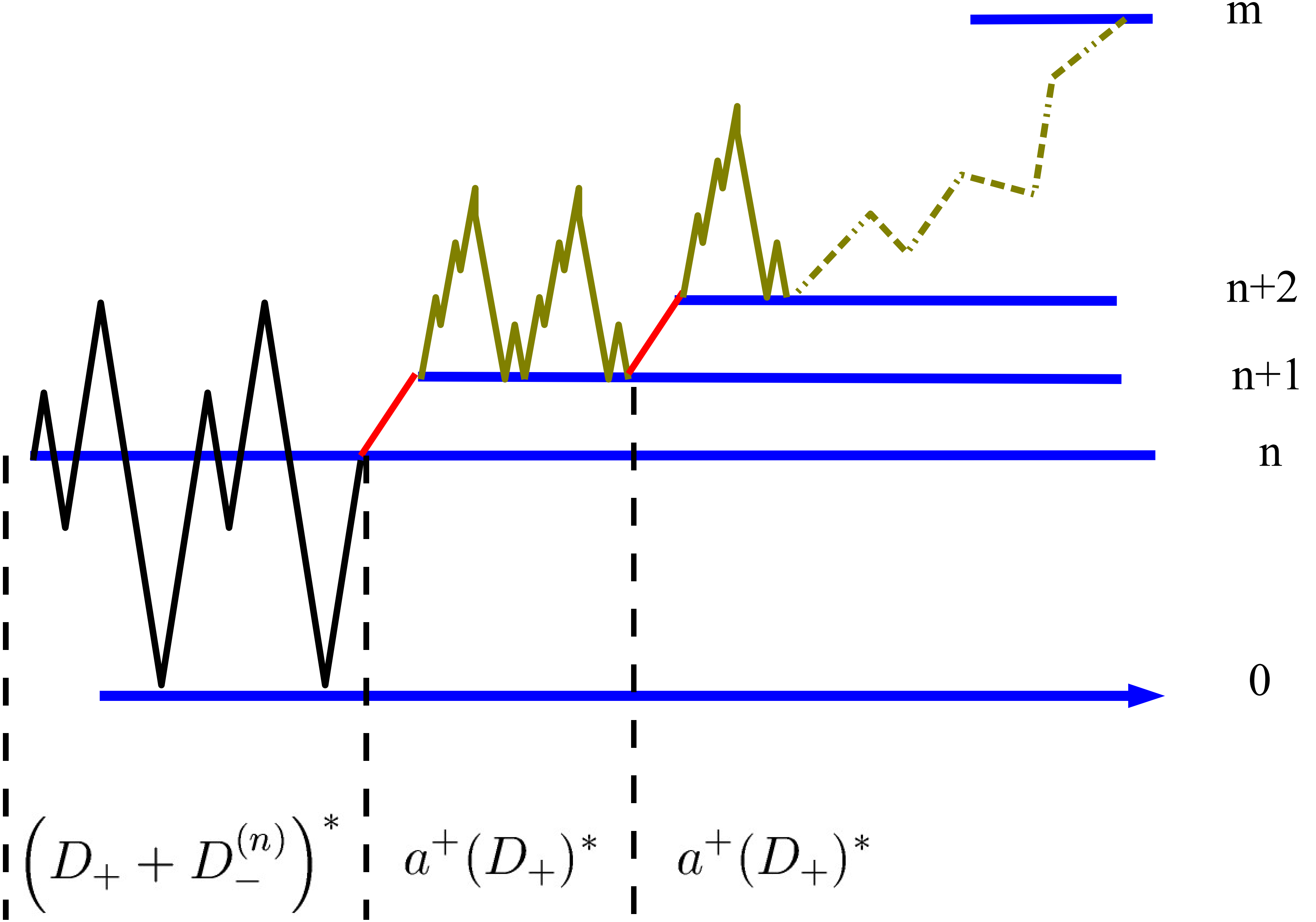
- The set of words which allow to pass from level « n » to level « m » in « i » steps is clearly.

$$W_k^{(i)} = \left\{ w \in \{a, a^+\}^* \middle| \pi_e(w) = k \text{ and } |w| = i \right\}$$

$$\text{with } k = m - n \text{ and } \pi_e(w) = |w|_{a^+} - |w|_a.$$

- The weight associated with this packet and the desired generating series are then

$$e_n.W^{(i)}_{m-n} = \omega^{(i)}_{n\to m}\ e_m\ ;\ T_{n\to n+k} := \sum_{i\geq 0} t^i \omega^{(i)}_{n\to n+k}$$


m
n+2
n+1
n
0
$\left(D_+ + D_-^{(n)}\right)^*$
$a^+(D_+)^*$
$a^+(D_+)^*$

The following selfreproducing formulas can be considered as noncommutative continued fraction expansions of the involved “D” codes

$$D_+ = a^+ \left(D_+\right)^* a \ ; \ D_-^{(n)} = a\left(D_-^{(n-1)}\right)^* a^+ \ ; \ D_-^{(0)} = \emptyset$$

Using a bit of analysis to extend the right action of the words to series of words and the representation

a ---> ?.ta ; a+ ---> ?.ta+

We get

$$T_{n\to n}[t] = \frac{1}{1 - \frac{t^2\alpha_{n+1}\beta_{n+1}}{1-\frac{t^2\alpha_{n+2}\beta_{n+2}}{1-\cdots}} - \frac{t^2\alpha_n\beta_n}{1-\frac{t^2\alpha_{n-1}\beta_{n-1}}{1-\cdots}}}$$

And, if one allows only the positive loops

$$T^{+}_{n\to n}[t] = \frac{1}{1 - \frac{t^2\alpha_{n+1}\beta_{n+1}}{1-\frac{t^2\alpha_{n+2}\beta_{n+2}}{1-\cdots}}}$$

Which solves, with two cases, the problem of the transfer packet.

Obtained and (re)obtained in many contexts

- *Theory of codes*
- *Orthogonal polynomials (with Motzkin paths)*
- *(Recently) for non-commutative probabilities*

General form serves for box ([n]) functions (Penson, Solomon et al.)

Another application to observations

(duality)

In many “Combinatorial” cases, we are concerned with the case A=k<∑> (non-commutative polynomials with coefficients in a field k).

Indeed, one has the following theorem (the beginning can be found in [ABE : Hopf algebras]) and the end is one of the starting points of Schützenberger's school of automata and language theory.

Theorem A: TFAE (the notations being as above)

i) $\Delta(c) \in A^* \otimes A^*$

ii) There are functions f_i, g_i $i=1,2..n$ such that

$$c(uv)=\sum_{i=1}^{n} f_i(u)\, g_i(v)$$

u,v words in Σ^* (the free monoid of alphabet Σ).

iii) There is a morphism of monoids $\mu: \Sigma^* \dashrightarrow k^{n \times n}$ (square matrices of size $n \times n$), a row λ in $k^{1 \times n}$ and a column ξ in $k^{n \times 1}$ such that, for all word w in A^*

$$c(w)=\lambda\mu(w)\xi$$

iv) (Schützenberger) (If Σ is finite) c lies in the rational closure of Σ within the algebra k<<A>>.

We can safely apply the first three conditions of **Theorem A** to *Ldiag*. The monoid of labelled diagrams is free, but with an infinite alphabet, so we cannot keep Schützenberger's equivalence at its full strength and have to take more "basic" functions. The modification reads

iv) (Σ is infinite) c is in the rational closure of the weighted sums of letters

$$\sum_{a \in \Sigma} p(a)\, a$$

within the algebra k<<A>>.

(Joint work with C. Tollu).

arXiv:0802.0249v1 [quant-ph]

In this case, *Schützenberger's* theorem (known as the theorem of Kleene-Schützenberger) could be rephrased in saying that functions in a Sweedler's dual are behaviours of finite (state and alphabet) automata.

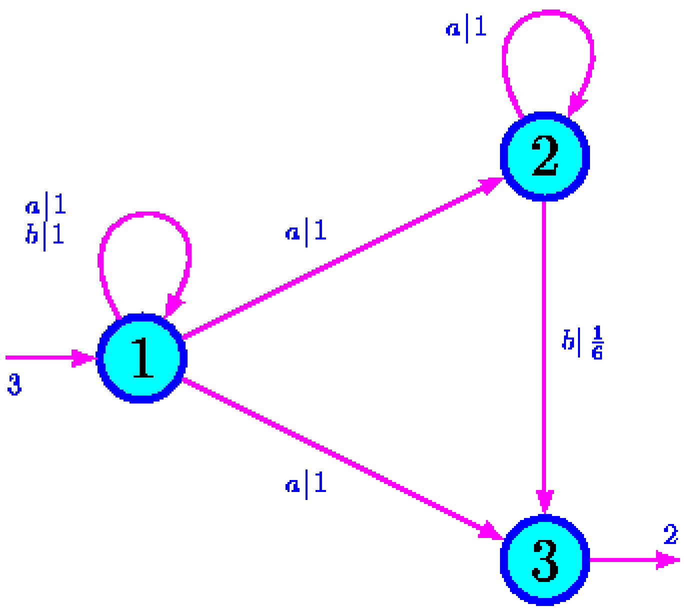
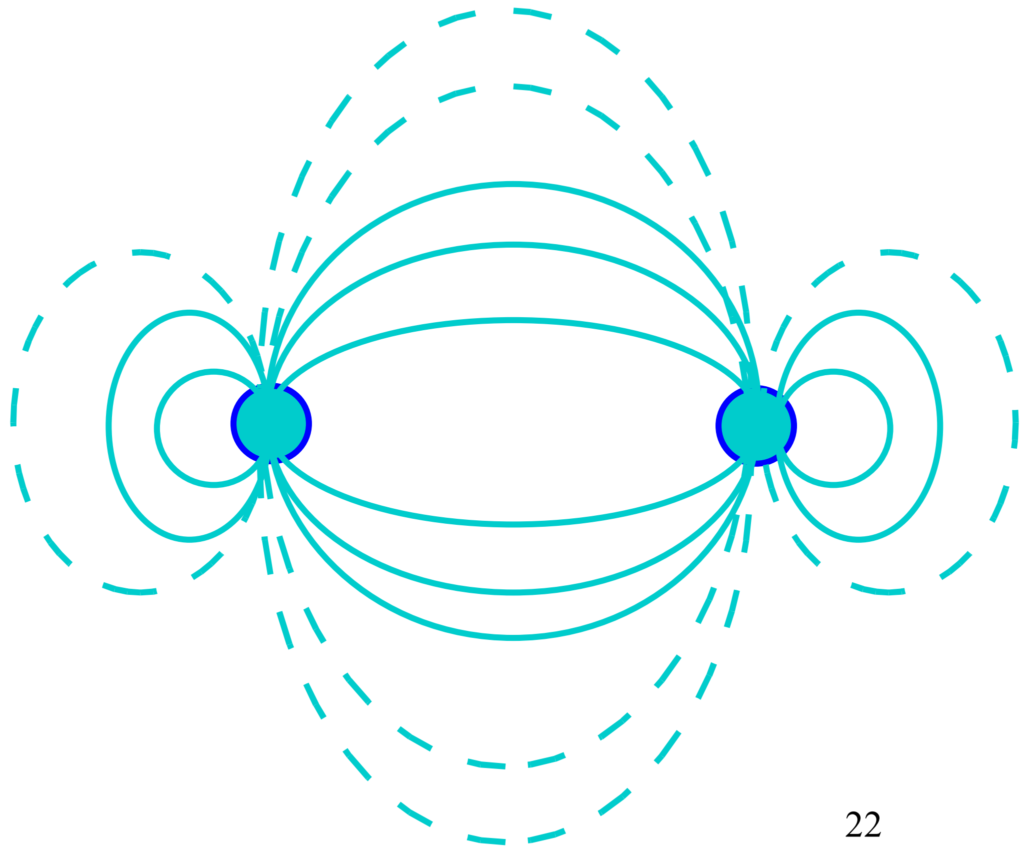


FIG. 1 – Un $\mathbb{Q}$-automate $\mathcal{A}$.

Le comportement de $\mathcal{A}$ est :

$$\text{comportement}(\mathcal{A}) = \sum_{a,b\in A} (a+b)^*(6+a^*b).$$

In our case, we are obliged to allow infinitely many edges.



Merci beaucoup

Thank you very much

Dziękuję bardzo

Vielen Dank