Multivariable Tangent and Secant q-derivative polynomials

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(Based on the paper by Foata-Han with the same title)

68e SLC, Ottrott March 26, 2012

Summary

- The derivative polynomials
- q-Calculus
- q-Derivative polynomials
- Main theorems

The derivative polynomials

$$
\begin{aligned}
D \tan u &= 1 + \tan^2 u \\
D^2 \tan u &= 2 \tan u \times D \tan u \\
&= 2 \tan u \times (1 + \tan^2 u) \\
&= 2 \tan u + 2 \tan^3 u \\
D^3 \tan u &= 2 + 8 \tan^2 u + 6 \tan^4 u \\
D^4 \tan u &= 16 \tan u + 40 \tan^3 u + 24 \tan^5 u \\
\ldots
\end{aligned}
$$

Knuth-Buckholtz (1967) :

$D^n \tan u$ is a polynomial in $\tan u$

The coefficients of the polynomial can be calculated by using a recurrence relation

The derivative polynomials

$$D^n \tan u = \sum_{m\geq 0} a(n,m) \tan^m u$$

$$a(0,m) = \delta_{1,m}$$
$$a(n+1,m) = (m-1)a(n,m-1) + (m+1)a(n,m+1)$$

$m =$	0	1	2	3	4	5	6	7
$n = 0$		**1**						
1	**1**		**1**					
2		**2**		**2**				
3	**2**		**8**		**6**			
4		**16**		**40**		**24**		
5	**16**		**136**		**240**		**120**	
6		**272**		**1232**		**1680**		**720**

The derivative polynomials

Again:

$$D^n \sec u := \sum_{m \geq 0} b(n,m) \tan^m u \sec u.$$

where $(b(n,m))$ satisfies the recurrence relation:

$$b_{0,m} = \delta_{0,m},$$

$$b(n+1,m) = mb(n,m-1) + (m+1)b(n,m+1).$$

The derivative polynomials

The coefficients $\mathbf{a}(\mathbf{n},\mathbf{m})$ and $b(n,m)$

$m =$	0	1	2	3	4	5	6		
$n = 0$	1	**1**						$\mathbf{1.2^0}$	1
1	**1**	1	**1**					$\mathbf{1.2^1}$	1
2	1	**2**	2	**2**				$\mathbf{1.2^2}$	3
3	**2**	5	**8**	6	**6**			$\mathbf{2.2^3}$	11
4	5	**16**	28	**40**	24	**24**		$\mathbf{5.2^4}$	57
5	**16**	61	**136**	180	**240**	120	**120**	$\mathbf{16.2^5}$	361

first column : tangent and secant numbers
diagonal : factorial number
right of the table : row sums

- row sums of $\mathbf{a(n,m)}$: tangent and secant numbers, $\times 2^n$
- row sums of $b(n,m)$: Springer numbers

The derivative polynomials: Generating function

Derivative polynomials:

$$A_n(x) := \sum_{m\geq 0} a(n,m)x^m$$

$$B_n(x) := \sum_{m\geq 0} b(n,m)x^m$$

The exponential generating functions for the derivative polynomials (Hoffman, 1995)

$$\sum_{n\geq 0} A_n(x)\frac{u^n}{n!} = \frac{x + \tan u}{1 - x\ \tan u}$$

$$\sum_{n\geq 0} B_n(x)\frac{u^n}{n!} = \frac{1}{\cos u - x\sin u}$$

The derivative poly. : Combinatorial interpretation

Hoffman (1999) has found a combinatorial interpretation of the derivative polynomials, in terms of $snakes$, a notion introduced by Arnold (1992).

This interpretation has been used again by Josuat-Vergès, 2011.

q-analog ?

q-Calculus

t-ascending factorial

$$(t;q)_n := \begin{cases} 1, & \text{if } n = 0 \\ (1-t)(1-tq)\cdots(1-tq^{n-1}), & \text{if } n \geq 1 \end{cases}$$

$$(t;q)_\infty := \lim_n (t;q)_n = \prod_{n\geq 0} (1-tq^n)$$

q-series

$$f(u) = \sum_{n\geq 0} f(n;q) \frac{u^n}{(q;q)_n}$$

q-derivative operator

$$D_q f(u) := \frac{f(u) - f(qu)}{u}$$

[Not the traditional $\frac{f(u)-f(qu)}{u(1-q)}$]

q-Calculus

q-*exponential* series

$$e_q(u) = \sum_{n\geq 0} \frac{u^n}{(q;q)_n} = \frac{1}{(u;q)_\infty}$$

(a result that goes back to Euler, 1748)

q-trigonometric series (Jackson, 1904):

$$\sin_q(u) := \frac{e_q(iu) - e_q(-iu)}{2i} = \sum_{n\geq 0} (-1)^n \frac{u^{2n+1}}{(q;q)_{2n+1}} ;$$

$$\cos_q(u) := \frac{e_q(iu) + e_q(-iu)}{2} = \sum_{n\geq 0} (-1)^n \frac{u^{2n}}{(q;q)_{2n}} ;$$

q-Calculus

q-*exponential* series

$$E_q(u) = \sum_{n \geq 0} q^{n(n-1)/2} \frac{u^n}{(q;q)_n} = (-u;q)_\infty$$

(a result that goes back to Euler, 1748)

q-trigonometric series (Jackson, 1904):

$$\mathrm{Sin}_q(u) := \frac{E_q(iu) - E_q(-iu)}{2i} = \sum_{n \geq 0} (-1)^n q^{n(2n+1)} \frac{u^{2n+1}}{(q;q)_{2n+1}}$$

$$\mathrm{Cos}_q(u) := \frac{E_q(iu) + E_q(-iu)}{2} = \sum_{n \geq 0} (-1)^n q^{n(2n-1)} \frac{u^{2n}}{(q;q)_{2n}}$$

q-Calculus

only one q-tangent

$$\tan_q(u) := \frac{\sin_q(u)}{\cos_q(u)} = \frac{\mathrm{Sin}_q(u)}{\mathrm{Cos}_q(u)} = \frac{\sum_{n\geq 0}(-1)^n u^{2n+1}/(q;q)_{2n+1}}{\sum_{n\geq 0}(-1)^n u^{2n}/(q;q)_{2n}}$$

two q-secants

$$\sec_q(u) := \frac{1}{\cos_q(u)} = \frac{1}{\sum_{n\geq 0}(-1)^n u^{2n}/(q;q)_{2n}}$$

$$\mathrm{Sec}_q(u) := \frac{1}{\mathrm{Cos}_q(u)} = \frac{1}{\sum_{n\geq 0}(-1)^n q^{n(2n-1)} u^{2n}/(q;q)_{2n}}$$

q-Calculus

Lemma.

$$D_q \tan_q(u) = 1 + \tan_q(u) \tan_q(qu)$$
$$D_q \sec_q(u) = \sec_q(qu)\ \tan_q(u)$$
$$D_q \operatorname{Sec}_q(u) = \operatorname{Sec}_q(u) \tan_q(qu)$$

What are

$$D_q^n \tan_q(u) = \ ?$$
$$D_q^n \sec_q(u) = \ ?$$
$$D_q^n \operatorname{Sec}_q(u) = \ ?$$

q-Derivative Polynomials

q-Leibniz:

$$D_q \prod_{1\le i\le n} f_i(u)$$

$$= \sum_{1\le i\le n} f_1(u)\cdots f_{i-1}(u)\cdot D_q f_i(u)\cdot f_{i+1}(qu)\cdots f_n(qu)$$

Remark: The LHS is symmetric in the f_i's, so does the RHS. However we cannot see the symmetry on the present writing form of the RHS, that depends on the order of the f_i's.

q-Derivative Polynomials

Proof.

Notation $f' := D_q f$.

$$
\begin{aligned}
&(f(x)g(x)h(x))' \\
=&\frac{1}{x}(f(x)g(x)h(x) - f(qx)g(qx)h(qx)) \\
=&\frac{1}{x}(f(x)g(x)h(x) - f(x)g(x)h(qx) \\
&+ f(x)g(x)h(qx) - f(x)g(qx)h(qx) \\
&+ f(x)g(qx)h(qx) - f(qx)g(qx)h(qx)) \\
=&f(x)g(x)h'(x) + f(x)g'(x)h(qx) + f'(x)g(qx)h(qx)
\end{aligned}
$$

q-Derivative Polynomials

We have two ways to calculate $\tan''(x) = (\tan'(x))' =$

$$
\begin{aligned}
=&(1+\tan(x)\tan(qx))' \\
=&\tan(x)(\tan(qx))' + \tan(x)'\tan(q^2x) \\
=&\tan(x)q(1+\tan(qx)\tan(q^2x)) + (1+\tan(x)\tan(qx))\tan(q^2x) \\
=&q\tan(x) + \tan(q^2x) + (1+q)\tan(x)\tan(qx)\tan(q^2x)
\end{aligned}
$$

and also

$$
\begin{aligned}
=&(1+\tan(qx)\tan(x))' \\
=&\tan(qx)(\tan(x))' + \tan(qx)'\tan(qx) \\
=&\tan(qx)(1+\tan(x)\tan(qx)) + q(1+\tan(qx)\tan(q^2x))\tan(qx) \\
=&(1+q)\tan(qx) + \tan(x)\tan(qx)^2 + q\tan(qx)^2\tan(q^2x)
\end{aligned}
$$

Notation $\tan := \tan_q$

q-Derivative Polynomials

$$\begin{aligned}
&\tan'''(x)\\
=&(q+q^2)\\
+&(q+q^2+q^3)\tan(x)\tan(qx)\\
+&(1+q+q^2)\tan(q^2x)\tan(q^3x)\\
+&q(1+q)\tan(x)\tan(q^3x)\\
+&(1+q)(1+q+q^2)\tan(x)\tan(qx)\tan(q^2x)\tan(q^3x)
\end{aligned}$$

and

q-Derivative Polynomials

$$\begin{aligned}\tan'''(x) = (q + q^2) \\ &+ (2q + 2q^2)\tan(qx)\tan(q^2x) \\ &+ (1 + q)\tan(qx)^2 \\ &+ (q^2 + q^3)\tan(q^2x)^2 \\ &+ (q + q^2)\tan(qx)^2\tan(q^2x)^2 \\ &+ q\tan(qx)^3\tan(q^2x) \\ &+ \tan(x)\tan(qx)^3 \\ &+ q^3\tan(q^2x)^3\tan(q^3x) \\ &+ q^2\tan(qx)\tan(q^2x)^3\end{aligned}$$

Main Theorems

Theorem. (Prototype)

Let

$$A_{n,k,a,b}(q) = \cdots$$

Then

$$D_q^n \tan_q(u) = \sum_{k,a,b} A_{n,k,a,b}(q) \left(\tan_q(q^{k+1}u)\right)^b \left(\tan_q(q^k u)\right)^a$$

Main Theorems

Theorem. (Prototype)

For each composition $\mathbf{c} = (c_0, c_1, c_2, \ldots, c_m)$ *of* n *let*

$$A_{n,\mathbf{c}}(q) = \cdots$$

Then,

$$D_q^n \tan_q(u) = \sum_{\mathbf{c}} A_{n,\mathbf{c}}(q) \tan_q(q^{\mathbf{c}} u)$$

where

$$\tan_q(q^{\mathbf{c}} u) := \tan_q(q^{c_0} u) \tan_q(q^{c_0+c_1} u) \cdots \tan_q(q^{c_0+c_1+\cdots+c_{m-1}} u).$$

Main Theorems

t-composition

$$\mathbf{c} = (c_0, c_1, \dots, c_{m-1}, c_m)$$

$$\begin{aligned} c_0 &\geq 0 \\ c_1, c_2, \dots, c_{m-1} &\geq 1 \\ c_m &\geq 0 \end{aligned}$$

(0) $c_0 + c_1 + \cdots + c_m = n$;

(1) either $m = 0$, so that $\mathbf{c} = (n)$ and n is an *odd* integer,

or $m \geq 1$ and both c_0, c_m are *even*;

(2) if $m \geq 2$, then all the parts c_1, c_2, ... , c_{m-1} are *odd*.

Notation $\mu\,\mathbf{c} := m$

Main Theorems

The set of all t-compositions of n is denoted by Θ_n.

Θ_0: $(0,0)$;

Θ_1: (1), $(0,1,0)$;

Θ_2: $(0,2)$, $(2,0)$, $(0,1,1,0)$;

Θ_3: (3), $(0,1,2)$, $(2,1,0)$, $(0,3,0)$, $(0,1,1,1,0)$;

Θ_4: $(0,4)$, $(2,2)$, $(4,0)$, $(0,3,1,0)$, $(0,1,3,0)$, $(0,1,1,2)$, $(2,1,1,0)$, $(0,1,1,1,1,0)$.

$(\operatorname{Card}\Theta_n) = (1,2,3,5,8,\ldots)$ the Fibonacci sequence

Main Theorems

A word $w = y_1 y_2 \cdots y_m$, whose letters are positive integers, is said to be *falling alternating*, or simply *alternating* , if

$$y_1 > y_2,\ y_2 < y_3,\ y_3 > y_4,\ \ldots$$

It is said to be *rising alternating* if

$$y_1 < y_2,\ y_2 > y_3,\ y_3 < y_4,\ \ldots$$

The length of each word w is denoted by λw

The *empty word* is denoted by ϵ

Main Theorems

tangent and *secant numbers* T_{2n+1} and E_{2n} occur as coefficients in the Taylor expansions of $\tan u$ and $\sec u$:

$$\tan u = \sum_{n\geq 0} \frac{u^{2n+1}}{(2n+1)!} T_{2n+1} = \frac{u}{1!}1 + \frac{u^3}{3!}2 + \frac{u^5}{5!}16 + \frac{u^7}{7!}272 + \cdots$$

$$\sec u = \frac{1}{\cos u} = \sum_{n\geq 0} \frac{u^{2n}}{(2n)!} E_{2n} = 1 + \frac{u^2}{2!}1 + \frac{u^4}{4!}5 + \frac{u^6}{6!}61 + \cdots$$

Désiré André (1879):

The number of falling (resp. rising) alternating permutations $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ of $1\,2\,\ldots\,n$ is equal to T_n when n is odd, and to E_n when n is even.

Main Theorems: t-permutations

Definition. A *t-permutation* of order n is defined to be a nonempty sequence $w = (w_0, w_1, \ldots, w_m)$ of words having the properties:

(i) the juxtaposition product $w_0 w_1 \cdots w_m$ is a permutation of $12 \cdots n$

(ii) $(\lambda w_0, \lambda w_1, \ldots, \lambda w_m)$ is a t-composition

(iii) w_0 is rising alternating, and $w_1, w_2, \ldots, w_m$ are (falling) alternating

Notation $\mu\, w := m$

Main Theorems: t-permutations

The set of all t-permutations of order n will be denoted by $\mathcal{T}_n$.

$\mathcal{T}_0$: (ϵ, ϵ) ;

$\mathcal{T}_1$: (1); $(\epsilon, 1, \epsilon)$;

$\mathcal{T}_2$: $(\epsilon, 21)$, $(12, \epsilon)$; $(\epsilon, 2, 1, \epsilon)$, $(\epsilon, 1, 2, \epsilon)$;

$\mathcal{T}_3$: (132), (231);
$(\epsilon, 3, 21)$, $(\epsilon, 312, \epsilon)$, $(\epsilon, 213, \epsilon)$, $(12, 3, \epsilon)$,
$(\epsilon, 1, 32)$, $(23, 1, \epsilon)$, $(13, 2, \epsilon)$, $(\epsilon, 2, 31)$;
$(\epsilon, 1, 2, 3, \epsilon)$, $(\epsilon, 1, 3, 2, \epsilon)$, $(\epsilon, 2, 1, 3, \epsilon)$, $(\epsilon, 2, 3, 1, \epsilon)$,
$(\epsilon, 3, 1, 2, \epsilon)$, $(\epsilon, 3, 2, 1, \epsilon)$.

Main Theorems: statistics

statistics on permutations:

The *descent set* of a permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ of $12\cdots n$, denoted by $\operatorname{DES}\sigma$, is defined to be the set of all i such that $1 \le i \le n-1$ and $\sigma(i) > \sigma(i+1)$;

$$\operatorname{ides}\sigma = \#\operatorname{DES}\sigma^{-1};$$
$$\operatorname{imaj}\sigma = \sum_i \sigma(i) \quad (\sigma(i) \in \operatorname{DES}\sigma^{-1});$$

Let $\operatorname{inv}\sigma$ be the *number of inversions* of σ, as being the number of pairs (i, j) such that $1 \le i < j \le n$ and $\sigma(i) > \sigma(j)$.

Main Theorems

statistics on t-permutations:

t-permutation $w = (w_0, w_1, \ldots, w_m)$

$$\mathrm{IDES}\, w := \mathrm{IDES}(w_0 w_1 \cdots w_m)$$
$$\mathrm{ides}\, w := \mathrm{ides}(w_0 w_1 \cdots w_m)$$
$$\mathrm{imaj}\, w := \mathrm{imaj}(w_0 w_1 \cdots w_m)$$
$$\mathrm{inv}\, w := \mathrm{inv}(w_0 w_1 \cdots w_m)$$

(the juxtaposition product is a permutation)

$$\min w := a \text{ if 1 is a letter in } w_a$$

Main Theorems

Example. With the t-permutation

$$w = (4\ 5,\ 11\ 1\ \mathbf{3},\ \mathbf{10}\ 7\ \mathbf{9},\ \mathbf{6},\ \mathbf{8}\ \mathbf{2}),$$

The inverse descents being reproduced in boldface, we have:

$$\begin{aligned} \operatorname{ides} w &= 6 \\ \operatorname{imaj} w &= \mathbf{3} + \mathbf{10} + \mathbf{9} + \mathbf{6} + \mathbf{8} + \mathbf{2} = \mathbf{38} \\ \operatorname{inv} w &= 27 \\ \min w &= 1 \qquad \text{since } w_1 = (11\ 1\ \mathbf{3}) \end{aligned}$$

Main Theorems

Theorem (Multivariable q-analog). *Let*

$$A_{n,k,a,b}(q) = \sum_{\substack{w \in \mathcal{T}_n \\ \operatorname{ides} w = k,\ \min w = a,\ a+b = \mu\, w}} q^{\operatorname{imaj} w}.$$

Then

$$D_q^n \tan_q(u) = \sum_{k,a,b} A_{n,k,a,b}(q) \left(\tan_q(q^{k+1}u)\right)^b \left(\tan_q(q^k u)\right)^a,$$

where $0 \leq k \leq n-1$, $0 \leq a+b \leq n+1$.

Main Theorems

Theorem (Composition q-analog). *For each t-composition $\mathbf{c}$ of n let*

$$A_{n,\mathbf{c}}(q) = \sum_{\substack{w \in \mathcal{T}_n, \\ (\lambda w_0, \lambda w_1, \ldots, \lambda w_m) = \mathbf{c}}} q^{\operatorname{inv} w}.$$

Then,

$$D_q^n \tan_q(u) = \sum_{\mathbf{c} \in \Theta_n} A_{n,\mathbf{c}}(q) \tan_q(q^{\mathbf{c}} u),$$

where

$$\tan_q(q^{\mathbf{c}} u) := \tan_q(q^{c_0} u) \tan_q(q^{c_0+c_1} u) \cdots \tan_q(q^{c_0+c_1+\cdots+c_{m-1}} u).$$

There is also a theory for $\sec_q$ and Sec_q. The underlying polynomials being $B_{n,k,a,b}(q)$ and $B_{b,\mathbf{c}(q)}$.

q-derivative polynomials

$$A_n(x,q) := \sum_{m\geq 0} x^m \sum_{k\geq 0,\, a+b=m} A_{n,k,a,b}(q)$$

$$B_n(x,q) := \sum_{m\geq 0} x^m \sum_{k\geq 0,\, a+b=m} B_{n,k,a,b}(q)$$

Main Theorems

Theorem (q-Hoffman).

$$\sum_{n\geq 0} A_n(x,q)\frac{u^n}{(q;q)_n} = \tan_q(u) + \frac{\sec_q(u)\times x\,\mathrm{Sec}_q(u)}{1-x\tan_q(u)},$$

$$\sum_{n\geq 0} B_n(x,q)\frac{u^n}{(q;q)_n} = \frac{\sec_q(u)}{1-x\tan_q(u)}.$$

Hoffman:

$$\sum_{n\geq 0} A_n(x)\frac{u^n}{n!} = \tan(u) + \frac{\sec(u)\times x\sec(u)}{1-x\tan(u)},$$

$$\sum_{n\geq 0} B_n(x)\frac{u^n}{n!} = \frac{\sec(u)}{1-x\tan(u)}.$$

arxiv: soon
http://www-irma.u-strasbg.fr/~guoniu/papers/

Thank you!