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Semi-skyline augmented fillings and a non-symmetric Cauchy identity in type A

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Cauchy identity

$$\prod_{(i,j)\in[n]\times[n]} (1 - x_i y_j)^{-1} = \sum_{\lambda \ partition \ \in\mathbb{N}^n} s_\lambda(x) s_\lambda(y).$$

$x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$

Definition

Let λ be a partition in $\mathbb{N}^n$. The Schur polynomial s_λ of shape λ in the variables $x = (x_1, x_2, \ldots, x_n)$ is

$$s_\lambda(x) = \sum_T x^T,$$

summed over all $SSYT$s of shape λ with entries in the alphabet [n].

Problem

$$\prod_{(i,j)\in[n]\times[n]} (1-x_i y_j)^{-1} = \sum_{\substack{(P,Q) SSYT \\ sh(P)=sh(Q) \\ cont(P),cont(Q)\in\mathbb{N}^n}} x^P y^Q = \sum_{\lambda \text{ partition } \in\mathbb{N}^n} s_\lambda(x) s_\lambda(y).$$

$$\prod_{(i,j)\in dg(n,n-1,\ldots,1)} (1-x_i y_j)^{-1} = \prod_{i+j\leq n+1} (1-x_i y_j)^{-1} = \sum_{(P,Q)\in\mathcal{F}'} x^P y^Q.$$

$$\mathcal{F}' \subseteq \{(P,Q) SSYT,\ sh(P)=sh(Q),\ cont(P), cont(Q) \in \mathbb{N}^n\}.$$

We want to characterize $\mathcal{F}'$.

A non-symmetric Cauchy identity

Amy M. Fu, Alain Lascoux (2009) have proved in a algebraic way, using Demazure operators in type A:

$$\prod_{(i,j)\in dg(n,n-1,\dots,1)} (1-x_i y_j)^{-1} = \prod_{i+j\leq n+1} (1-x_i y_j)^{-1} = \sum_{\nu\in\mathbb{N}^n} \widehat{K}_\nu(x) K_{\omega\nu}(y).$$

Demazure operator (isobaric divided difference) in type A

$$\pi_i : f \mapsto \pi_i(f) := \frac{x_i f - x_{i+1} s_i(f)}{x_i - x_{i+1}}, \quad 1 \leq i < n, \;\; \widehat{\pi}_i := \pi_i - 1.$$

The Demazure character (key polynomials) $K_{\sigma\lambda}$ corresponding to a partition λ and reduced decomposition of a permutation $\sigma = s_{i_1} s_{i_2} \dots s_{i_k}$ is

$$K_{\sigma\lambda} = \pi_{i_1} \pi_{i_2} \dots \pi_{i_k}(x^\lambda).$$

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Lascoux's combinatorial interpretation using double crystal graphs.

Lascoux has used the crystal version of RSK and the combinatorial interpretation of Demazure operators in terms of crystal operators.

$$\prod_{i+j\leq n+1}(1-x_iy_j)^{-1}=\sum_{\nu\in\mathbb{N}^n}\widehat{K}_\nu(x)K_{\omega\nu}(y).$$

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We want to characterize $\mathcal{F}'$ in terms of semi-skyline augmented filling.

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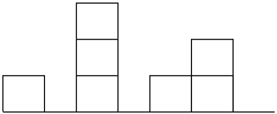
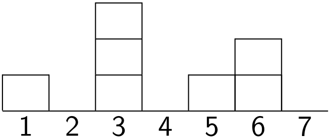
Semi-skyline augmented filling. (Haglund, Haiman, Loehr. 2005)

Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \mathbb{N}^n$.
The *column diagram* of γ is a figure $dg'(\gamma)$ consisting of n columns, with γ_i boxes in column i.
The *augmented diagram* of γ is a *column diagram* with a basement consisting of the numbers 1 through n in strictly increasing order, denoted by $\widehat{dg}(\gamma)$.
The *column diagram* and *augmented diagram* for $\gamma = (1, 0, 3, 0, 1, 2, 0)$ are

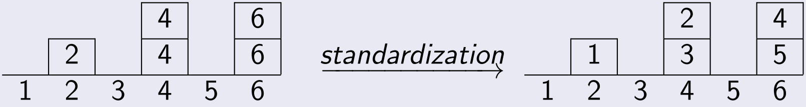


$dg'(\gamma)$



$\widehat{dg}(\gamma)$

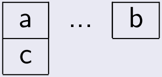
The standardization of an augmented filling



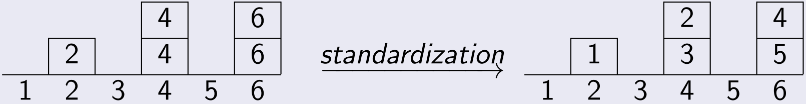
4 6 2 4 6 $\longrightarrow$ 2 4 1 3 5

An inversion triple of type 1.

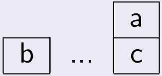
The triple $\{a, b, c\}$ is an inversion triple of type 1 if and only if after standardization, the ordering from smallest to largest induces a counterclockwise orientation.



An inversion triple of type 2.



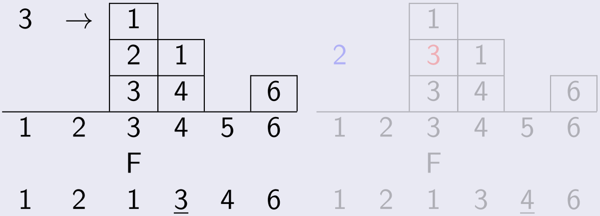
The triple $\{a, b, c\}$ is an inversion triple of type 2 if and only if after standardization, the ordering from smallest to largest induces a clockwise orientation.



SSAF

A semi-skyline augmented filling (SSAF) is any augmented filling F that is weakly decreasing along columns, from bottom to top, and every triple is an inversion triple.

An analogue of Schensted row insertion in SSAF.



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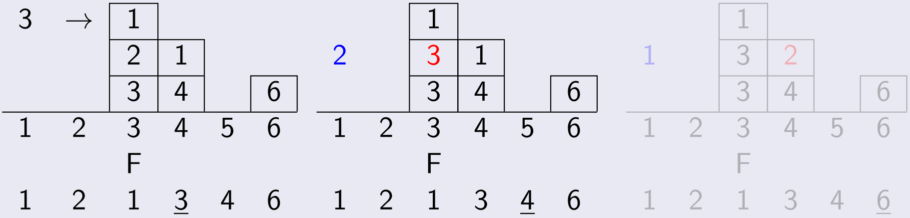

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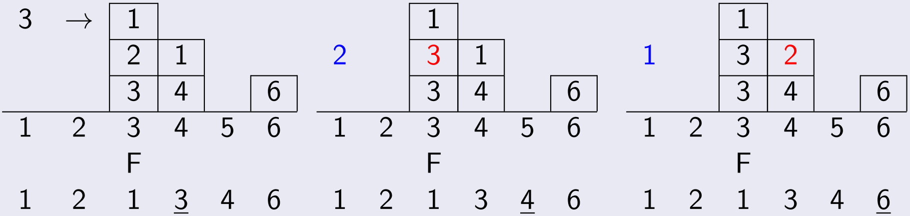

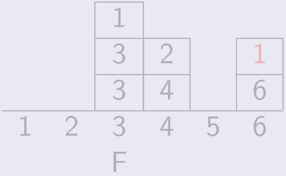

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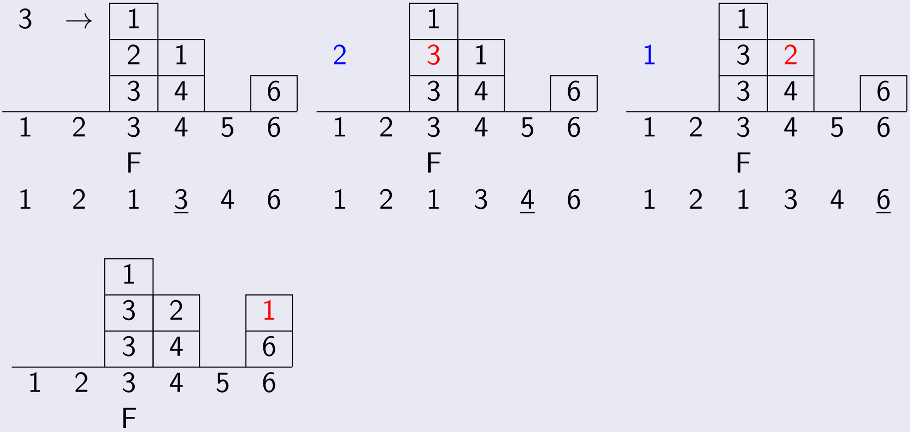

A weight preserving, shape rearranging bijection between SSYT and SSAF.(S. Mason, 2008)

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Example

$$T = \begin{matrix} 4 & & & \\ 2 & 5 & & \\ 1 & 3 & 3 & 3 \end{matrix} \quad \rightarrow \; col(T) = 421\ 53\ 3\ \underline{3}$$

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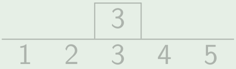

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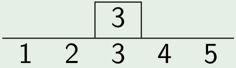

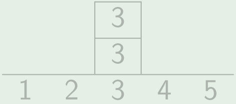

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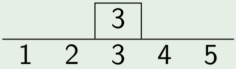

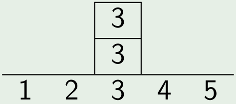

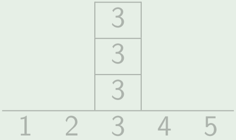

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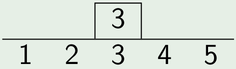

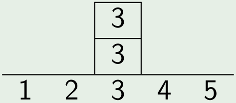

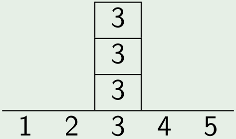

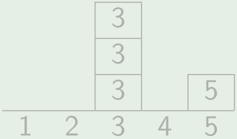

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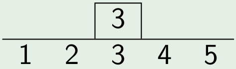

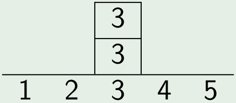

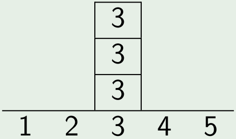

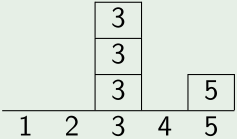

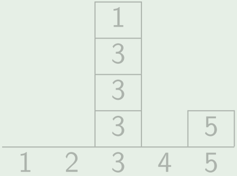

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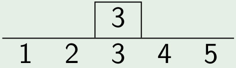

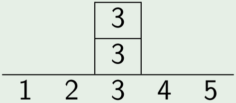

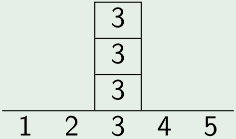

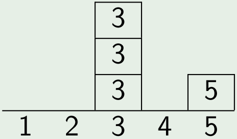

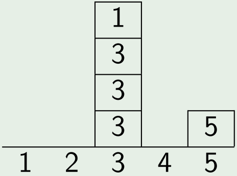

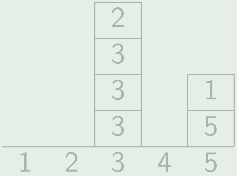

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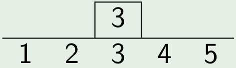

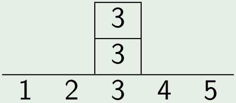

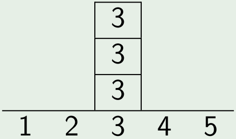

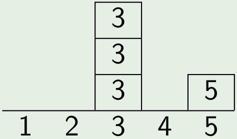

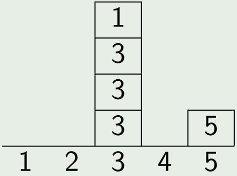

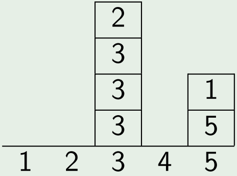

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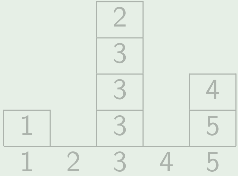

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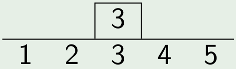
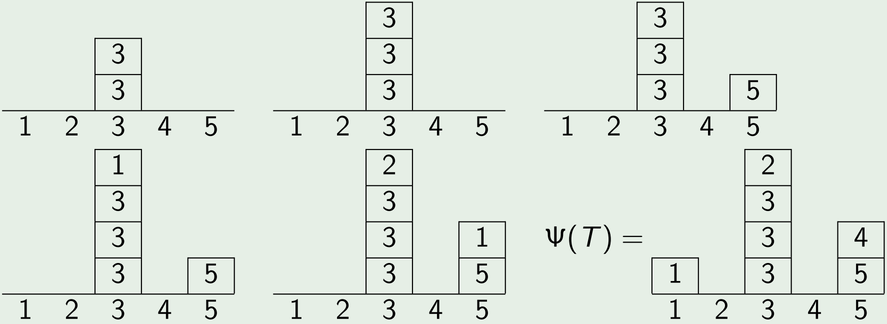




Right key of SSYT (S.Mason 2009)

$$P(SSYT) \underset{\rightarrow}{\Psi} F(SSAF),$$

$$k_+(P) = key(sh(F)).$$

Remark

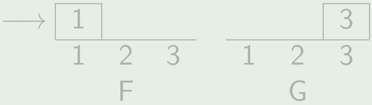
The original definition of right key of a tableau is due to Lascoux and Schützenberger (1988).

An analogue of the RSK for semi-skyline augmented filling. (S.Mason 2008)

Bijection Φ between words in commutative biletters and pairs (F, G) of semi-skyline augmented fillings whose shapes are rearrangements of the same partition.

Example

$$w_A = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 3 & 2 & 2 & 1 & 1 \end{pmatrix}$$



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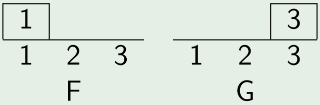

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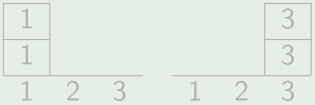

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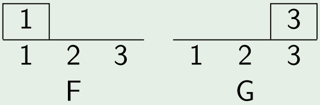

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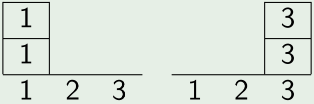

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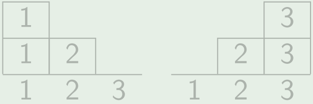

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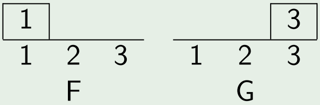
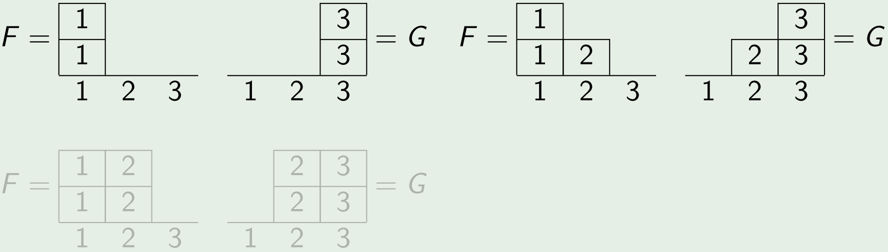
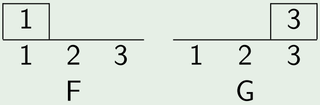
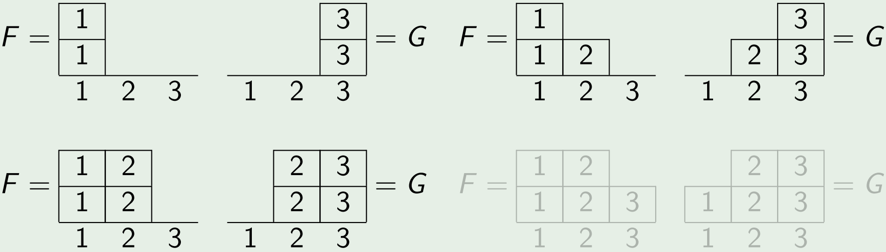




Example

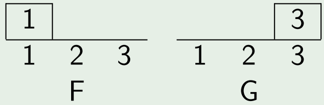
$$w_A = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 3 & 2 & 2 & 1 & 1 \end{pmatrix} \longrightarrow$$

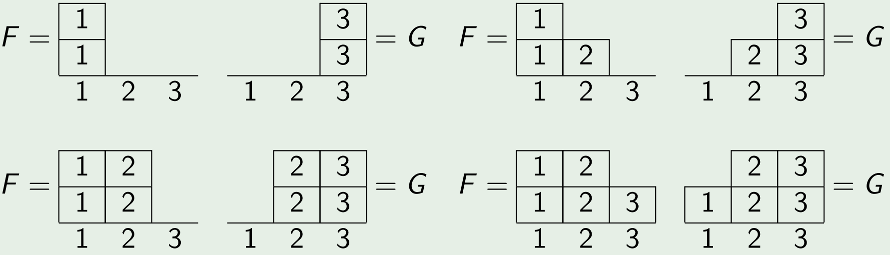


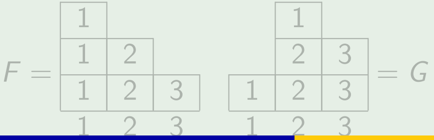


Example

$$w_A = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 3 & 2 & 2 & 1 & 1 \end{pmatrix} \longrightarrow$$



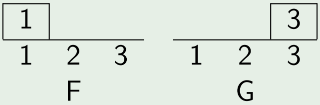


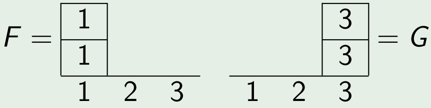


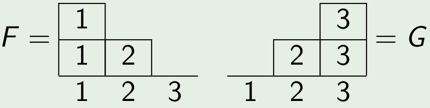
$sh(F)^+ = sh(G)^+ = (3, 2, 1)$

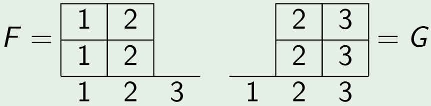
Example

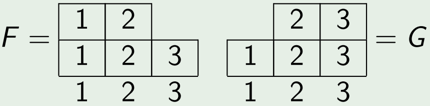
$$w_A = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 3 & 2 & 2 & 1 & 1 \end{pmatrix} \longrightarrow$$

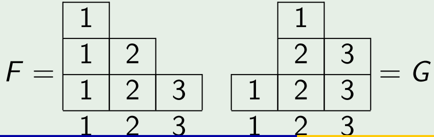




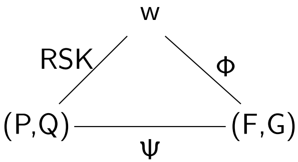








$$sh(F)^+ = sh(G)^+ = (3, 2, 1)$$



$$sh(F)^+ = sh(G)^+ = sh(P) = sh(Q)$$

$$key(sh(F)) = k_+(P), \; key(sh(G)) = k_+(Q)$$

$$cont(F) = cont(P), \; cont(G) = cont(Q)$$

Problem

Cauchy identity in terms of SSAF

$$\prod_{(i,j)\in[n]\times[n]} (1 - x_i y_j)^{-1} = \sum_{\substack{(F,G) SSAF \\ sh(F)^+ = sh(G)^+ \\ cont(F), cont(G) \in \mathbb{N}^n}} x^F y^G$$

Non-symmetric Cauchy kernel

$$\prod_{i+j\leq n+1} (1 - x_i y_j)^{-1} = \sum_{(F,G)\in\mathcal{F}} x^F y^G$$

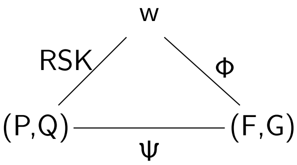
$$\mathcal{F} \subseteq \{(F,G) SSAF,\ sh(F)^+ = sh(G)^+,\ cont(F), cont(G) \in \mathbb{N}^n\}.$$

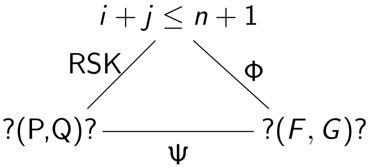
Non-symmetric Cauchy kernel in terms of SSYT

$$\prod_{i+j\leq n+1}(1-x_iy_j)^{-1}=\sum_{(P,Q)\in\mathcal{F}'}x^Py^Q=\sum_{(F,G)\in\mathcal{F}}x^Fy^G$$

$$\mathcal{F}'\subseteq\{(P,Q)SSYT,\ sh(P)=sh(Q),\ cont(P),cont(Q)\in\mathbb{N}^n\}.$$

$$\mathcal{F}\subseteq\{(F,G)SSAF,\ sh(F)^+=sh(G)^+,\ cont(F),cont(G)\in\mathbb{N}^n\}.$$





Theorem (E.)

Let A be a biword in lexicographic order, let $\Phi(A) = (F, G)$. For each biletter $\begin{pmatrix} i \\ j \end{pmatrix}$ in A we have $i + j \leq n + 1$ if and only if $key(sh(G)) \leq key(\omega sh(F))$.

Corollary

Let A be a biword in lexicographic order, let $A \ \underrightarrow{RSK}\ (P, Q)$.
For each biletter $\begin{pmatrix} i \\ j \end{pmatrix}$ in A we have $i + j \leq n + 1$ if and only if $k_+(Q) \leq evac(k_+(P))$.

$$k_+(Q) \leq evac(k_+(P)) \Longleftrightarrow key(sh(G)) \leq key(\omega sh(F)).$$

w	$\overset{\Phi}{\leftrightarrow}$	(F, G)	$\overset{\Psi}{\leftrightarrow}$	(P, Q)
$i + j \leq n + 1$		$key(sh(G)) \leq key(\omega sh(F))$		$K_+(P) = key(\alpha)$
		$sh(F) = \alpha, sh(G) = \beta$		$K_+(Q) = key(\beta)$
		$key(\beta) \leq key(\omega\alpha)$		$key(\beta) \leq key(\omega\alpha)$
		$\beta \leq_B \omega\alpha$		$K_+(Q) \leq evac(K_+(P)$

$$\prod_{i+j\leq n+1}(1-x_iy_j)^{-1}=\sum_{(F,G)\in\mathcal{F}}x^Fy^G=\sum_{\nu\in\mathbb{N}^n}\sum_{\substack{(F,G)\in SSAF\\ sh(F)=\nu\\ sh(G)\leq\omega\nu}}x^Fy^G.$$

$$\prod_{i+j\leq n+1}(1-x_iy_j)^{-1}=\sum_{(P,Q)\in\mathcal{F}'}x^Py^Q=\sum_{\nu\in\mathbb{N}^n}\sum_{\substack{(P,Q)\in SSYT\\ sh(P)=sh(Q)=\nu^+\\ K_+(P)=key(\nu)\\ K_+(Q)=key(\beta)\\ \beta\leq\omega\nu}}x^Py^Q.$$

Or the shorter way:

$$\prod_{i+j\leq n+1}(1-x_iy_j)^{-1}=\sum_{\substack{(P,Q)\in SSYT\\ K_+(Q)\leq evac(K_+(P))\\ cont(P),cont(Q)\in\mathbb{N}^n}}x^Py^Q.$$

$$\prod_{i+j\leq n+1}(1-x_iy_j)^{-1}=\sum_{(F,G)\in\mathcal{F}}x^Fy^G=\sum_{\nu\in\mathbb{N}^n}\sum_{\substack{(F,G)\in SSAF\\ sh(F)=\nu\\ sh(G)\leq\omega\nu}}x^Fy^G.$$

$$\prod_{i+j\leq n+1}(1-x_iy_j)^{-1}=\sum_{\nu\in\mathbb{N}^n}\sum_{\substack{(P,Q)\in SSYT\\ sh(P)=sh(Q)=\nu^+\\ K_+(P)=key(\nu)\\ K_+(Q)=key(\beta)\\ \beta\leq\omega\nu}}x^Py^Q.$$

$$\prod_{i+j\leq n+1}(1-x_iy_j)^{-1}=\sum_{\nu\in\mathbb{N}^n}\widehat{K}_\nu(x)K_{\omega\nu}(y).$$

Combinatorial structure for key polynomials

$$K_\nu(x) = \sum_{\substack{P \ SSYT \\ sh(P)=\nu^+ \\ K_+(P)\leq key(\nu)}} x^P = \sum_{\substack{F \ SSAF \\ sh(F)\leq\nu}} x^F.$$

and

$$\widehat{K}_\nu(x) = \sum_{\substack{P \ SSYT \\ sh(P)=\nu^+ \\ K_+(P)=key(\nu)}} x^P = \sum_{\substack{F \ SSAF \\ sh(F)=\nu}} x^F.$$

.

Lascoux and Schützenberger have studied the combinatorial structure of Key polynomials in terms of SSYT in (1988) and S.Mason has studied it in terms of SSAF in (2009).

Theorem (E.)

Let A be a biword in lexicographic order, let $\Phi(A) = (F, G)$. For each biletter $\begin{pmatrix} \text{i} \\ \text{j} \end{pmatrix}$ in A we have $i + j \leq n + 1$ if and only if $key(sh(G)) \leq key(\omega sh(F))$.

sketch of proof:

By induction on the number of biletters in A and using lemma below:

Lemma

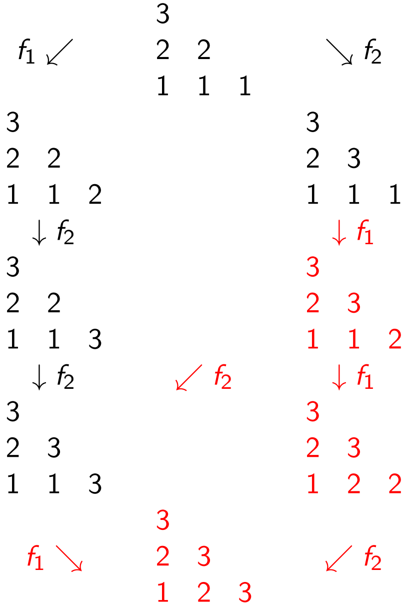
Given a partition λ, let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ be two rearrangements of λ, such that $key(\beta) \leq key(\alpha)$.
Let $k, k' \in \{1, \ldots, n\}$ such that $\beta_{k'}$ is the first entry of β such that $\alpha_k = \beta_{k'}$. If $\alpha' = (\alpha_1, \alpha_2, \ldots, \alpha_k + 1, \ldots, \alpha_n)$ and $\beta' = (\beta_1, \beta_2, \ldots, \beta_{k'} + 1, \ldots, \beta_n)$, then $key(\beta') \leq key(\alpha')$

Work in progress with R. Mamede

$$\Omega^A := \frac{1}{\prod_{i+j\leq n+1}(1-x_iy_j)} = \sum_{\nu\in\mathbb{N}^n} \widehat{K}_\nu(x)K_{\omega\nu}(y),$$

$$\Omega^B := \frac{\prod_{1\leq i<j\leq n}(1-x_ix_j)\prod_{i=1}^n(1+x_i)}{\prod_{i,j=1}^n(1-x_iy_j)\prod_{i=1}^n\prod_{j=i}^n(1-\frac{x_i}{y_j})} = \sum_{\nu\in\mathbb{N}^n} \widehat{K}_\nu(x)K^B_{-\nu}(y),$$

$$\Omega^C := \frac{\prod_{1\leq i<j\leq n}(1-x_ix_j)}{\prod_{i,j=1}^n(1-x_iy_j)\prod_{i=1}^n\prod_{j=i}^n(1-\frac{x_i}{y_j})} = \sum_{\nu\in\mathbb{N}^n} \widehat{K}_\nu(x)K^C_{-\nu}(y),$$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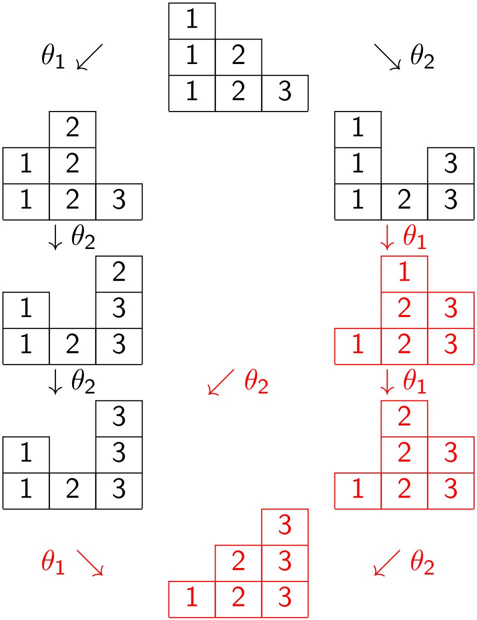
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THANK YOU

Proof by Lascoux

Let $\Xi_n := \sum_{\sigma\in S_n} \widehat{\pi}^x_\sigma \pi^y_{\omega\sigma}$.

$$\Xi_n \frac{1}{(1-x_1y_1)(1-x_1x_2y_1y_2)\dots(1-x_1\dots x_ny_1\dots y_n)} = \frac{1}{\prod_{i+j\le n+1}(1-x_iy_j)}$$

as

$$\frac{1}{(1-x_1y_1)(1-x_1x_2y_1y_2)\dots(1-x_1\dots x_ny_1\dots y_n)}$$

is the generating function of dominant monomials $x^\lambda y^\lambda$ in n variables $x_1y_1,\dots,x_ny_n$,

$$\prod_{i+j\le n+1}(1-x_iy_j)^{-1} = \sum \Xi_n x^\lambda y^\lambda = \sum_{\sigma\in S_n} \hat{\pi}^x_\sigma(x^\lambda)\pi^y_{\omega\sigma}(y^\lambda) = \sum_{\nu\in\mathbb{N}^n} \widehat{K}_\nu(x)K_{\omega\nu}(y$$