Rough paths, Hopf algebras and Renormalization

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Plan

Signature of a regular path

$X := (X_t(1), \ldots, X_t(d)) : \mathbb{R} \to \mathbb{R}^d$ smooth path with d components

Signature of X:

$$\mathbf{X}^{ts}(i_1, \ldots, i_n) := \int_s^t dX_{x_1}(i_1) \int_s^{x_1} dX_{x_2}(i_2) \ldots \int_s^{x_{n-1}} dX_{x_n}(i_n).$$

Solution of differential equations

$$dY_t = \sum_{j=1}^{d} V_j(Y_t)dX_t(j)$$

Formal solution:

$$Y_t = Y_s + \sum_{j=1}^{\infty} \sum_{1\le i_1,\dots,i_j\le d} [V_{i_1}\cdots V_{i_j}\cdot \mathrm{Id}](Y_s) \ \cdot\ \mathbf{X}^{ts}(i_1,\dots,i_j).$$

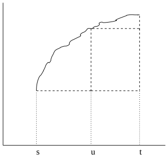
Euler scheme of rank N:

Replace with truncated series ($j \le N$) $\rightsquigarrow$

$$Y_t = \Phi(\mathbf{X}^{ts}(i_1),\dots,\mathbf{X}^{ts}(i_1,\dots,i_N); Y_s)$$

Compose $\rightsquigarrow Y_t \simeq \Phi(\mathbf{X}^{t,\frac{n-1}{n}t}; \cdots \Phi(\mathbf{X}^{\frac{2t}{n},\frac{t}{n}}; \Phi(\mathbf{X}^{\frac{t}{n},0}; Y_0)\cdots).$

Algebraic properties: rank 2



Chen property: $\mathbf{X}^{ts}(i_1, i_2) = \mathbf{X}^{tu}(i_1, i_2) + \mathbf{X}^{us}(i_1, i_2) + \mathbf{X}^{tu}(i_1)\mathbf{X}^{us}(i_2)$.

Shuffle property: $\mathbf{X}^{ts}(i_1, i_2) + \mathbf{X}^{ts}(i_2, i_1) = \mathbf{X}^{ts}(i_1)\mathbf{X}^{ts}(i_2)$.

Axiomatization (1)

$X = (X_t(1), \ldots, X_t(d))$ is α-Hölder if $\sup_{s,t\in[0,T]} \frac{|X_t - X_s|}{|t-s|^\alpha} < \infty$.
Main example: fractional Brownian motion (fBm) with Hurst index $\alpha^+ > \alpha$.

Definition: A geometric α-rough path over X is some functional

$$J_X^{ts} = (J_X^{ts}(i_1) = X_t(i_1) - X_s(i_1), J_X^{ts}(i_1, i_2), \ldots, J_X^{ts}(i_1, \ldots, i_N))$$

such that $\sup_{s,t\in[0,T]} \frac{|J_X^{ts}(i_1,\ldots,i_k) - \mathbf{X}^{\varepsilon,ts}(i_1,\ldots,i_k)|}{|t-s|^{k\alpha}} \to_{\varepsilon\to 0} 0, \quad 2 \leq k \leq N$ for some smooth approximation X^ε such that $X^\varepsilon \to X$ in α-Hölder norm.

Axiomatization (2)

Definition: A α-rough path over X is some functional

$$J_X^{ts} = (J_X^{ts}(i_1) = X_t(i_1) - X_s(i_1), J_X^{ts}(i_1, i_2), \ldots, J_X^{ts}(i_1, \ldots, i_N))$$

such that:
(i) (Hölder regularity) $\sup_{s,t\in[0,T]} \frac{|J_X^{ts}(i_1,\ldots,i_k)|}{|t-s|^{k\alpha}} < \infty, \quad 2 \leq k \leq N$;
(ii) (Algebraic properties): Chen and shuffle properties for $\mathbf{X}^{ts}(i_1, \ldots, i_n)$, $2 \leq n \leq N$.

The two main statements

1st Theorem. Existence of rough paths.

T. Lyons, N. Victoir (2007).

2nd Theorem. Any α-rough path is a geometric α^--rough path: so any rough path may be obtained by some (mysterious) approximation procedure.

Challenges

1. Forgetting about regularity: Classify all formal rough paths.

Regularization scheme-independent algorithm called Fourier normal ordering (FNO): Formal rough paths are exactly determined by tree data, J. U. (2009), Loïc Foissy and J. U. (2010).

2. Find explicit rough paths with natural properties
ex. X Gaussian $\rightsquigarrow$ find $J_X^{ts}(i_1, \ldots, i_k)$ in k-th chaos.

Several regularization schemes using FNO: Fourier domain regularization, general case (J. U. 2009), fBm case (J. U. 2009; D. Nualart and S. Tindel, 2010); Feynman integral renormalization in fBm case (J. U. 2010).

3. Better: find explicit geometric rough paths obtained by natural approximation.

For fBm with $1/6 < \alpha < 1/4$: non-Gaussian construction by singular measure penalization, solved by constructive field theory, using cluster expansion and renormalization (J. Magnen and J.U., 2010).

4. Use these constructions to develop stochastic calculus for fBm with $\alpha < 1/4$.

Plan

1. Introduction
2. An appetizer: Lévy area of fBm
3. Classification of formal rough paths by tree data
4. Renormalization of rough paths
5. From constructive field theory to fractional stochastic calculus

In direct space

$$\mathbb{E}B(s)B(t) = \frac{1}{2}(|s|^{2\alpha} + |t|^{2\alpha} - |t-s|^{2\alpha}) = \lim_{\varepsilon\to 0} \mathbb{E}B^{\varepsilon}(s)B^{\varepsilon}(t) :=$$

$$c_\alpha \lim_{\varepsilon\to 0} \int_0^s du \int_0^t dv \left[(-\mathrm{i}(u-v)+\varepsilon)^{2\alpha-2} + c.c. \right].$$

$\mathrm{Var}\mathbf{B}^{\varepsilon,ts}(1,2)$ contains

$$\int_s^t du \int_s^u dv (-\mathrm{i}(u-v)+\varepsilon)^{2\alpha-2}(+\mathrm{i}(u-v)+\varepsilon)^{2\alpha}.$$

Problem: $\int_s^u dv |u-v|^{(2\alpha-2)+2\alpha} = \infty$ if $\alpha \leq 1/4$.

L. Coutin, Z. Qian (2002) – J. U. (2009,2010)

In Fourier space

Harmonizable representation: $B_t^\varepsilon = \int \frac{e^{it\xi}-1}{i\xi} |\xi|^{\frac{1}{2}-\alpha} e^{-\varepsilon|\xi|} dW_\xi$.

$$\mathbf{B}^{\varepsilon,ts}(1,2) = \int |\xi_1\xi_2|^{\frac{1}{2}-\alpha} e^{-\varepsilon(|\xi_1|+|\xi_2|)} dW_{\xi_1}(1) dW_{\xi_2}(2) \,.\, \int_s^t du \int_s^u dv e^{i(u\xi_1+v\xi_2)}.$$

1st step: split $\int_s^u dv e^{iv\xi_2}$ into $\left(\int^u - \int^s\right) dv e^{iv\xi_2} = -\frac{i}{\xi_2}\left(e^{iu\xi_2} - e^{is\xi_2}\right)$

Increment term:

$\int_s^t du \frac{e^{iu(\xi_1+\xi_2)}}{\xi_2} = -i\left(\frac{e^{it(\xi_1+\xi_2)}}{(\xi_1+\xi_2)\xi_2} - \frac{e^{is(\xi_1+\xi_2)}}{(\xi_1+\xi_2)\xi_2}\right)$

Boundary term: $\int_s^t du e^{iu\xi_1} \frac{e^{is\xi_2}}{\xi_2} = -i \left(\frac{e^{it\xi_1} - e^{is\xi_1}}{\xi_1} \right) \cdot \left(\frac{e^{is\xi_2}}{\xi_2} \right)$.

Fourier normal ordering

Formal classification: a Lévy area is defined up to an increment:

$$\mathbf{B}^2_{ts} - \mathbf{B}^2_{tu} - \mathbf{B}^2_{us} = \left[\mathbf{B}^2_{ts} - (G(t) - G(s))\right] - \left[\mathbf{B}^2_{tu} - (G(t) - G(u))\right] - \left[\mathbf{B}^2_{us} - (G(u) - G(s))\right].$$

Silly decomposition: $\int_s^t dB_1(u) \int_s^u dB_2(v) = \int_s^t \left(dB_1(u) \int^u dB_2(v) \right) - \left(\int_s^t dB_1(u) \right) \cdot \left(\int^s dB_2(u) \right)$.

Sum of $G(t) - G(s)$ divergent increment and of only α-Hölder, infra-red divergent boundary term.

Good decomposition: $\int^s dB_2(u)$ becomes remainder if

$$dB_1 \otimes dB_2 \rightsquigarrow \mathcal{P}^+_{1,2}(dB_1 \otimes dB_2) = \mathcal{F}^{-1} \left(\mathbf{1}_{|\xi_1|<|\xi_2|} \mathcal{F}(dB_1 \otimes dB_2)(\xi_1, \xi_2) \right).$$

$\rightsquigarrow$ If $|\xi_1| > |\xi_2|$, apply Fubini,

$$\int_s^t dB_1(u) \int_s^u dB_2(v) = \int_s^t dB_2(v) \int_v^t dB_1(u) = -\int_s^t dB_2(v) \int_s^v dB_1(u) + \left(\int_s^t dB_2(v) \right) \cdot \left(\int_s^t dB_1(u) \right).$$

Theorem 1 (boundary term) $\mathcal{P}^+_{1,2}$ $\left(\int_s^t dB_1(u) \right) \cdot \left(\int^s dB_2(u) \right)$ and $\mathcal{P}^-_{1,2}$ $\left(\int_s^t dB_2(u) \right) \cdot \left(\int^s dB_1(u) \right)$ are $2\alpha^-$-Hölder.

Regularization of increment term

"Cheat" : $\mathcal{P}^+_{1,2} G \rightsquigarrow 0$!

Generalization : set all tree data to zero (D. Nualart - S. Tindel)

$\mathcal{P}^+_{1,2} G(t) = \int dW_{\zeta_1} \frac{e^{it\zeta_1}}{\zeta_1} a(\zeta_1)$, $a(\zeta_1) = \int_{\zeta_1/2}^{\infty} \frac{d\tilde{W}_{\zeta_2}}{\zeta_2} |\zeta_1 - \zeta_2|^{\frac{1}{2}-\alpha} |\zeta_2|^{\frac{1}{2}-\alpha}$.

Divergent like $\int d\zeta_2 |\zeta_2|^{-4\alpha}$ when $|\zeta_2| = |\xi_2| \gg |\zeta_1| = |\xi_1 + \xi_2|$, i.e. $\xi_1 \approx -\xi_2$.

Renormalization: $a(\zeta_1) \rightsquigarrow a(\zeta_1) - a(0)$: convergent.

Plan

Shuffle property and shuffle algebra

$$\mathbf{X}^{ts}(i_1,\ldots,i_{n_1})\mathbf{X}^{ts}(j_1,\ldots,j_{n_2}) = \sum_{\boldsymbol{k}\in\mathrm{Sh}(\boldsymbol{i},\boldsymbol{j})} \mathbf{X}^{ts}(k_1,\ldots,k_{n_1+n_2})$$

Sh=shuffles

Ex. $\mathbf{X}^{ts}(i_1,i_2).\mathbf{X}^{ts}(j_1) = \mathbf{X}^{ts}(i_1,i_2,j_1) + \mathbf{X}^{ts}(i_1,j_1,i_2) + \mathbf{X}^{ts}(j_1,i_1,i_2)$.

Hopf algebraic interpretation.

$\mathbf{Sh}^d := \{$words with letters in $1,\ldots,d\} \simeq \{$decorated trunk trees$\}$

Product=shuffle product

$\mathbf{X}^{ts}$ has shuffle property $\Longleftrightarrow$ $\mathbf{X}^{ts}$ character of $\mathbf{Sh}^d$.

Chen property and coalgebra structures

$$\mathbf{X}^{ts}(i_1,\ldots,i_n) = \mathbf{X}^{tu}(i_1,\ldots,i_n) + \mathbf{X}^{us}(i_1,\ldots,i_n) + \sum_k \mathbf{X}^{tu}(i_1,\ldots,i_k)\mathbf{X}^{us}(i_{k+1},\ldots,i_n).$$

Hopf algebraic interpretation.

Coproduct of $\mathbf{Sh}^d$: $\Delta((i_1,\ldots,i_n)) = \sum_k (i_1,\ldots,i_k) \otimes (i_{k+1},\ldots,i_n)$.

$\mathbf{X}^{ts}$ has Chen property $\Longleftrightarrow \mathbf{X}^{ts} = \mathbf{X}^{tu} * \mathbf{X}^{us}$.

Permutations and Connes-Kreimer algebra of trees

$\int \ldots \int_{t>t_1>\ldots>t_n>s} dX_{i_1}(t_1)\ldots dX_{i_n}(t_n) = I^{ts}_{dX_{i_1}\otimes\ldots\otimes dX_{i_n}}(\mathbb{T}_n) =$
$\sum_{\mathrm{trees}\,\mathbb{T}} I^{ts}_{dX_{i_{\sigma(1)}}\otimes\ldots\otimes dX_{i_{\sigma(n)}}}(\mathbb{T}) = I^{ts}_{dX_{i_{\sigma(1)}}\otimes\ldots\otimes dX_{i_{\sigma(n)}}}(\mathbb{T}^{\sigma})$

Hence $I^{ts}_{\mu}(\mathbb{T}_n) = \sum_{\sigma\in\Sigma_n} I^{ts}_{\mu^{\sigma}}(\mathbb{T}^{\sigma}),\ \mu^{\sigma} := \mathcal{P}^{+}(\mu\circ\sigma).$

Similarly $\int^t dX_{i_1}(t_1)\int^{t_1} dX_{i_2}(t_2)\ldots\int^{t_{n-1}} dX_{i_n}(t_n) =$
$\mathrm{SkI}^t_{dX_{i_1}\otimes\ldots\otimes dX_{i_n}}(\mathbb{T}_n) = \mathrm{SkI}^t_{\mu}(\mathbb{T}_n) = \sum_{\sigma\in\Sigma_n}\mathrm{SkI}^t_{\mu^{\sigma}}(\mathbb{T}^{\sigma}).$

Example. $\sigma = (231)$:

$$\int_s^t dX_{i_1}(t_1)\int_s^{t_1} dX_{i_2}(t_2)\int_s^{t_2} dX_{i_3}(t_3) = \int_s^t dX_{i_2}(t_2)\int_s^{t_2} dX_{i_3}(t_3)\int_{t_2}^t dX_{i_1}(t_1)$$

$$= -\int_s^t dX_{i_2}(u_1)\int_s^{u_1} dX_{i_3}(u_2)\int_s^{u_1} dX_{i_1}(u_3) + \int_s^t dX_{i_2}(u_1)\int_s^{u_1} dX_{i_3}(u_2)\,.\int_s^t dX_{i_1}(u_3)$$

$\longrightarrow$ (after permuting indices): $\mathbb{T}^{\sigma} = -\,{}^{3}\mathrm{V}_{1}^{\,2} + \mathrm{I}^{2}_{1}\ \bullet_{3}$

A Hopf algebra isomorphism

Definition. Hopf algebra $\mathbf{H}_{ho}$ of heap-ordered trees.

Definition. Hopf algebra **FQSym** of quasi-symmetric functions

$$\sigma \cdot \tau = \sum_{\varepsilon \in Sh(k,l)} (\sigma \otimes \tau) \circ \varepsilon; \qquad \Delta(\sigma) = \sum_{k=0}^{n} \sum_{\sigma = \varepsilon^{-1} \circ (\sigma_1 \otimes \sigma_2)} \sigma_1 \otimes \sigma_2$$

Theorem (Loïc Foissy, J. U.)
$\Theta : \mathbf{H}_{ho} \mapsto \mathbf{FQSym}$, $\mathbb{F} \mapsto \sum_{\sigma \in S_{\mathbb{F}}} \sigma$ is a Hopf algebra isomorphism.

Link with our construction: $\mathbb{T}^{\sigma} = \Theta^{-1}(\sigma)$!

General rough path construction by Fourier normal ordering

Theorem. Let $\phi^t_{\mathbb{T}} : \mathcal{P}^{\mathbb{T}} Meas(\mathbb{R}^n) \to \mathbb{R}$, $\mu \mapsto \phi^t_\mu(\mathbb{T})$ $(t \in \mathbb{R}, \mathbb{T} \in \mathbf{H}(n))$ linear and such that:

- $\phi^t_{dX(i)}(\mathfrak{t}_1) - \phi^s_{dX(i)}(\mathfrak{t}_1) = X_t(i) - X_s(i)$.
- $\phi^t_{\mu_1}(\mathbb{T}_1)\phi^t_{\mu_2}(\mathbb{T}_2) = \phi^t_{\mu_1 \otimes \mu_2}(\mathbb{T}_1.\mathbb{T}_2)$.

Then:

1. $\chi^t_X(\mathbb{T}_n) := \sum_{\sigma \in \Sigma_n} \phi^t_{(dX_{i_1} \otimes \ldots \otimes dX_{i_n})^\sigma}(\mathbb{T}^\sigma)$ is a character of $\mathbf{Sh}^d$.
2. $J^{ts}_X(\mathbb{T}_n) := \chi^t_X * (\chi^s_X \circ S)(\mathbb{T}_n)$ is a rough path over X.
3. $J^{ts}_X(\ell(1), \ldots, \ell(n)) = \sum_{\sigma \in \Sigma_n} (\phi^t * (\phi^s \circ \bar{S}))_{\mu^\sigma_{(X,\ell)}}(\mathbb{T}^\sigma)$.

Example

Choose zero tree data, i.e. $\phi^t_\mu(\mathbb{T}) = 0$ if $\mathbb{T}$ has at least two vertices. Each forest in $\mathbb{T}^\sigma$ contains at least one tree with ≥ 2 vertices except $\mathbb{T}^{\sigma_0}$, where $\sigma_0 = \begin{pmatrix} 1 & \ldots & m \\ m & \ldots & 1 \end{pmatrix}$.

Corollary.

$$J^{ts}_B(\mathbb{T}_n) = c_\alpha^{n/2} \sum_{k=0}^n (-1)^{n-k} \int \ldots \int_{|\xi_1|>\ldots>|\xi_k|,|\xi_k|<\ldots<|\xi_n|}$$
$$\left[\prod_{j=1}^{k-1} e^{\mathrm{i}t\xi_j} \frac{|\xi_j|^{\frac{1}{2}-\alpha}}{\mathrm{i}\xi_j} dW_{\xi_j}(i_j) \right]$$
$$(e^{\mathrm{i}t\xi_k} - e^{\mathrm{i}s\xi_k}) \frac{|\xi_k|^{\frac{1}{2}-\alpha}}{\mathrm{i}\xi_k} dW_{\xi_k}(i_k) \left[\prod_{j=k+1}^{n} e^{\mathrm{i}s\xi_j} \frac{|\xi_j|^{\frac{1}{2}-\alpha}}{\mathrm{i}\xi_j} dW_{\xi_j}(i_j) \right]$$

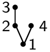
is a rough path over fBm.

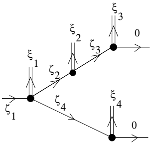
J. U. (2009), D. Nualart and S. Tindel (2010).

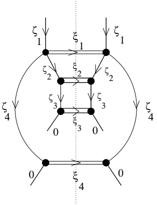
Plan

Feynman amplitudes for fBm.

Associate to a skeleton integral $\mathrm{SkI}_B(\mathbb{T})$ a Feynman half-diagram and a Feynman diagram.

Example. $\mathbb{T} =$ 





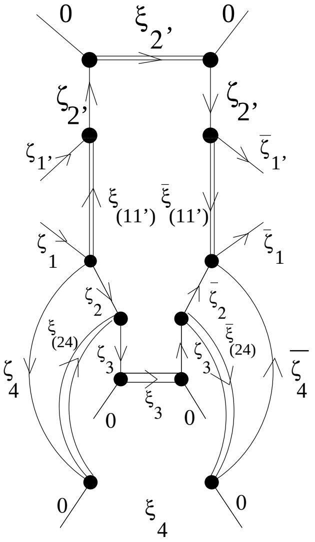
Feynman rules $\rightsquigarrow$ Diagram evaluation $A_{G^{\frac{1}{2}}(\mathbb{T})}(\zeta_1)$ or $A_{G(\mathbb{T})}(\zeta_1)$.

$A_{G^{\frac{1}{2}}(\mathbb{T})}$: $1/\zeta,\ |\xi|^{\frac{1}{2}-\alpha}$; $\qquad A_{G(\mathbb{T})}$: $1/\zeta,\ |\xi|^{1-2\alpha}$

Skeleton integral. $\mathrm{SkI}^t(\mathbb{T}) := \int \prod_{v\in V(\mathbb{T})} dW_{\xi_v}(i_v) \int \frac{e^{it\zeta_1}}{\zeta_1} d\zeta_1 A_{G^{\frac{1}{2}}(\mathbb{T})}(\zeta_1)$

$$\mathrm{Var}\left(\mathrm{SkI}^t(\mathbb{T}) - \mathrm{SkI}^s(\mathbb{T})\right) = \int \frac{d\zeta_1}{\zeta_1^2} |e^{it\zeta_1} - e^{is\zeta_1}|^2 A_{G(\mathbb{T})}(\zeta_1)$$

More complicated Feynman diagrams



BPHZ renormalization

Theorem (J. U., 2010).

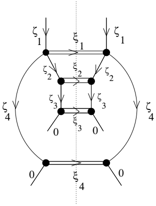
Let

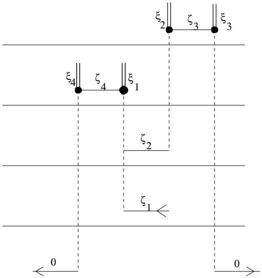
1. $$\mathcal{R}A_{G^{\frac{1}{2}}(\mathbb{T})}(\zeta_1) := \sum_{\mathbb{F}\in\mathcal{F}^{div}(G(\mathbb{T}))} \prod_{g\in\mathbb{F}} (-\tau_g) A_{G^{\frac{1}{2}}(\mathbb{T})}(\zeta)$$

2. $$\phi^t(\mathbb{T}) := \mathcal{P}^{\mathbb{T}} \int \prod_{v\in V(\mathbb{T})} dW_{\xi_v}(i_v) \frac{e^{it\zeta_1}}{\zeta_1} d\zeta_1 \mathcal{R}A_{G^{\frac{1}{2}}(\mathbb{T})}(\zeta_1).$$

Then J^{ts} is a rough path over B.

Example





A Gallavotti-Nicolò tree.

$$A_G(\zeta_1) = \int d\zeta_2 d\zeta_3 d\zeta_4 \left(|\zeta_2 - \zeta_3|^{\frac{1}{2}-\alpha} |\zeta_3|^{-\frac{1}{2}-\alpha} \cdot |\zeta_1 - \zeta_2 - \zeta_4|^{\frac{1}{2}-\alpha} |\zeta_4|^{-\frac{1}{2}-\alpha}. \cdot \zeta_2^{-1} \right)^2 .$$

$$|\xi_2|^{\frac{1}{2}-\alpha} = |\zeta_2 - \zeta_3|^{\frac{1}{2}-\alpha} \rightsquigarrow |\zeta_2 - \zeta_3|^{\frac{1}{2}-\alpha} \; -|\zeta_3|^{\frac{1}{2}-\alpha} = O(\zeta_2 \; \cdot |\zeta_3|^{-\frac{1}{2}-\alpha})$$

$$|\xi_1|^{\frac{1}{2}-\alpha} = |\zeta_1 - \zeta_2 - \zeta_4|^{\frac{1}{2}-\alpha} \rightsquigarrow \left(|\zeta_1 - \zeta_2 - \zeta_4|^{\frac{1}{2}-\alpha} - |\zeta_1 - \zeta_4|^{\frac{1}{2}-\alpha} \right) - \left(|\zeta_2 + \zeta_4|^{\frac{1}{2}-\alpha} - |\zeta_4|^{\frac{1}{2}-\alpha} \right)$$

$$= O(\zeta_1 \zeta_2 \; \cdot \; |\zeta_4|^{-3/2-\alpha}).$$

Plan

Singular penalizations

Definition (stationary field associated to fBm).

$$\phi_{1,2}(t) = \int \frac{e^{\mathrm{i}t\xi}}{\mathrm{i}\xi} |\xi|^{\frac{1}{2}-\alpha} dW_{1,2}(\xi), \quad \mathbb{E}|\mathcal{F}\phi_{1,2}(\xi)|^2 = \frac{1}{|\xi|^{1+2\alpha}}.$$

Associated Gaussian measure: $d\mu(\phi)$

Idea: penalize trajectories with many small area bubbles by replacing $d\mu(\phi)$ with $\frac{1}{Z(\lambda)} e^{-\frac{1}{2}\lambda^2 \int \mathcal{L}_{int}(t)dt}$ where $\lambda \ll 1$ and $\mathcal{L}_{int}$ quadratic in the Lévy area, $\mathcal{A}$.

"Trick": $e^{-\frac{1}{2}\int \lambda^2 \mathcal{A}^2} = \int e^{\mathrm{i}\lambda \int \mathcal{A}\sigma} d\mu(\sigma)$

Associated Gaussian measure : $d\mu(\sigma)$, $\mathbb{E}|\mathcal{F}\sigma(\xi)|^2 = \frac{1}{|\xi|^{1-4\alpha}}$

Cultural note : field theory

Classical language in elementary particle physics and in statistical physics.

Multi-scale Fourier analysis $\rightsquigarrow$ integrating w.r. to highest Fourier scales yields an effective theory at low frequency (=at large distances) with renormalized parameters

Examples:

$\lambda \rightsquigarrow \lambda^j$ effective parameter for $2^j \lesssim |\xi| \lesssim 2^{j+1}$;

$\frac{1}{|\xi|^{1-4\alpha}} \rightsquigarrow \frac{1}{|\xi|^{1-4\alpha}+\sum_{k\geq j} b^k}$, b^j=effective mass of the σ-field

Other examples:

• Weakly self-avoiding paths or ϕ^4 theory (D=4): free theory at large distances ($\lambda^j \to 0$ quand $j \to -\infty$)

• Quantum chromodynamics: free theory at small distances.

$(\phi, \partial\phi, \sigma)$-model

Ultra-violet cut-off: $|\xi| \lesssim 2^\rho$

$$Z(\lambda) = \int d\mu^{\to\rho}(\phi) d\mu^{\to\rho}(\sigma) e^{-\mathrm{i}\lambda \int \mathcal{P}^+(\partial\phi_1(x)\phi_2(x))\sigma(x)dx}$$

Renormalization: $b^j \approx \lambda^2 2^{j(1-4\alpha)}$ for every j
The interaction introduces a screening mass $\approx \infty$
$\rightsquigarrow$ by integration by parts:

$$\langle |\mathcal{F}(\partial A^\pm)(\xi)|^2 \rangle_\lambda = \frac{1}{\lambda^2} |\xi|^{1-4\alpha} \left[1 - |\xi|^{1-4\alpha} \langle |(\mathcal{F}\sigma_+)(\xi)|^2 \rangle_\lambda \right]. \quad (5.1)$$

Perturbative "proof"

Feynman diagrams: formal expansion

$e^{-\mathrm{i}\mathcal{L}_{int}} = \sum \frac{(-\mathrm{i}\mathcal{L}_{int})^n}{n!} \rightsquigarrow$ Wick formula

Bubble:

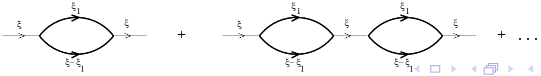
$$-|\xi|^{1-4\alpha} \cdot (-\mathrm{i}\lambda)^2 \int_{|\xi_1|<|\xi-\xi_1|}^{\Lambda} d\xi_1 \left\{ \left(\mathbb{E}[|\mathcal{F}\sigma_+(\xi)|^2] \right)^2 \mathbb{E}[|\mathcal{F}(\partial\phi_1)(\xi_1)|^2]\, \mathbb{E}[|\mathcal{F}\phi_2(\xi-\xi_1)|^2] \right\}$$

$$= \lambda^2 |\xi|^{4\alpha-1} \int_{|\xi_1|<|\xi-\xi_1|}^{\Lambda} d\xi_1 |\xi_1|^{1-2\alpha} |\xi-\xi_1|^{-1-2\alpha} \sim_{\Lambda\to\infty} K\lambda^2 (\Lambda/|\xi|)^{1-4\alpha}, \quad (5.2)$$

Bubble series: $\frac{1}{|\xi|^{1-4\alpha}} \rightsquigarrow \frac{1}{|\xi|^{1-4\alpha+b}}$, $b \approx \lambda^2 \Lambda^{1-4\alpha}$

$$\frac{1}{\lambda^2}|\xi|^{1-4\alpha} \left[1 - \frac{1}{1+K'\lambda^2(\Lambda/|\xi|)^{1-4\alpha}} \right] = \frac{1}{\lambda^2}|\xi|^{1-4\alpha} \cdot \frac{K'\lambda^2(\Lambda/|\xi|)^{1-4\alpha}}{1+K'\lambda^2(\Lambda/|\xi|)^{1-4\alpha}}$$

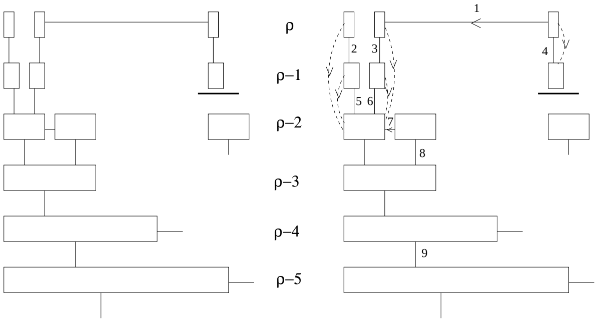
$$\to_{\Lambda\to\infty} \frac{1}{\lambda^2}|\xi|^{1-4\alpha}. \quad (5.3)$$



Constructive proof (I)

Multi-scale vertical slicing $\psi = \sum_j \psi^j$, $supp(\mathcal{F}\psi^j) \subset [2^{j-1}, 2^{j+1}]$
$\rightsquigarrow$ Horizontal slicing : one degree of freedom per dyadic interval Δ^j of length 2^{-j}

Cluster expansion: finite-order expansion within each interval Δ^j, and approximate decoupling of degrees of freedom $\rightsquigarrow$ polymers $\mathbb{P}$.



Constructive proof (II)

$$Z_V^{\to\rho}(\lambda) = \sum_n \frac{1}{n!} \sum_{\mathbb{P}_1,\ldots,\mathbb{P}_n \text{ non-overlapping}} F_{HV}(\mathbb{P}_1)\ldots F_{HV}(\mathbb{P}_n),$$

$\ln Z_V^{\to\rho}(\lambda) = |V| \sum_{j=0}^{\rho} 2^j f_V^{j\to\rho}$, where $f_V^{j\to\rho} \to_{|V|\to\infty} O(\lambda)$

Renormalization: The local parts of diverging diagrams are resummed into an exponential scale after scale $\Leftrightarrow$ σ-covariance renormalized by the mass counterterm b^j,

$$\langle|\sigma^j(\xi)|^2\rangle \rightsquigarrow \frac{1}{|\xi|^{1-4\alpha} + \sum_{k\geq j} b_k} = O(\lambda^{-2} 2^{-\rho(1-4\alpha)}) \to_{\rho\to\infty} 0.$$

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