A Pieri formula for double Grothendieck polynomials

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SLC 68

Grothendieck polynomials

We generate a family of polynomials by applying operators to a starting point:

$$G_{(\omega)} := \prod_{\substack{i=1\ldots n \\ j=1\ldots n-i}} (1 - y_j x_i^{-1}),$$

$$\omega = [n, n-1, \ldots, 1]$$

Divided differences

For $1 \leq i < n$

$$f\partial_i := \frac{f - f^{s_i}}{x_i - x_{i+1}}$$

$$f\pi_i := x_i\partial_i = \frac{x_i.f - x_{i+1}.f^{s_i}}{x_i - x_{i+1}}$$

$$f\hat{\pi}_i := \partial_i x_{i+1} = \frac{x_{i+1}.f - x_{i+1}.f^{s_i}}{x_i - x_{i+1}}$$

Braid relations

$$\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1} \qquad \hat{\pi}_i \hat{\pi}_{i+1} \hat{\pi}_i = \hat{\pi}_{i+1} \hat{\pi}_i \hat{\pi}_{i+1}$$

$$\pi_i \pi_j = \pi_j \pi_i \qquad \hat{\pi}_i \hat{\pi}_j = \hat{\pi}_j \hat{\pi}_i, |i - j| > 1$$

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Quadratic relations

$$\pi_i \pi_i = \pi_i$$

$$\hat{\pi}_i \hat{\pi}_i = -\hat{\pi}_i$$

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Quadratic relations

$$\pi_i \pi_i = \pi_i \qquad \hat{\pi}_i \hat{\pi}_i = -\hat{\pi}_i$$

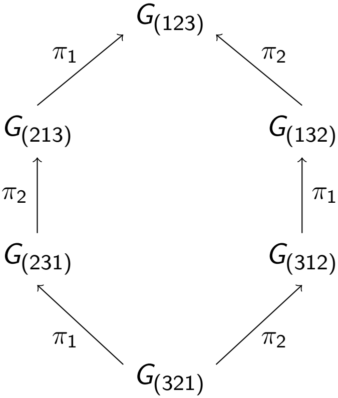
And...

$$\hat{\pi}_i = \pi_i - 1$$

Grothendieck polynomials

$$G_{(\sigma s_i)} := G_{(\sigma)}\pi_i,$$

if $\sigma(i) > \sigma(i+1)$.



Pieri formula

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$$\equiv G_{(\sigma)} - \frac{y_1 \dots y_k}{y_{\sigma_1} \dots y_{\sigma_k}} \sum \pm G_\mu \quad (Lenart - Postnikov)$$

where the sum is over chains of the k-Bruhat order

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New result:

$$G_{(\sigma)} G_{(s_k)} \equiv G_{(\sigma)} - \frac{y_1 \dots y_k}{y_{\sigma_1} \dots y_{\sigma_k}} \sum_{\sigma \leq \mu \leq \eta(\sigma,k)} \pm G_\mu$$

Theorem

(Lascoux)

$$G_{(\sigma)}\frac{y_{\sigma_1}\cdots y_{\sigma_k}}{x_1\cdots x_k} \equiv G_{(\omega)}\hat{\pi}_{\omega\zeta}\pi_{\zeta^{-1}\sigma}$$

where $\sigma \in \mathfrak{S}_{\mathfrak{n}}$, $1 \leq k < n$, and ζ is a specific representative element of the class of σ in $\mathfrak{S}_n/(\mathfrak{S}_k \times \mathfrak{S}_{n-k})$.

$$\mu = s_{i_1}\dots s_{i_m} \rightarrow \pi_\mu = \pi_{i_1}\dots\pi_{i_m}$$

$$\left.\begin{array}{l}\sigma = 43678215 \\ k = 3\end{array}\right\}\zeta = 643|87521$$

Formal bases

$$K := \{K_\sigma ; \sigma \in \mathfrak{S}_n\}$$

$$K_\sigma \pi_i = \begin{cases} K_{\sigma s_i} & \text{if } \sigma_i > \sigma_{i+1} \\ K_\sigma & \text{otherwise} \end{cases}$$

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$$\omega = [n, n-1, \ldots, 2, 1]$$

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$$\rightarrow K_\omega \hat{\pi}_{\omega\zeta} \pi_{\zeta^{-1}\sigma}$$

Change of basis

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$$\sigma = s_1 s_2 s_1$$

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Change of basis

$$
\begin{aligned}
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&= (\hat{\pi}_1 + 1)(\hat{\pi}_2 + 1)(\hat{\pi}_1 + 1) \\
&= \hat{\pi}_1\hat{\pi}_2\hat{\pi}_1 + \hat{\pi}_1\hat{\pi}_2 + \hat{\pi}_1\hat{\pi}_1 + \hat{\pi}_1 + \hat{\pi}_2\hat{\pi}_1 + \hat{\pi}_2 + \hat{\pi}_1 + 1
\end{aligned}
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Change of basis

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Change of basis

$$\begin{aligned}\sigma &= s_{i_1} \dots s_{i_m} \\ \pi_\sigma &= \pi_{i_1} \dots \pi_{i_m} \\ &= (\hat{\pi}_{i_1} + 1)(\hat{\pi}_{i_2} + 1) \dots (\hat{\pi}_{i_m} + 1) \\ &= \sum_{\mu} \hat{\pi}_\mu\end{aligned}$$

where a reduced decomposition of μ is a subword of a reduced decomposition of σ.(Lascoux)

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$\rightarrow \mu \leq \sigma$ for the Bruhat order of permutations

Bruhat order

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An order on permutations, graded by the number of inversions

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Let $\sigma \in \mathfrak{S}_n$, μ is a successor of σ iff :

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- $\ell(\mu) = \ell(\sigma) + 1$

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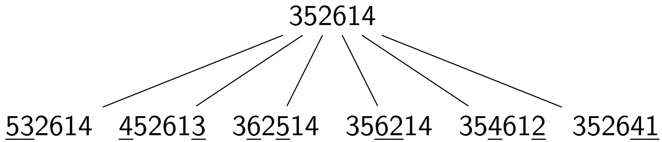
k-Bruhat

$\tau = (a, b), a \leq k, b > k$
τ is a *k-Bruhat transposition* for σ.

Example:



Example:

352614

$\underline{53}2614$ $\underline{4}5261\underline{3}$ $3\underline{6}2\underline{5}14$ $35\underline{62}14$ $35\underline{4}61\underline{2}$ $3526\underline{41}$

Counterexample: $\underline{6}52\underline{3}14$ is not a successor, added inversions:

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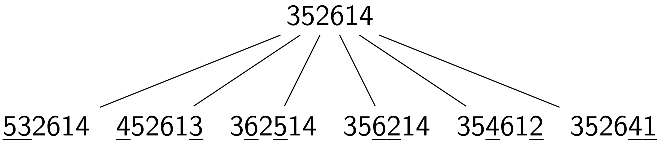

Counterexample: 652314 is not a successor, added inversions: (6,3)

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Successors : $..b...d.. \to ..d...b..$ where values between b and d are $< b$ or $> d$

Example:

$35|2614$

$\underline{53}|2614$ $\underline{4}5|261\underline{3}$ $3\underline{6}|2\underline{5}14$ $35|\underline{62}14$ $35|\underline{4}61\underline{2}$ $35|26\underline{41}$

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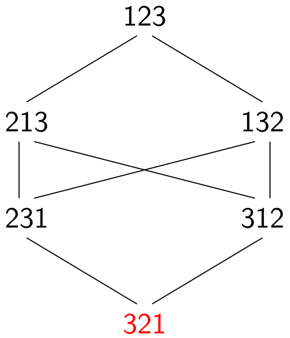
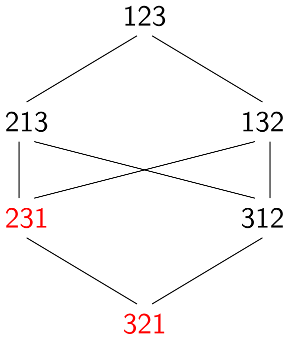
k-Successors

Change of basis : K, $\hat{K}$



$$\hat{K}_{321} = K_{321}$$

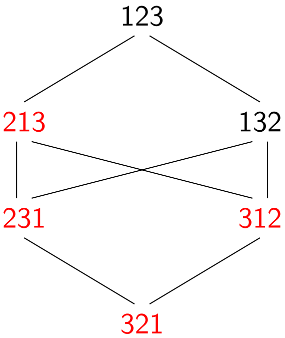
Change of basis : K, $\hat{K}$



$$\hat{K}_{321} = K_{321}$$

$$\hat{K}_{231} = \hat{K}_{321}\hat{\pi}_1 = K_{321}(\pi_1 - 1) = K_{231} - K_{321}$$

Change of basis : K, $\hat{K}$



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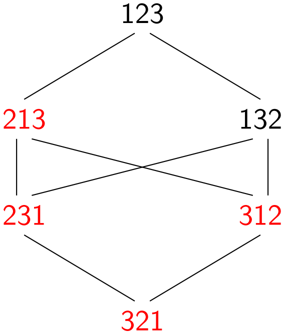
$$\hat{K}_{213} = \hat{K}_{231}\hat{\pi}_2 = K_{231}\hat{\pi}_2 - K_{321}\hat{\pi}_2$$

$$= K_{213} - K_{231} - K_{312} + K_{321}$$

Change of basis : K, $\hat{K}$



$$\begin{aligned}
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\hat{K}_{231} &= \hat{K}_{321}\hat{\pi}_1 = K_{321}(\pi_1 - 1) \\
&= K_{231} - K_{321} \\
\hat{K}_{213} &= \hat{K}_{231}\hat{\pi}_2 = K_{231}\hat{\pi}_2 - K_{321}\hat{\pi}_2 \\
&= K_{213} - K_{231} - K_{312} + K_{321}
\end{aligned}$$

$$\hat{K}_\sigma = \sum_{\mu \geq \sigma} (-1)^{\ell(\mu) - \ell(\sigma)} K_\mu$$

$$K_\sigma = \sum_{\mu \geq \sigma} \hat{K}_\mu$$

Expansion in the K basis of:

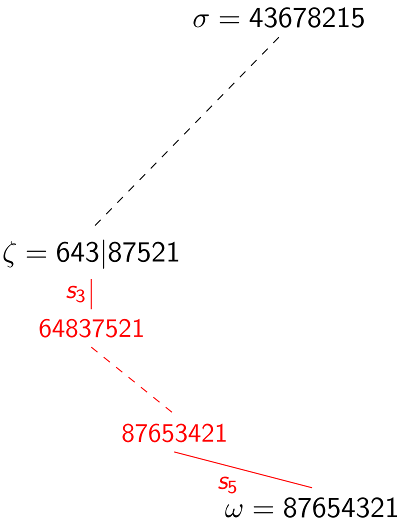
$$K_\omega \hat{\pi}_{\omega\zeta} \pi_{\zeta^{-1}\sigma}$$

$\sigma = 43678215$

K_ω

$\zeta = 643|87521$

$\omega = 87654321$



$K_\omega \hat{\pi}_{\omega\zeta}$

$\sigma = 43678215$

s_2

46378215

64387251

s_6

$\zeta = 643|87521$

s_3

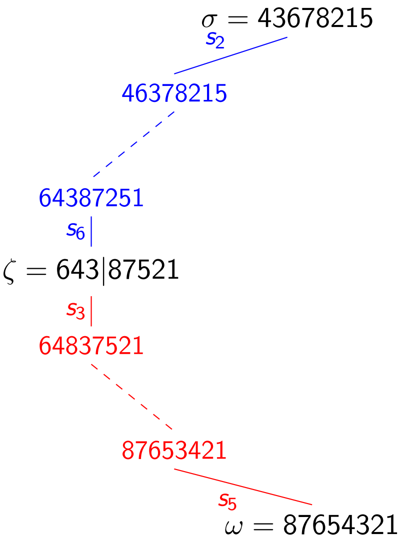
64837521

87653421

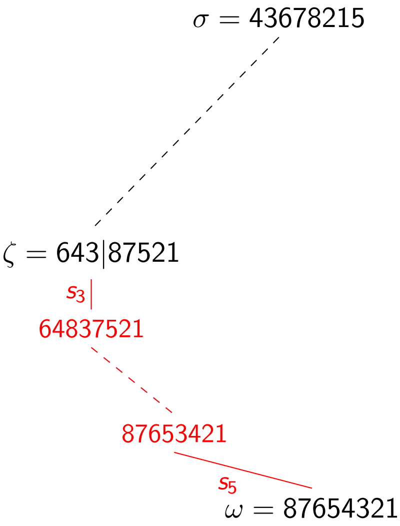
s_5

$\omega = 87654321$

$$K_\omega \hat{\pi}_{\omega\zeta} \pi_{\zeta^{-1}\sigma}$$



$$K_\omega \hat{\pi}_{\omega\zeta} \pi_{\zeta^{-1}\sigma} = ?$$



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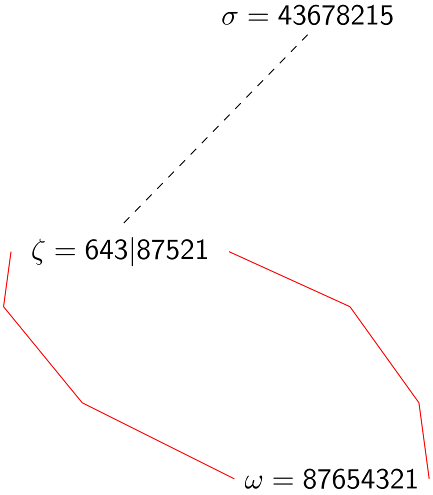

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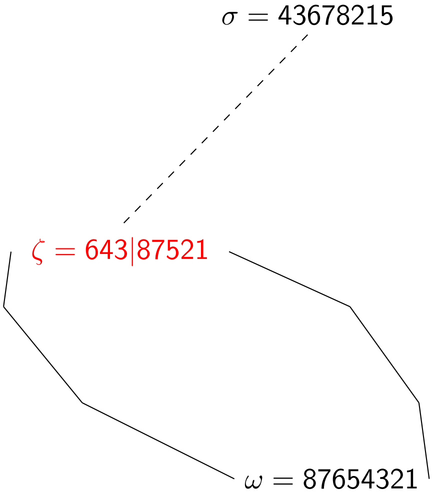

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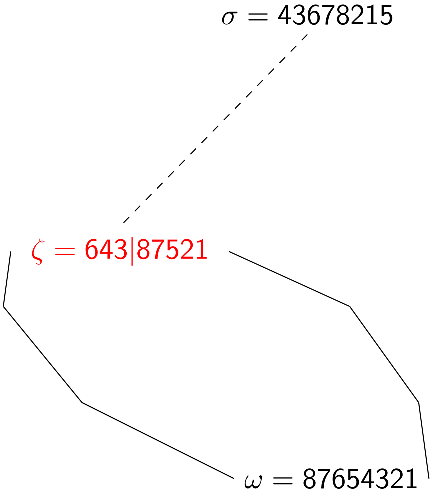

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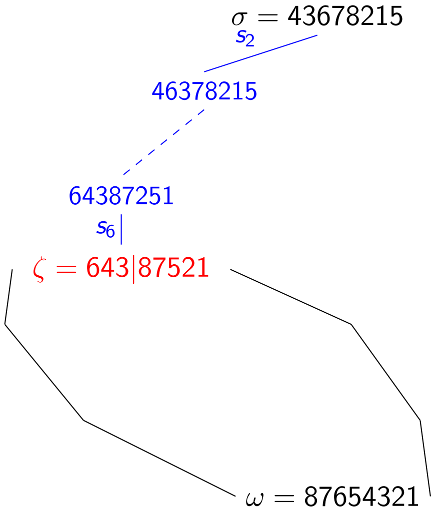
$\hat{K}_\zeta$



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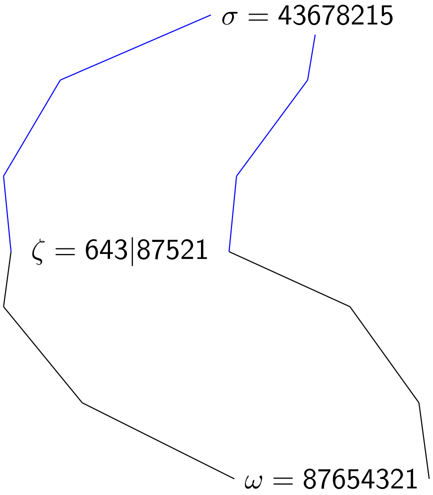
$$\hat{K}_\zeta \pi_{\zeta^{-1}\sigma}$$



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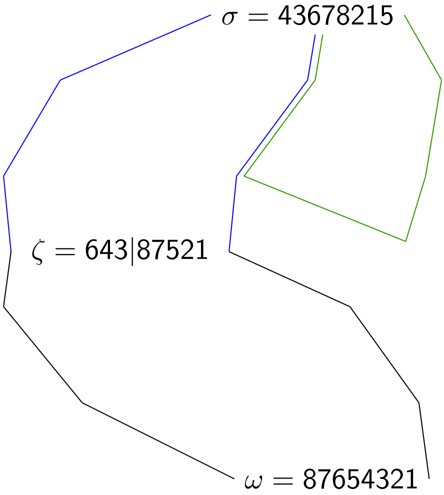

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$$= \hat{K}_\zeta$$

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$$= \sum_\nu \pm K_\nu$$

A sum on chains of the k-Bruhat order

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Let $W_{\sigma,k}$ be the list of k-Bruhat transpositions (a, b) of σ in decreasing order on $\sigma(a)$ then increasing order on $\sigma(b)$.

$$\sigma = 1362|54$$

$$W_{\sigma,k} =$$

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$$W_{\sigma,k} = ((2, 6)$$

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$$W_{\sigma,k} = (\tau_1, \tau_2, \ldots, \tau_m)$$

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$$\mathfrak{E}_{\sigma,k} := K_\sigma \cdot (1 - \tau_1) \cdot (1 - \tau_2) \cdots (1 - \tau_m)$$

with:

$$K_\mu \cdot \tau = \begin{cases} K_{\mu\tau} \text{ if } \tau \text{ is a Bruhat transposition of } \mu, \\ 0 \text{ otherwise.} \end{cases}$$

$$+K_{1362|54}$$

$$W_{\sigma,k} = ((2,6),(2,5),(4,6),(4,5))$$

$+K_{1362|54}$

(2,6)

$-K_{1462|53}$

$$W_{\sigma,k} = ((2,6),(2,5),(4,6),(4,5))$$

$$+K_{1362|54}$$

(2,6)

$$-K_{1462|53}$$

(2,5)

$$+K_{1562|43}$$

$$W_{\sigma,k} = ((2,6),(2,5),(4,6),(4,5))$$

$+K_{1362|54}$

(2,6)

$-K_{1462|53}$

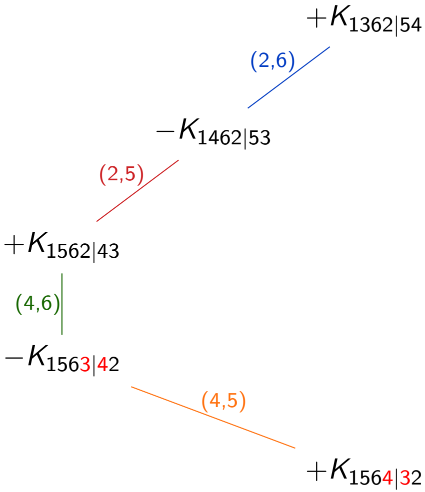
(2,5)

$+K_{1562|43}$

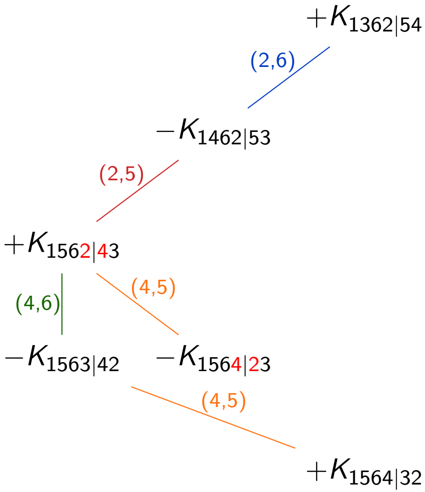
(4,6)

$-K_{1563|42}$

$$W_{\sigma,k} = ((2,6),(2,5),(4,6),(4,5))$$



$$W_{\sigma,k} = ((2,6),(2,5),(4,6),(4,5))$$



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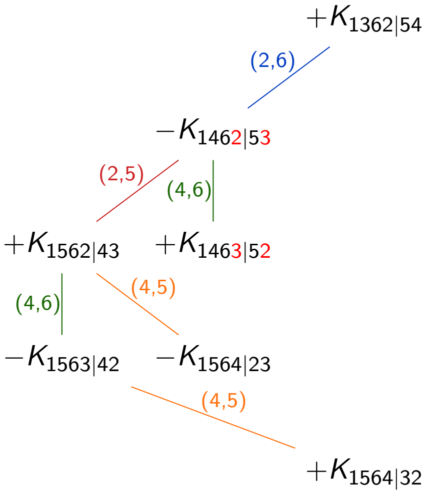

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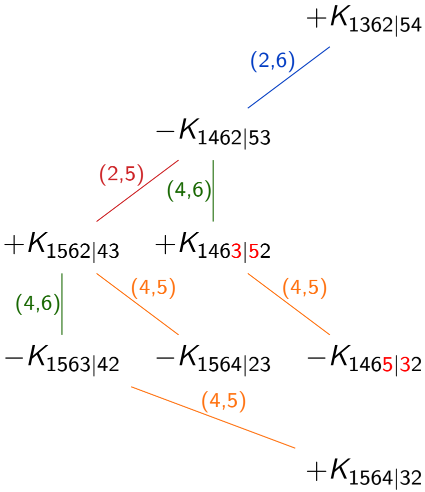

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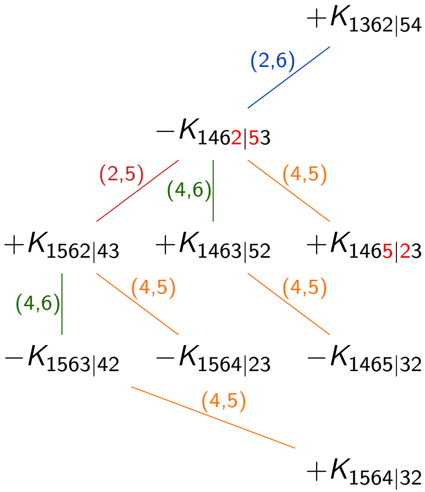

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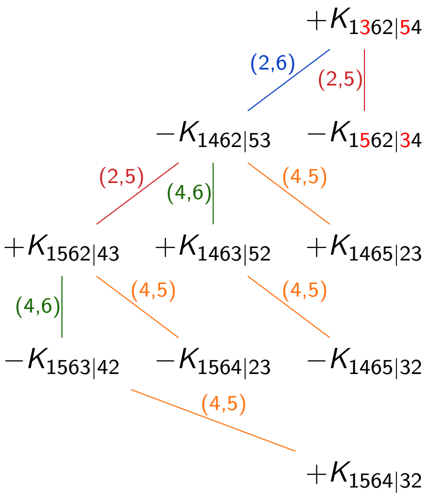

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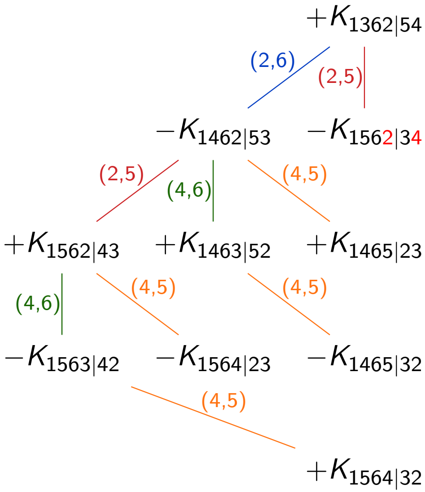

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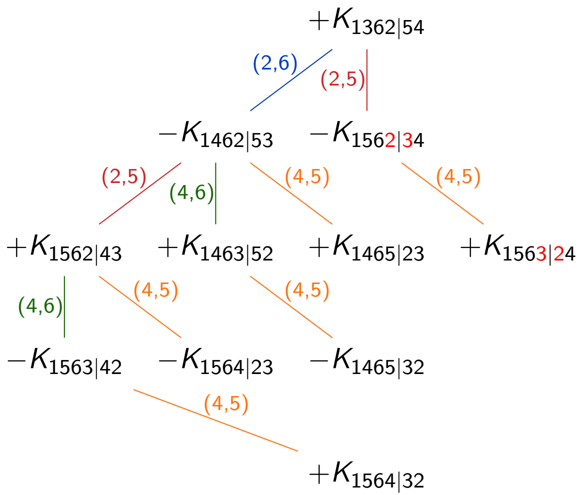

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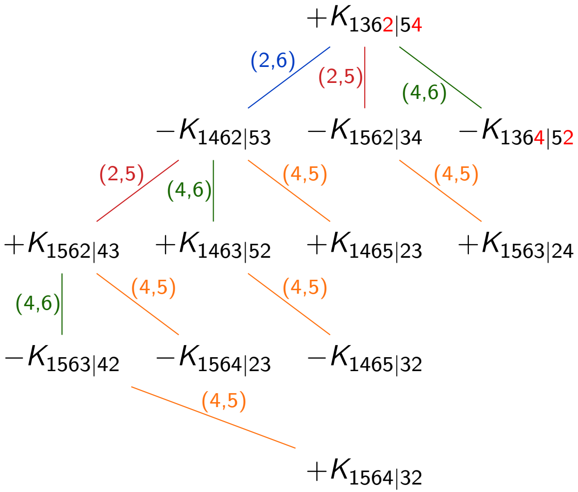

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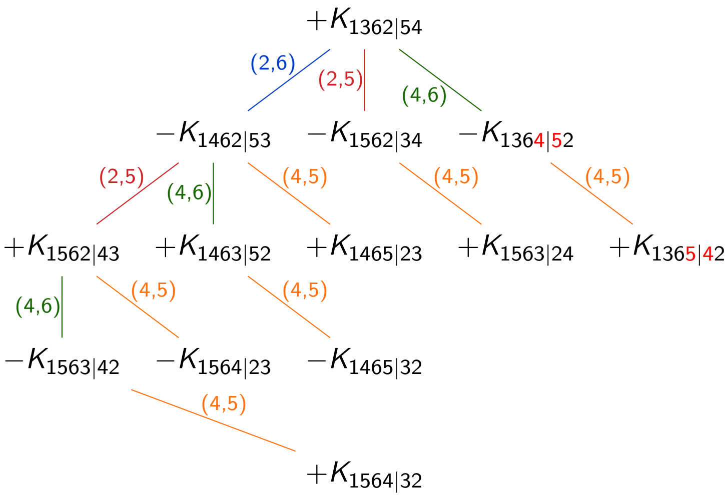

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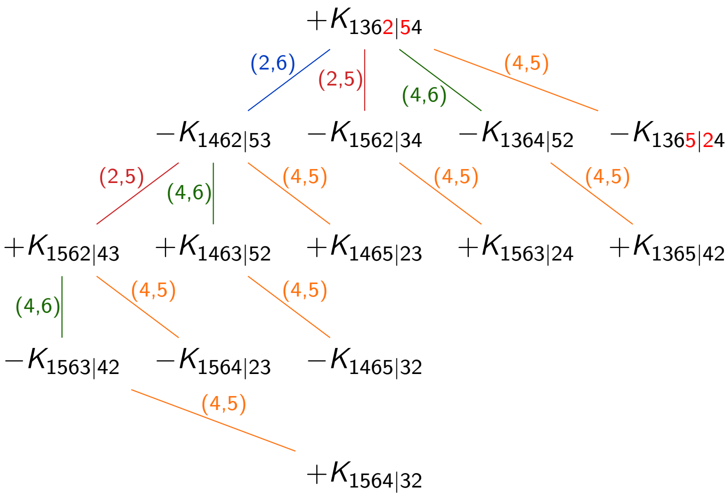

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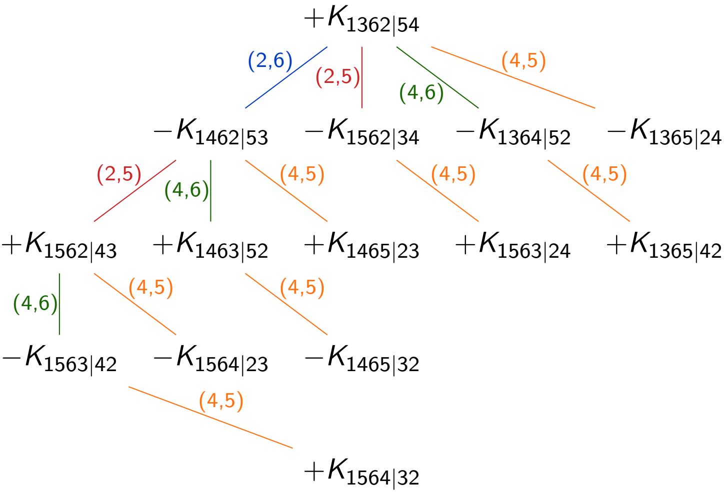

$$W_{\sigma,k} = ((2,6),(2,5),(4,6),(4,5))$$

$$K_\omega \hat{\pi}_{\omega\zeta} \pi_{\zeta^{-1}\sigma} = \mathfrak{E}_\sigma$$

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Direct proof :

$$K_\omega \hat{\pi}_{\omega\zeta} \pi_{\zeta^{-1}\sigma} = \left(\sum_{\mu \geq \zeta} \pm K_\mu \right) \pi_{\zeta^{-1}\sigma}$$

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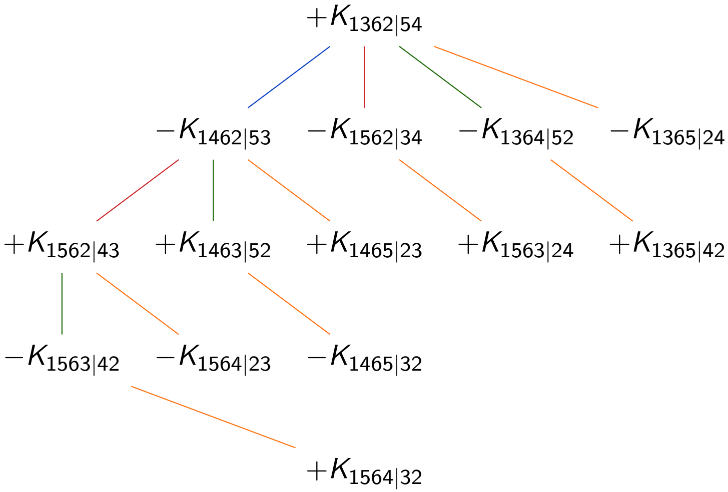
$$= \mathfrak{E}_{\zeta,k} \pi_{\zeta^{-1}\sigma}$$

Interval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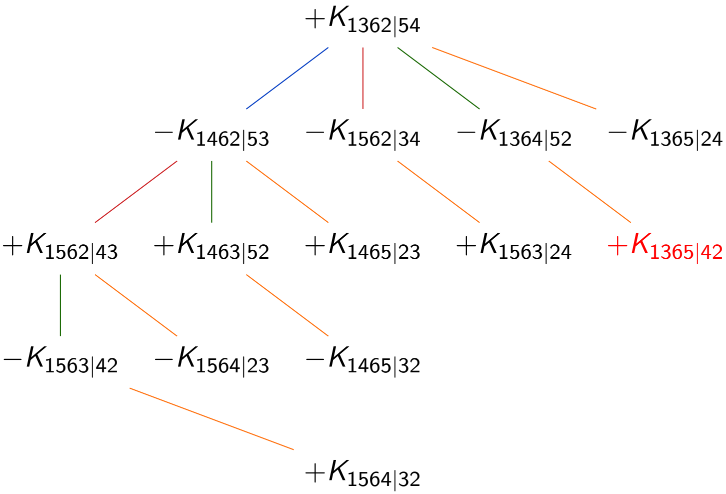
Interval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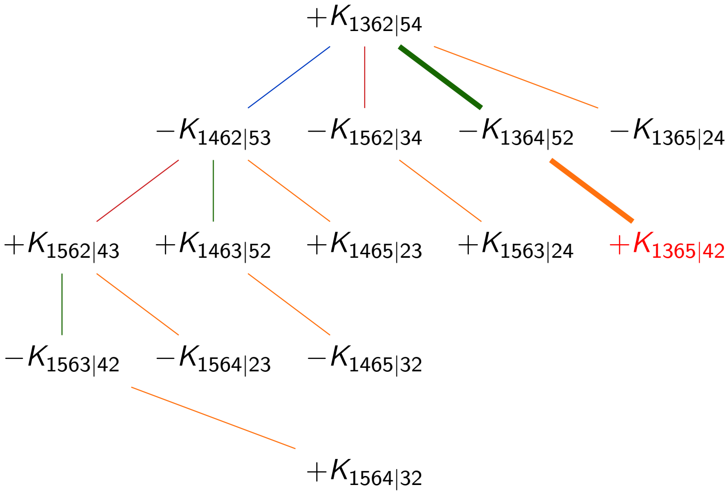
Interval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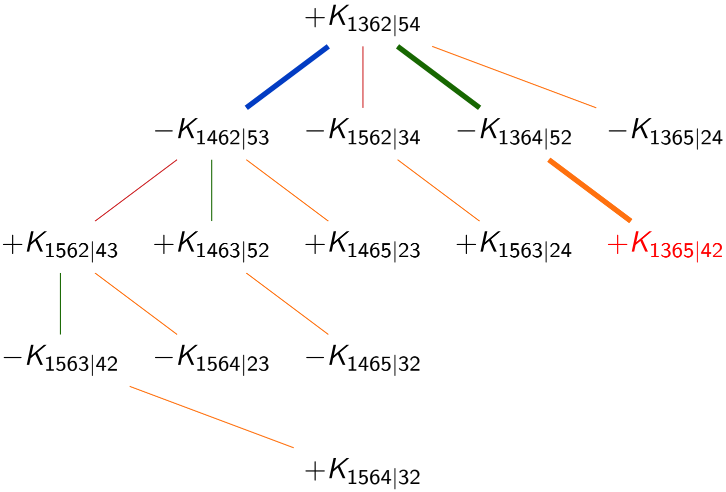
Interval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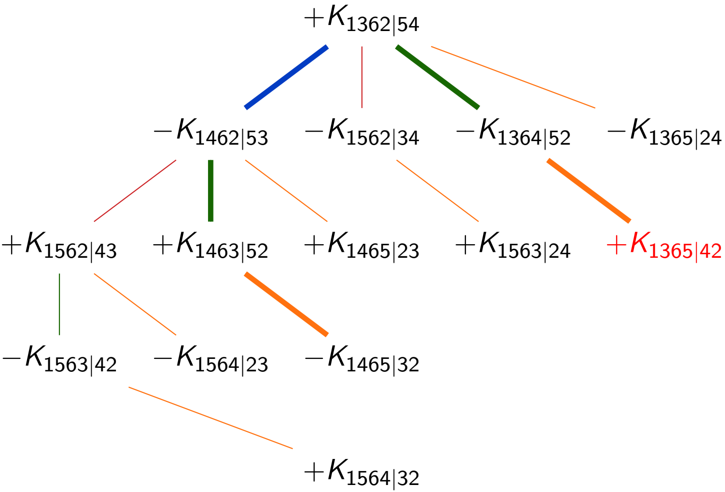
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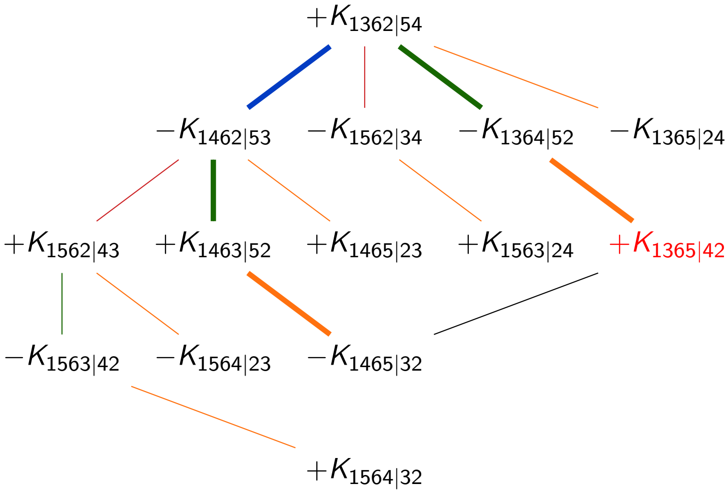
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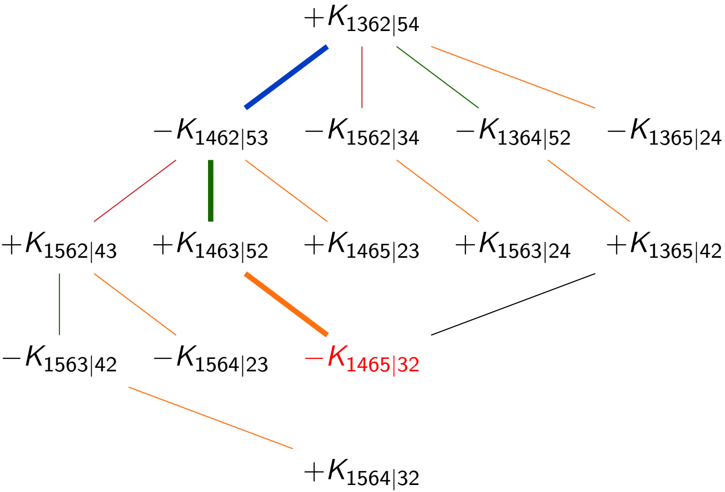
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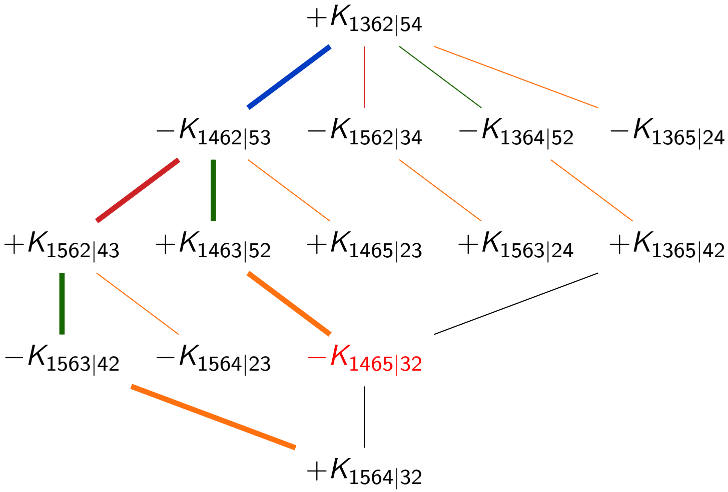
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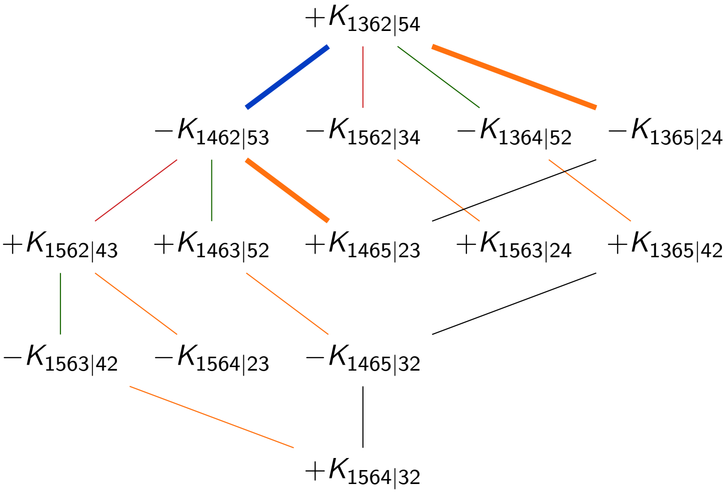
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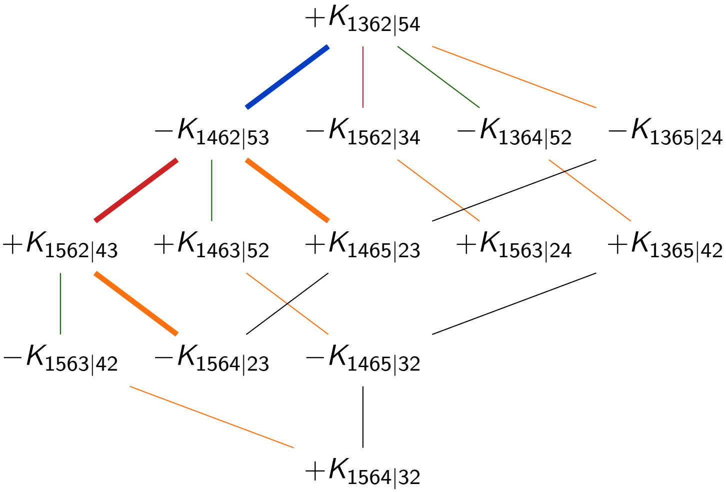
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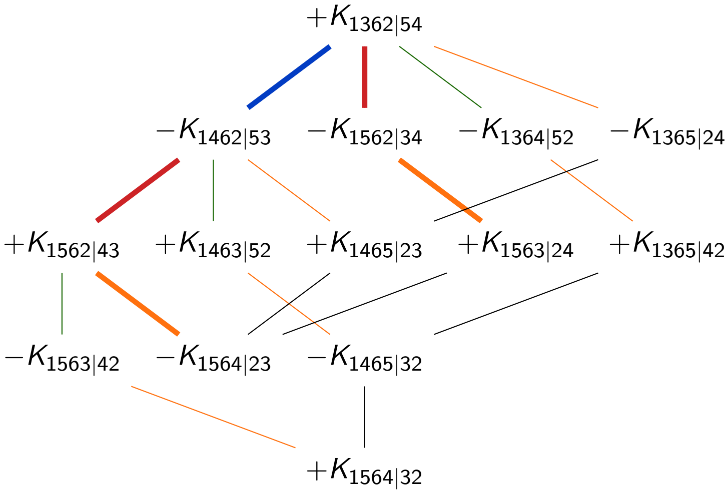
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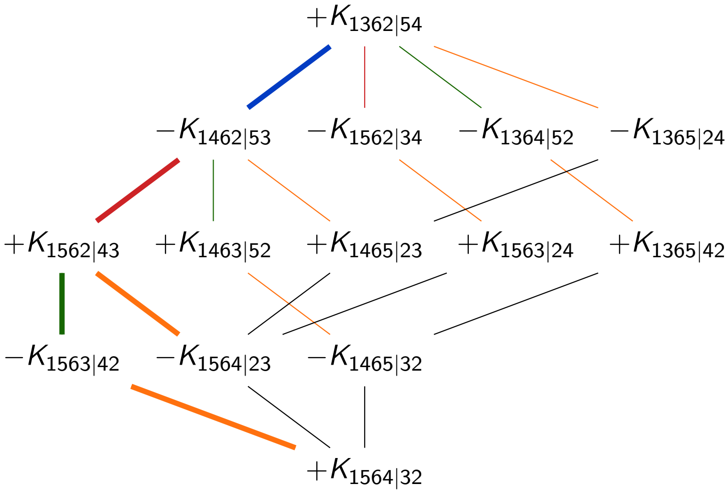
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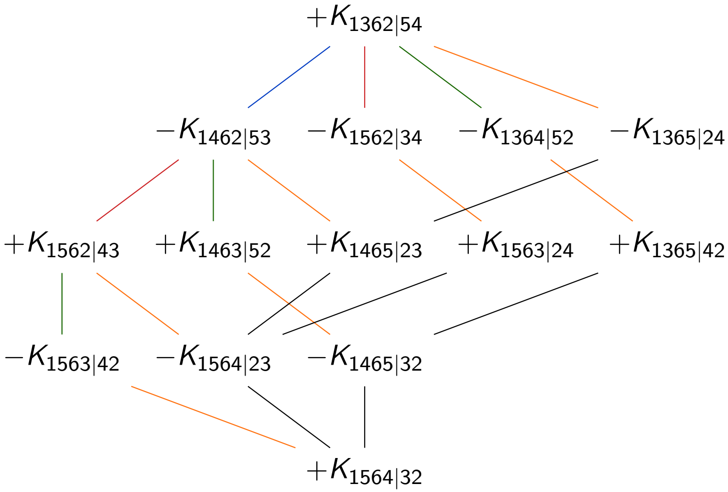
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