Universal behavior of context-free grammars: complete characterization of the critical exponent

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Plan

1. Introduction
2. Algebraic grammar
   - Definitions
   - Asymptotics
   - Critical exponent of strongly connected graph
   - Critical exponent of non-strongly connected graph
3. Conclusion

Motivations

- Grammars are a fundamental structure in computer science: information theory, language theory, compilation, bioinformatics, combinatorics (Schützenberger methodology)...
- Challenge: is it easy to test if a given generating function is $\mathbb{N}$-algebraic? (i.e. is it associated to a context-free grammar?)
- Can we find an easy criterion?

$1-(1-4z)^{1/3}+O((1-4z)^{2/3})$, $1-(1-4z)^{1/4}+O((1-4z)^{3/4})$.

Flajolet and Sedgewick, Analytic Combinatorics, p.493:
"It would at least be desirable to determine directly,
from a positive (but *reducible*) system,
the type of singular behaviour of the solution,
but the systematic research involved in such a programme
is yet to be carried out."

Plan

Combinatorial object	Specification	Grammar
Super-bicolored trees	$\mathcal{B} = 2z\mathcal{T}\,\mathrm{Seq}(\mathcal{B})$	$\begin{cases} \mathcal{B} \to a \cdot \mathcal{T} \cdot \mathcal{U} \mid b \cdot \mathcal{T} \cdot \mathcal{U} \\ \mathcal{U} \to \mathcal{B} \cdot \mathcal{U} \mid \epsilon \\ \mathcal{T} \to a \cdot \mathcal{T} \cdot \mathcal{A} \mid a \\ \mathcal{A} \to \mathcal{T} \cdot \mathcal{A} \mid \epsilon \end{cases}$

- Closure properties: union, concatenation, star, shuffle...
- Non closure properties: complement, intersection
  ($\Rightarrow$ no minus sign!)
- The generating function of an algebraic grammar is called $\mathbb{N}$-algebraic. (reminiscent of the set of $\mathbb{N}$-rational generating functions associated to automata.)

Algebraic branches

Combinatorial object	Algebraic grammar	Functional equation	Branches
Binary trees	$T \to a \cdot T \cdot T \mid a$	$T(z) = z + zT^2(z)$	$\begin{cases} \frac{1+\sqrt{1-4z^2}}{2z} \\ \frac{1-\sqrt{1-4z^2}}{2z} \end{cases}$

Algebraic grammar with m non-terminals (well-founded system):

1. Polynomial system of equations $\{Y_j = P_j(z, Y_1, \ldots, Y_m)\}_j$,
2. P_j have nonnegative coefficients,
3. $\overrightarrow{Y^{(0)}} = (0, \ldots, 0)$, $\overrightarrow{Y^{(t+1)}} = P_j(\overrightarrow{Y^{(t)}})$, $\overrightarrow{Y} = \lim_{t \to \infty} \overrightarrow{Y^{(t)}}$

$\Rightarrow$ unique branch analytic with nonnegative Taylor coefficients around 0.

Asymptotics of algebraic grammars

Theorem: asymptotics of the function via Puiseux expansion [Newton-Puiseux]

Let $f(z)$ the branch of an algebraic equation $P(z, f(z)) = 0$, near its singularities ρ, $f(z)$ admits a convergent fractional series expansion of the form: $f(z) = \sum_{k \geq k_0} c_k \,.\, (z-\rho)^{\frac{k}{K}}$ with $k_0 \in \mathbb{Z}$ and $K \geq 1$.

Theorem: asymptotics of coefficients via singularity analysis [Darboux, Flajolet–Odlyzko]

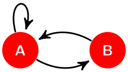
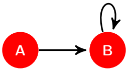
$f(z) \sim c_0 + C.(z-\rho)^{\alpha}$

$$\Rightarrow f_n \sim C \frac{1}{\Gamma(-\alpha)} \rho^{-n} n^{\gamma} \text{ with } \gamma = -\alpha - 1$$

What are the possible values of the "critical exponent" γ?

It is just known: $\gamma \in \mathbb{Q}/\{-\mathbb{N}\}$. (+problem also tackled by Schaeffer & Bousquet-Mélou)

A strongly connected graph is a directed graph that has a path from each vertex to every other vertex.

Language	Grammar	Dependency graph
$\mathcal{L} = \{(a^n.b^{2n})^*,\ n \geq 0\}$	$A \to aAB \mid \epsilon$ $B \to bbA \mid \epsilon$	 strongly connected graph.
$\mathcal{L} = \{a^n b^m a^m b^n,\ n \geq 0, m \geq 0\}$	$A \to aAb \mid B$ $B \to bBa \mid \epsilon$	 non-strongly connected graph.

Critical exponent of strongly connected graph

Theorem of [Drmota–Lalley–Woods, 97–93–97]

Let $\{Y_j = P_j(z, Y_1, \ldots, Y_m)\}_j$ a system satisfying:

1. all polynomials P_i have positive coefficients and $\frac{\partial^2 P_i}{\partial Y_i} \neq 0$;
2. the system admits a fixed point $\vec{Y}$;
3. the dependency graph of the system is strongly connected.

Then:

1. $Y(z) = C_0 - C.(1 - \frac{z}{\rho})^{\frac{1}{2}} + O(1 - \frac{z}{\rho})$ for $z \sim \rho$,
2. $[z^n]Y(z) \sim C.\frac{1}{2\sqrt{\pi}}.\rho^{-n}.n^{-\frac{3}{2}}$ with ρ, C_0, C algebraic numbers.

Examples

One single non-terminal $\Rightarrow$ universality of the $-3/2$ exponent (e.g., simple families of trees [Meir and Moon]).

$$\begin{cases} \mathcal{A} \to a \cdot \mathcal{A} | b \cdot \mathcal{B} \cdot \mathcal{B} | c \\ \mathcal{B} \to b \cdot \mathcal{A} | a \cdot \mathcal{B} \cdot \mathcal{B} | c \end{cases} \qquad \begin{cases} A(z) = zA(z) + zB(z)^2 + z \\ B(z) = zA(z) + zB(z)^2 + z \end{cases}$$

- Algebraic equation: $zA(z)^2 + (z-1)A(z) + z = 0$
- Branches of $A(z)$:

$$-\frac{z - 1 + \sqrt{-3z^2 - 2z + 1}}{2z}, \quad \frac{-z + 1 + \sqrt{-3z^2 - 2z + 1}}{2z}$$

- The singularities: $-1,\ 0, 1/3$
- Puiseux expansion: $A(z) = \frac{5}{2} - (3(1-3z))^{1/2} + O(1-3z)$

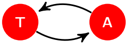
Sketch of proof [Drmota–Lalley–Woods Theorem, 97–93–97]

- Each component solution is an algebraic function with a positive radius of convergence
  (proof: combinatorial reason),
- each component has a unique branch with positive coefficients
  (proof: by construction, well-founded system),
- the Y_j's have the same dominant singularity
  (proof: if not, contradiction between $\lim_{x \to \rho} \partial_x^m Y_j(x) = \infty$),
- the Y_j's have a square root behavior
  (proof: via the implicit function theorem + Taylor expansion).

Critical exponent of non-strongly connected graph

general trees
$\mathcal{T} = z\,\mathrm{Seq}(\mathcal{T})$

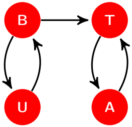
$$\begin{cases} \mathcal{T} \to a \cdot \mathcal{T} \cdot \mathcal{A} | a \\ \mathcal{A} \to \mathcal{T} \cdot \mathcal{A} | \epsilon \end{cases}$$



strongly connected.

super-bicolored trees:
$\mathcal{B} = \mathcal{T}\,[(z+z).\mathcal{T}]$
$\mathcal{B} = 2z\mathcal{T}\,\mathrm{Seq}(\mathcal{B})$

$$\begin{cases} \mathcal{B} \to a \cdot \mathcal{T} \cdot \mathcal{U} | b \cdot \mathcal{T} \cdot \mathcal{U} \\ \mathcal{U} \to \mathcal{B} \cdot \mathcal{U} | \epsilon \\ \mathcal{T} \to a \cdot \mathcal{T} \cdot \mathcal{A} | a \\ \mathcal{A} \to \mathcal{T} \cdot \mathcal{A} | \epsilon \end{cases}$$



non-strongly connected.

Branch: $B(z) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4z + 4z\sqrt{1-4z}}$

Puiseux expansion: $B(z) = \frac{1}{2} - \frac{1}{2}(1-4z)^{1/4} + O((1-4z)^{1/2})$

Can be automatized via the Algolib Maple library [Flajolet–Salvy–Zimmermann].

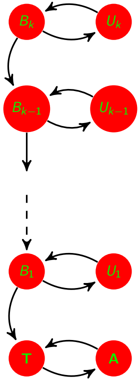
Explicit construction of critical behavior in $1/2^k$

- The critical behavior of bicolored trees $\mathcal{T}$ is $1/2$

- super-bicolored trees: $\mathcal{B} = \mathcal{T}[(z+z).\mathcal{T}]$
  - algebraic grammar,
  - schema of critical composition,
  - $\frac{1}{2^2}$ is the critical behavior.

- k-super-bicolored trees: $\mathcal{B}_k = \mathcal{T}[(z+z).\mathcal{B}_{k-1}]$
  - algebraic grammar,
  - schema of k nested critical compositions,
  - $\frac{1}{2^{k+1}}$ is the critical behavior.

Explicit construction of critical behavior in $1/2^k$ (proof)



- k-super-bicolored trees:
  $\mathcal{B}_k = \mathcal{T}\left[(z+z).\mathcal{B}_{k-1}\right]$
  $= ((z+z).\mathcal{B}_{k-1}.\,\mathrm{Seq}(\mathcal{B}_k)$
- Grammar of $\mathcal{B}_k$ is algebraic:
  $$\begin{cases} \mathcal{B}_k \to a \cdot \mathcal{B}_{k-1} \cdot \mathcal{S}_k | b \cdot \mathcal{B}_{k-1} \cdot \mathcal{S}_k \\ \mathcal{S}_k \to \mathcal{B}_k \cdot \mathcal{S}_k | \epsilon \end{cases}$$

The dependency graph of $\mathcal{B}_k$ is a non strongly connected graph.

- The composition is always a critical composition: (i.e. $h(z) = f(g(z))$ with $\rho_h = \rho_g$ and $\rho_f = g(\rho_g)$).

  $B_k(z) = T(2zB_{k-1}(z)) = \frac{1}{2}\sqrt{1 - 8zB_{k-1}(z)}.$

A first generalization of the Drmota–Lalley–Woods theorem

Let $\{Y_j = P_j(z, Y_1, \ldots, Y_m)\}_j$ a system satisfying:

1. all polynomials P_i have positive coefficients and $\frac{\partial^2 P_i}{\partial Y_i} \neq 0$;
2. the system admits a fixed point $\vec{Y}$;
3. ~~the dependency graph of the system is strongly connected~~.

Then:

1. $Y(z) = \sum_{i\geq 0} c_i \,.\, (1 - \frac{z}{\rho})^{\frac{i}{2^k}}$,
2. $[z^n]Y(z) \sim C.\frac{1}{2\sqrt{\pi}}.\rho^{-n}.n^{-\frac{1}{2^k}-1}$ with ρ, C_0, C algebraic numbers.

A second generalization of the Drmota–Lalley–Woods theorem

Let $\{Y_j = P_j(z, Y_1, \ldots, Y_m)\}_j$ a system satisfying:

1. all polynomials P_i have positive coefficients and ~~$\frac{\partial^2 P_i}{\partial Y_i} \neq 0$~~ $\frac{\partial P_j}{\partial Y_i} \neq 0$ and $\frac{\partial^2 P_j}{\partial Y_i \partial Y_h} \neq 0$ for all (i, j);
2. the system admits a fixed point $\vec{Y}$;
3. ~~the dependency graph of the system is strongly connected~~.

Then:

1. $Y(z) = \sum_{i \geq 0} c_i \,.\, (1 - \frac{z}{\rho})^{\frac{i}{2^k} - d}$,
2. $[z^n]Y(z) \sim C . \frac{1}{2\sqrt{\pi}} . \rho^{-n} . n^{-\frac{1}{2^k} - d - 1}$ with ρ, C algebraic numbers.

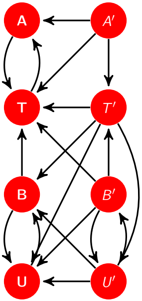
Proof: closure by sum, product and substitution.

Example 1

- The generating function of super-bicolored trees:
  $B(z) = \frac{1}{2} - \frac{1}{2}(1-4z)^{1/4} + O((1-4z)^{3/4})$

  $\frac{\partial}{\partial z} B(z) = \frac{1}{2}(1-4z)^{-3/4} + O((1-4z)^{-1/4})$

$$
\left\{
\begin{array}{c}
\mathcal{B}' \to \mathcal{T} \cdot \mathcal{U} | a \cdot \mathcal{T}' \cdot \mathcal{U} | a \cdot \mathcal{T} \cdot \mathcal{U}' | b \cdot \mathcal{T}' \cdot \mathcal{U} | b \cdot \mathcal{T} \cdot \mathcal{U}' \\
\mathcal{B} \to a \cdot \mathcal{T} \cdot \mathcal{U} | b \cdot \mathcal{T} \cdot \mathcal{U} \\
\mathcal{T}' \to \mathcal{T}\mathcal{A} | a \cdot \mathcal{T}' \cdot \mathcal{A} | a \cdot \mathcal{T} \cdot \mathcal{A}' | \epsilon \\
\mathcal{T} \to a \cdot \mathcal{T} \cdot \mathcal{A} | a \\
\mathcal{U}' \to \mathcal{B}' \cdot \mathcal{U} | \mathcal{B} \cdot \mathcal{U}' | \epsilon \\
\mathcal{U} \to \mathcal{B} \cdot \mathcal{U} | \epsilon \\
\mathcal{A}' \to \mathcal{T}' \cdot \mathcal{A} | \mathcal{T} \cdot \mathcal{A}' | \epsilon \\
\mathcal{A} \to \mathcal{T} \cdot \mathcal{A} | \epsilon
\end{array}
\right.
$$

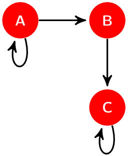


non-strongly connected.

The dependency graph is non-strongly connected.

Example 2

$$\begin{cases} \mathcal{A} \to a \cdot \mathcal{A} \cdot \mathcal{A} \cdot \mathcal{A} | b \cdot \mathcal{A} \\ \mathcal{B} \to a | a \cdot \mathcal{B}\mathcal{C} | b \cdot \mathcal{B}\mathcal{C} \\ \mathcal{C} \to a | b \cdot \mathcal{C}\mathcal{C} \end{cases}$$



$$\begin{cases} A(z) = \frac{z^4(1-4z^2)^{-3/2}}{1-z} \\ B(z) = z(1-4z^2)^{-1/2} \\ C(z) = \frac{1-(1-4z^2)^{1/2}}{2z} \end{cases}$$

$$\begin{cases} A(z) = \frac{\sqrt{2}}{32}(1-2z)^{-3/2} + \dots \\ B(z) = \frac{\sqrt{2}}{4}(1-2z)^{-1/2} + \dots \\ C(z) = 1 - \sqrt{2}(1-2z)^{1/2} + \dots \end{cases}$$

$$[z^n]A(z) \sim \frac{\sqrt{2}}{56\sqrt{\pi}} 2^n n^{1/2}$$

Counter-example

- walks in the quarter-plane: Kreweras (1965), Gessel (2001) [Bostan & Kauers, 2010]: algebraic GF, asymptotics compatible with $\mathbb{N}$-algebraicity . . . but non $\mathbb{N}$-algebraic (proof via Ogden's pumping lemma).

- Some families of maps: algebraicity proven via the kernel method (Tutte, Brown, Bousquet-Mélou, . . .) but non $\mathbb{N}$-algebraic because their critical exponents are not in our set of dyadic possible exponents.

Plan

Conclusion

- We gave a characterization of the critical exponents for $\mathbb{N}$-algebraic functions (generating function associated to an algebraic grammar).
- Other work done: Perron-Frobenius "multivariate" generalisation= the possible limit laws in non strongly connected context-free linear grammars (=automata) ? Answer = "we can asymptotically get <u>any</u> limit law" [Banderier–Bodini–Ponty–Tafat, 2011] + applications to bioinformatics, Boltzmann random generation.
- Work in progress: the possible limit laws in non strongly connected context free-grammars? (Gaussian case: [Bender, Drmota, Soria, Flajolet, Hwang], what else?)
- A description of the ring of the algebraic constants C, ρ, ... in $f_n \sim C\rho^{-n}/\Gamma(\gamma+1)n^{\gamma}$.

Conclusion (implementations)

- an effective Soittola-like theorem for $\mathbb{N}$-algebraic functions:
  - Input: algebraic equation
  - Output: context-free grammar.

  (decidable in the case of 2 non-terminals = genus 0 [Abhyankar])
  (decidable for $\mathbb{N}$-rational functions [Soittola, implementation: Koutschan & Strehl])

- Future implementation (in SageMath):
  - Input: language, pattern,
  - Output: asymptotics, limit law.