Key polynomials of type C and crystal graphs

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Outline

Key polynomials /Demazure characters

- Key polynomials were introduced by Demazure for all Weyl groups (1974). They were studied combinatorially in the case of the symmetric group (type A) by Lascoux and Schützenberger (1988).

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- If $\mathfrak{g}$ is a simple Lie algebra with Weyl group W, $U_q(\mathfrak{g})$ its quantum group, and $V(\lambda)$ the integrable representation with highest weight λ and u_λ the highest weight vector, for a given $w \in W$ the Demazure module is defined to be

$$V_w(\lambda) := U_q(\mathfrak{g})^{>0}.u_{w(\lambda)},$$

and the Demazure character is the character of $V_w(\lambda)$.

$V(\lambda) \longrightarrow$ crystal basis $\longrightarrow$ crystal graph (colored oriented graph).

Key polynomials /Demazure characters

- It was conjectured by Littelmann (1991) and proved by Kashiwara (1993) that the intersection of a crystal basis of V_λ with $V_w(\lambda)$ is a crystal basis for $V_w(\lambda)$. The resulting subset $B_w(\lambda) \subseteq B(\lambda)$ is called Demazure crystal.
- The Demazure character is defined by the Demazure crystal $B_w(\lambda)$.

Key polynomials of type A (L-S, 1988)

- $\mathfrak{S}_n$ is generated by the permutations $s_i = (i\, i+1)$, $i = 1, \dots, n-1$, The generators s_i act on vectors $v = [v_1, \dots, v_n] \in \mathbb{N}^n$ by

$$s_i v = [v_1, \dots, v_{i+1}, v_i \dots, v_n], \text{ for } i = 1, \dots, n-1,$$

and induce an action of $\mathfrak{S}_n$ on $\mathbb{Z}[x_1, x_2, .., x_n]$ by considering vectors v as exponents of monomials $x^v = x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n}$.

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- Two families of Demazure operators: For $i = 1, \dots, n-1$, $\pi_i, \widehat{\pi}_i : \mathbb{Z}[x_1, x_2, .., x_n] \longrightarrow \mathbb{Z}[x_1, x_2, .., x_n]$, $\widehat{\pi}_i = \pi_i - 1$ where
$$\pi_i f = \frac{x_i f - x_{i+1} f^{s_i}}{x_i - x_{i+1}} \text{ and } \widehat{\pi}_i f = \frac{x_{i+1} f - x_{i+1} f^{s_i}}{x_i - x_{i+1}} = \pi_i f - f.$$

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- Two families of key polynomials: For λ a partition (at most n parts) and $w = s_{i_N} \ldots s_{i_2} s_{i_1}$ a reduced decomposition in $\mathfrak{S}_n$, one defines the type A key polynomials indexed by $w\lambda$
$$\kappa_{w\lambda}(x) = \pi_{i_N} \pi_{i_2} \ldots \pi_{i_1} x^\lambda \text{ and } \widehat{\kappa}_{w\lambda}(x) = \widehat{\pi}_{i_N} \widehat{\pi}_{i_2} \ldots \widehat{\pi}_{i_1} x^\lambda.$$
Examples: monomial x^λ and $\kappa_{\omega\lambda}(x) = s_\lambda(x)$ Schur polynomial.

$\mathfrak{gl}_n$-crystal operators and Demazure operators

- Choose the alphabet $A_n = \{1 < 2 < ... < n\}$. Consider a nonempty word w on this alphabet and let $i \in \{1, \ldots, n-1\}$.
  - Pick the subword consisting only of letters $i, i+1$.
  - Encode

$$i \longrightarrow (\qquad\qquad i+1 \longrightarrow).$$

  - Ignore successively all the factors ( ) to construct a new subword

$$\rho_i =)^r\ (^s.$$

  - Define $f_i : A_n^* \longrightarrow A_n^* \cup \{0\}$:
    If $s = 0$, $f_i(w) = 0$. If $s > 0$, $f_i(w)$ is obtained by changing the leftmost parentheses) of $\rho_i =)^r\ (^s$ into)

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    If $r = 0$, $e_i(w) = 0$. If $r > 0$, $e_i(w)$ is obtained by changing the rightmost parentheses) of $\rho_i =)^r\ (^s$ into (

$$i+1 \longrightarrow i.$$

$\mathfrak{gl}_n$-crystal operators

- f_2 : 212

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$\mathfrak{gl}_n$-crystal operators

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- $f_2(212) = 312$

$$)(\quad))$$

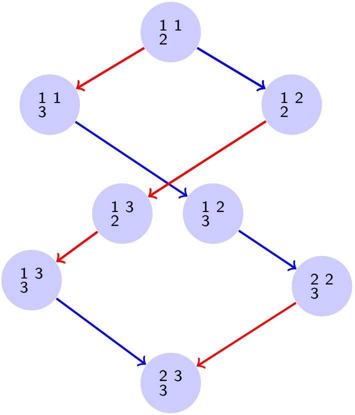
$$f_2^2(212) = 313$$

$$e_2(313) = 312 \qquad e_2 e_2(312) = 212.$$

$\mathfrak{gl}_n$-crystal graph

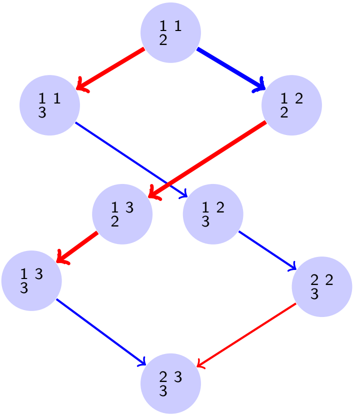
- Oriented colored graph with colors $\{1, \ldots, n-1\}$. An arrow $a \xrightarrow{i} b$ iff $f_i(a) = b \Leftrightarrow e_i(b) = a$.
- $n = 3$, $\lambda = (210)$, $B(210)$ $\mathfrak{gl}_3$-crystal

  $1 =$ — $2 =$ —



Type A Demazure-crystal graph

- The Demazure crystal $B_{s_2 s_1}(210) \subset B(210)$
- $1 =$ blue $2 =$ red



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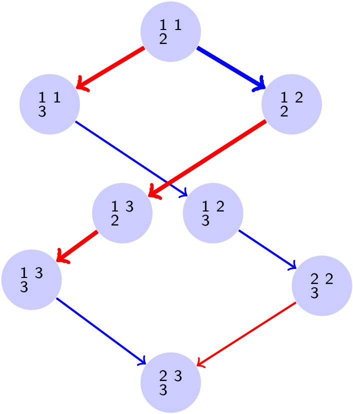

$$f_{s_1}(K) = \{K, f_1(K)\}$$

(blue) : $\pi_1(x^{210}) = \sum_{T \in f_{s_1}(K)} x^{wt(T)}$
$= x^{210} + x^{120}$

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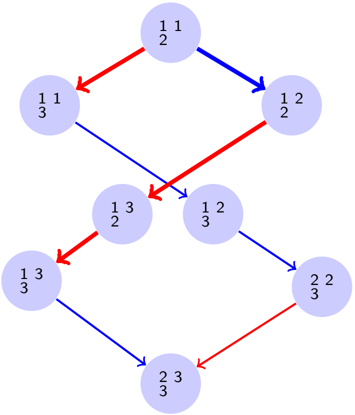

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$$f_{s_2 s_1}(K) = \{f_2^{m_2} f_1^{m_1}(K) : m_1, m_2 \geq 0\}$$
$$= \{K, f_2(K)\} \cup \{f_1(K), f_2 f_1(K), f_2^2 f_1(K)\}$$

(red) : $\pi_2(x^{210}) + \pi_2(x^{120}) = \sum_{T \in f_{s_2 s_1}(K)} x^{wt(T)}$
$= (x^{210} + x^{201}) + (x^{120} + x^{111} + x^{102})$

Type A Demazure-crystal graph

- The Demazure crystal $B_{s_2 s_1}(210) \subset B(210)$
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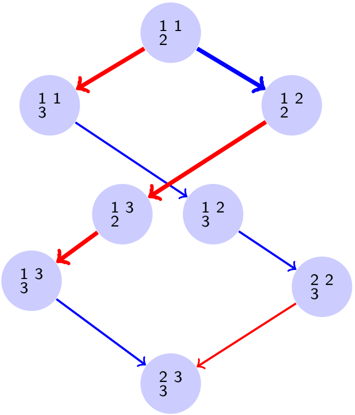

$$f_{s_1}(K) = \{K, f_1(K)\}$$

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$$f_{s_2 s_1}(K) = \{f_2^{m_2} f_1^{m_1}(K) : m_1, m_2 \geq 0\} = \{K, f_2(K)\} \cup \{f_1(K), f_2 f_1(K), f_2^2 f_1(K)\}$$

$$\text{(red)}: \ \pi_2(x^{210}) + \pi_2(x^{120}) = \sum_{T \in f_{s_2 s_1}(K)} x^{wt(T)} = (x^{210} + x^{201}) + (x^{120} + x^{111} + x^{102})$$

$$\kappa_{s_2 s_1 \lambda}(x) = \pi_2 \pi_1(x^{\lambda}) = \sum_{T \in B_{s_2 s_1}(210)} x^{wt(\lambda)} = \widehat{\kappa}_{\lambda} + \widehat{\kappa}_{s_1 \lambda} + \widehat{\kappa}_{s_2 \lambda} + \widehat{\kappa}_{s_2 s_1 \lambda} = \sum_{\nu < s_2 s_1} \widehat{\kappa}_{\nu \lambda}$$

Tableau criterion for Bruhat order on $\mathfrak{S}_n$

- $\sigma,\ \mu \in \mathfrak{S}_n,\ v = (n, n-1, \dots, 2, 1) \in \mathbb{N}^n$

$$\sigma \geq \mu \text{ iff } key(\sigma v) \geq key(\mu v)$$

- $key(v_1, v_2, \dots, v_n)$=semistandard tableau of shape the decreasing rearrangement of $(v_1, \dots, v_n)$ whose first v_i columns contain the letter i
- $n = 4$
  $key(s_2 s_3 s_1 s_2(4,3,2,1)) = key(2,1,4,3)$,
  $key(s_2 s_3 s_1(4,3,2,1)) = key(3,1,4,2)$

$$key(2,1,4,3) = \begin{matrix} 1 & 1 & 3 & 3 \\ 2 & 3 & 4 & \\ 3 & 4 & & \\ 4 & & & \end{matrix} \quad \geq key(3,1,4,2) = \begin{matrix} 1 & 1 & 1 & 3 \\ 2 & 3 & 3 & \\ 3 & 4 & & \\ 4 & & & \end{matrix}$$

$$s_2 s_3 s_1 s_2 \geq s_2 s_3 s_1$$

Characterisation of the tableaux in the $\mathfrak{gl}_n$-Demazure crystal, L-S (1988)

- key polynomial in n variables $x = (x_1, \dots, x_n)$

$$\kappa_{w\lambda}(x) = \sum_{T \in B_w(\lambda)} x^{wt T} = \sum_{key(\beta) \leq key(w\lambda)} \widehat{\kappa}_\beta.$$

$$\widehat{\kappa}_\beta(x) = \sum_{\substack{SSYT T \\ wt T \in \mathbb{N}^n \\ sh(T)=\lambda \\ K^+(T)=key(\beta)}} x^{wt T},$$

$$\kappa_{w\lambda}(x) = \sum_{\substack{SSYT T \\ wt T \in \mathbb{N}^n \\ sh(T)=\lambda \\ K^+(T)\leq key(w\lambda)}} x^{wt T}.$$

Hyperoctahedral group $\mathfrak{S}_n^B$

- The hyperoctahedral group $\mathfrak{S}_n^B$ ($2^n n!$ elements) is the Weyl group of the symplectic group $\mathbb{S}_p(2n,\mathbb{C})$. $\mathfrak{S}_n^B$ is generated by the sign permutations $s_i = (i, i+1)(\overline{i}, \overline{i+1})$, $i = 1, \ldots, n-1$ and $s_n = (n, \overline{n})$, which satisfy the relations:

  1. $s_i^2 = 1,\ i = 1, \ldots, n;$
  2. $s_i s_j = s_j s_i$ if $|i-j| \geq 2;$
  3. $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},\ i = 1, \ldots, n-2;$
  4. $s_{n-1} s_n s_{n-1} s_n = s_n s_{n-1} s_n s_{n-1}.$

$$[123\bar{4}] = s_4,\ \ [1243] = s_3,\ \ [124\bar{3}] = s_3 s_4,$$

$$[12\bar{4}3] = s_4 s_3,\ \ [12\bar{3}4] = s_3 s_4 s_3,\ \ [12\bar{3}\bar{4}] = s_4 s_3 s_4 s_3,\ \ [\bar{1}\bar{2}\bar{3}\bar{4}]$$

$\mathfrak{S}_n^B$

- The generators s_i act on vectors $v = [v_1, \dots, v_n] \in \mathbb{Z}^n$ by

$$s_i v = [v_1, \dots, v_{i-1}, v_{i+1}, v_i \dots, v_n], \text{ for } i = 1, \dots, n-1,$$

and

$$s_n v = [v_1, \dots, v_{n-1}, -v_n].$$

This induces an action of $\mathfrak{S}_n^B$ on the Laurent polynomials ring $\mathbb{Z}[x_1^{\pm}, \dots, x_n^{\pm}]$ by considering vectors $v \in \mathbb{Z}^n$ as exponents of monomials $x^v = x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n}$.

Type C Demazure operators

- Two families of Demazure operators: For $i = 1, \ldots, n$, $\pi_i^C, \widehat{\pi}_i^C : \mathbb{Z}[x^{\pm}, n] \longrightarrow \mathbb{Z}[x^{\pm}, n]$, $\widehat{\pi}_i^C = \pi_i^C - 1$ where

$$\pi_i^C f = \frac{x_i f - x_{i+1} f^{s_i}}{x_i - x_{i+1}} \quad \text{and} \quad \widehat{\pi}_i^C f = \frac{x_{i+1} f - x_{i+1} f^{s_i}}{x_i - x_{i+1}}, \quad i \neq n$$

and

$$\pi_n^C f = \frac{x_n^2 f - f^{s_n}}{x_n^2 - 1} \quad \text{and} \quad \widehat{\pi}_n^C f = \frac{f - f^{s_n}}{x_n^2 - 1}.$$

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- Two families of key polynomials: For λ a partition (at most n parts) and $w = s_{i_N}^C \dots s_{i_2}^C s_{i_1}^C$ reduced decomposition, one defines type C key polynomials indexed by $w\lambda$

$$\kappa_{w\lambda}^C(x) = \pi_{i_1}^C \pi_{i_2} \dots \pi_{i_N}^C x^\lambda \;\text{ and }\; \widehat{\kappa}_{w\lambda}^C(x) = \widehat{\pi}_{i_1}^C \widehat{\pi}_{i_2}^C \dots \widehat{\pi}_{i_n}^C x^\lambda.$$

  Examples: monomial x^λ and $\kappa_{\omega\lambda}^C(x) = sp_\lambda(x)$ symplectic Schur polynomial.

Symplectic crystal operators

- Choose the alphabet $C_n = \{1 < 2 < ... < n < \bar{n} < ... < \bar{2} < \bar{1}\}$. Consider a nonempty word w on this alphabet and let $i \in \{1, \ldots, n\}$.
  - Pick the subword consisting only of letters $\overline{i+1}, \bar{i}, i, i+1$.
  - Encode
$$\overline{i+1}, i \longrightarrow + \qquad \bar{i}, i+1 \longrightarrow -$$
  - Ignore all the successive factors $+-$ to construct a new subword
$$\rho_i = -^r +^s$$
  - Define $f_i : C_n^* \longrightarrow C_n^* \cup \{0\}$:
If $s = 0$, $f_i(w) = 0$. If $s > 0$, $f_i(w)$ is the word obtained by changing the left most symbol $+$ of $\rho_i = -^r +^s$ into $-$ where
$$i \longrightarrow i+1, \qquad \overline{i+1} \longrightarrow \bar{i},$$
and when $i = n$, $\quad n \longrightarrow \bar{n}$.
If $r = 0$, $e_i(w) = 0$. If $r > 0$, $e_i(w)$ is obtained by changing the rightmost symbol $-$ of $\rho_i =)^r\ (^s$ into $+$ where
$$i+1 \longrightarrow i \qquad \bar{i} \longrightarrow \overline{i+1},$$
and when $i = n$, $\quad \bar{n} \longrightarrow n$.

Example

- $$\omega = 12\bar{1}\bar{1}\bar{2}2121\bar{1}\bar{2}$$

$$+ - - - + - + + - +$$

Ignore all factors $+ \quad -$

$$- \quad - \quad +$$

Example

- $$\omega = 12\bar{1}\bar{1}\bar{2}212 1\bar{1}\bar{2}$$

  $$+ - - - + - + + - +$$

  Ignore all factors $+ \quad -$

  $$- \quad - \quad + \qquad\qquad - \quad - \quad -$$

  $$\bar{2} \longrightarrow \bar{1}$$

  $\omega = (12)\bar{1}\bar{1}(\bar{2}2)(12)(1\bar{1})\bar{1}.$

Symplectic tableaux (De Concini, Kashiwara-Nakashima)

- Admissible columns on the alphabet $C_n = \{1 < 2 < \ldots < n < \bar{n} < \ldots < \bar{2} < \bar{1}\}$. $n = 6$

$$P = \begin{array}{c} 2 \\ 4 \\ 6 \\ \bar{6} \\ \bar{4} \end{array} \quad \begin{array}{cccccc} \emptyset & 2 & \emptyset & 4 & \emptyset & 6 \\ \emptyset & \emptyset & \emptyset & \bar{4} & \emptyset & \bar{6} \end{array} \qquad Q = \begin{array}{c} 2 \\ 3 \\ 4 \\ \bar{4} \\ \bar{2} \end{array} \quad \begin{array}{cccccc} \emptyset & 2 & 3 & 4 & \emptyset & \emptyset \\ \emptyset & \bar{2} & \emptyset & \bar{4} & \emptyset & \emptyset \end{array}$$

- A column is an admissible column iff the diagram is such that there is a matching which sends each full slot to an empty slot to its left.
- Splitting column form

$$P = \begin{array}{c} 2 \\ 4 \\ 5 \\ \bar{6} \\ \bar{5} \end{array} \quad \begin{array}{cccccc} \emptyset & 2 & \emptyset & 4 & 5 & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset & \bar{5} & \bar{6} \end{array} \qquad (\ell T, rT) = \begin{array}{cc} 2 & 2 \\ {\color{red}3} & 4 \\ 4 & 5 \\ \bar{6} & \bar{6} \\ \bar{5} & {\color{red}\bar{3}} \end{array}$$

- A SSYT tableau T with admissible columns $T_1, T_2, \ldots, T_c$ is a symplectic tableau if the split column form of T is semistandard.

Symplectic tableaux

- $$P = \begin{array}{llll} 1 & 2 & 2 & \bar{1} \\ 4 & 4 & \bar{3} & \\ \bar{4} & \bar{2} & \bar{1} & \\ \bar{3} & & & \end{array} \qquad (\ell P, rP) = \begin{array}{llllllll} 1 & 1| & 1 & 2| & 2 & 2| & \bar{1} & \bar{1} \\ 2 & 4| & 4 & 4| & \bar{3} & \bar{3}| & & \\ \bar{4} & \bar{3}| & \bar{2} & \bar{1}| & \bar{1} & \bar{1}| & & \\ \bar{3} & \bar{2}| & & & & & & \end{array}$$

Symplectic tableaux

- $$P = \begin{array}{cccc} 1 & 2 & 2 & \bar{1} \\ 4 & 4 & \bar{3} & \\ \bar{4} & \bar{2} & \bar{1} & \\ \bar{3} & & & \end{array} \qquad (\ell P, rP) = \begin{array}{cccccccc} 1 & 1| & \color{red}{1} & 2| & 2 & 2| & \bar{1} & \bar{1} \\ \color{red}{2} & 4| & 4 & 4| & \bar{3} & \bar{3}| & & \\ \bar{4} & \bar{3}| & \bar{2} & \color{red}{\bar{1}|} & \bar{1} & \bar{1}| & & \\ \bar{3} & \color{red}{\bar{2}|} & & & & & & \end{array}$$

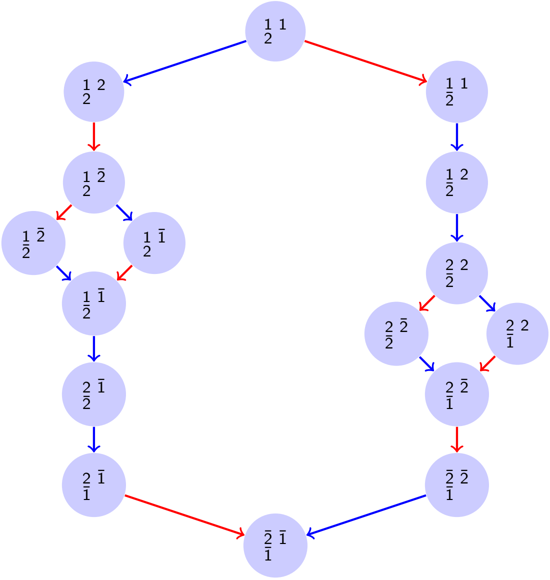
- $wt(P) = (d_1, \ldots, d_n)$, d_i is the number of letters i minus the number of letters $\bar{i}$,

$$wt(P) = (-1, 1, -2, 1).$$

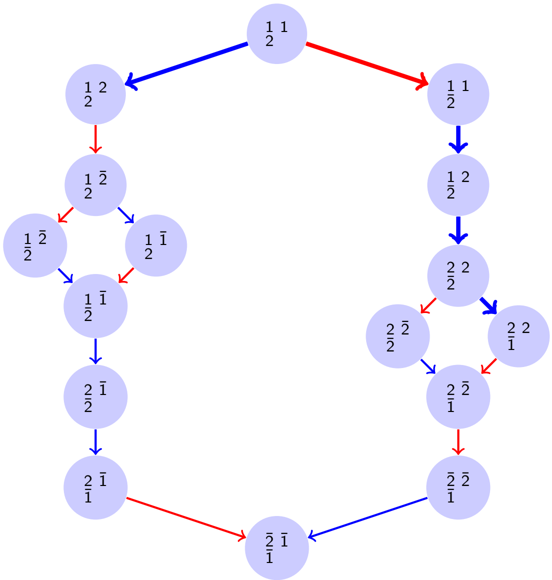
Symplectic key tableaux

- $(v_1, v_2, \ldots, v_n) \in \mathbb{Z}^n$, alphabet $C_n = \{1 < 2 < \ldots < n < \bar{n} < \ldots < \bar{2} < \bar{1}\}$.
- $key(v_1, v_2, \ldots, v_n)$=semistandard tableau of shape $(|v_1|, \ldots, |v_n|)^+$ whose first $|v_i|$ columns contain the letter i if $v_i \geq 0$, and $-i$ if $v_i < 0$.
- $$key(-1,0,3,-2,5,0,-3) = \begin{matrix} 3 & 3 & 3 & 5 & 5 \\ 5 & 5 & 5 & & \\ \bar{7} & \bar{7} & \bar{7} & & \\ \bar{4} & \bar{4} & & & \\ \bar{1} & & & & \end{matrix}$$
- A column of a key either contains j or $\bar{j}$ but not both. The columns of a key are admissible and its splitting form is a semistandard tableau.

Symplectic crystal graph $\mathfrak{sp}_2$



Symplectic Demazure crystal $B_{s_1 s_2}(21)$



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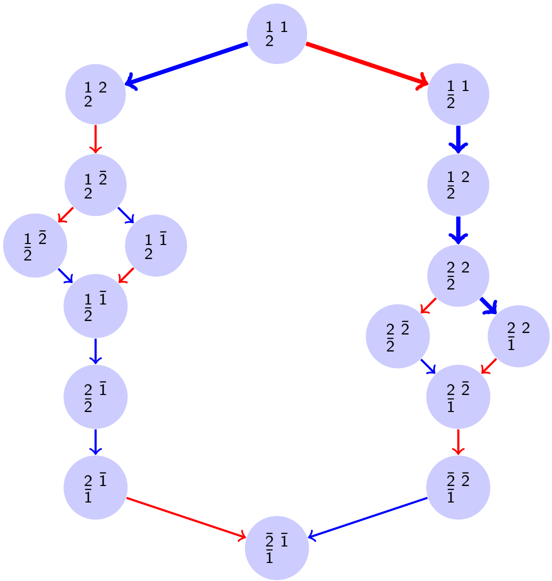
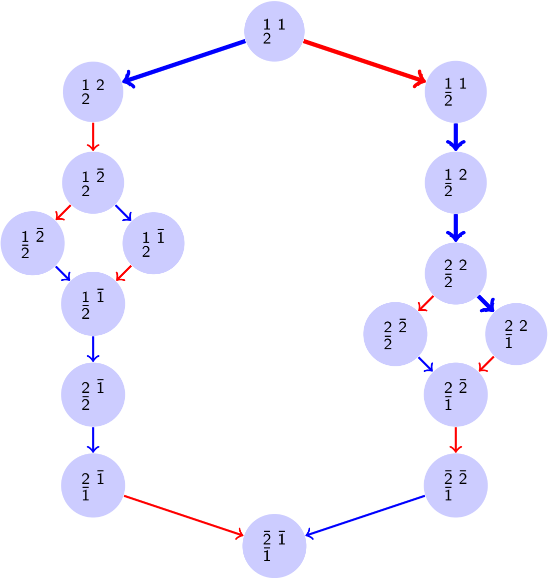


$-$: $\pi_2(x^{21}) = x^{21} + x^{2\bar{1}}$

Symplectic Demazure crystal $B_{s_1 s_2}(21)$

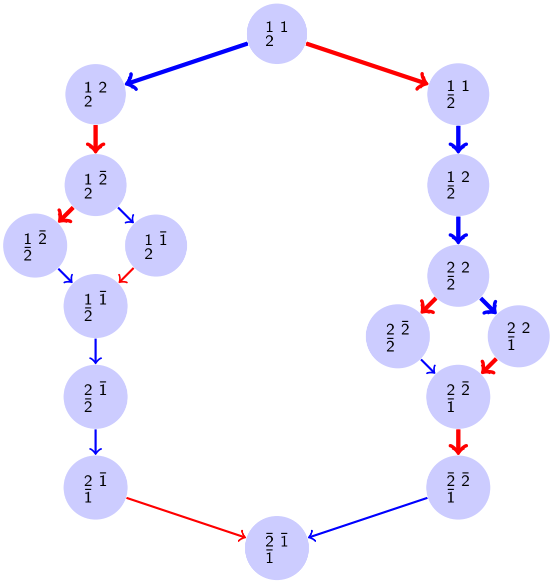


(red) : $\pi_2(x^{21}) = x^{21} + x^{2\bar{1}}$

(blue) : $\pi_1(x^{21}) = x^{21} + x^{12}$

$\pi_1(x^{2\bar{1}}) = x^{2\bar{1}} + x^{10} + x^{01} + x^{\bar{1}2}$

Symplectic Demazure crystal $B_{s_2 s_1 s_2}(21)$



red : $\pi_2(x^{21}) = x^{21} + x^{2\bar{1}}$

blue : $\pi_1(x^{21}) = x^{21} + x^{12}$
$\pi_1(x^{2\bar{1}}) = x^{2\bar{1}} + x^{10} + x^{01} + x^{\bar{1}2}$

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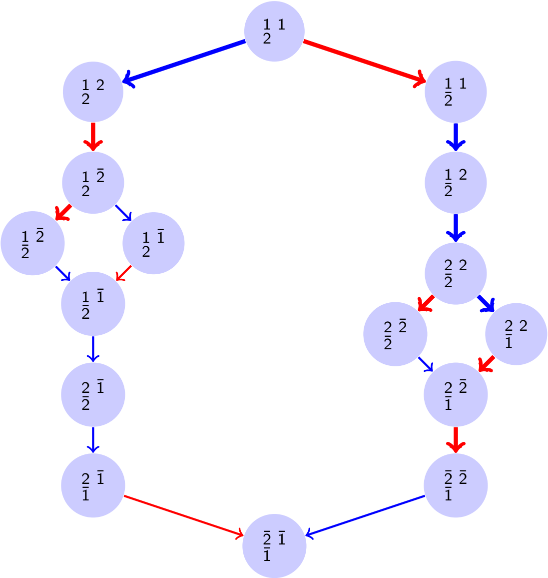

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$\pi_2(x^{01}) = x^{01} + x^{0\bar{1}}$

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Symplectic Demazure crystal of a $\mathfrak{sp}_2$-crystal

- Key polynomial of type C

$$\kappa^C_{s_1 s_2 \lambda}(x_1, x_2) = \sum_{T \in B_{s_1 s_2}(21)} x^{wtT} = x^{(21)} + x^{(2\bar{1})} + x^{(12)} + x^{(10)} + x^{(01)} x^{(\bar{1},2)} =$$

$$= x_1^2 x_2 + x_1^2 x_2^{-1} + x_1 x_2^2 + x_1 + x_2 + x_1^{-1} x_2^2.$$

$wtT = (d_1, \ldots, d_n) \in \mathbb{Z}^n$, where d_i is the number of letters i in T minus the number of letters $\bar{i}$ in T.

Tableau criterion for Bruhat order on $\mathfrak{S}_n^B$

- $\sigma,\ \mu \in \mathfrak{S}_n^B,\ \sigma \leq^C \mu \Rightarrow B_\sigma \subseteq B_\mu$
- $v = (n, n-1, \dots, 2, 1) \in \mathbb{N}^n$

$$\sigma \leq^C \mu \text{ iff } key^C(\sigma v) \leq key^C(\mu v)$$

- $n = 4,\ \mu = s_1 s_2 s_4 s_3 s_4 s_3 \geq^C \sigma = s_2 s_3 s_4$

$$key^C(\mu(4321)) \geq key^C(\sigma(4321))$$

$$key^C(\bar{2}, 43\bar{1}) = \begin{matrix} 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & \\ \bar{4} & \bar{1} & & \\ \bar{1} & & & \end{matrix} \geq key^C(4\bar{1}32) = \begin{matrix} 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & \\ 4 & 4 & & \\ \bar{2} & & & \end{matrix}$$

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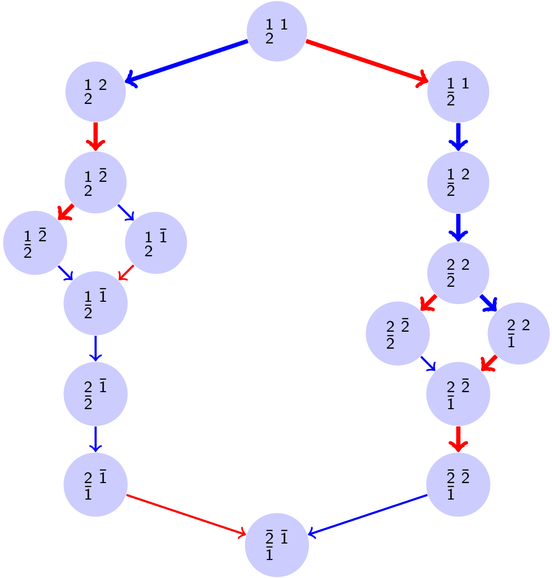

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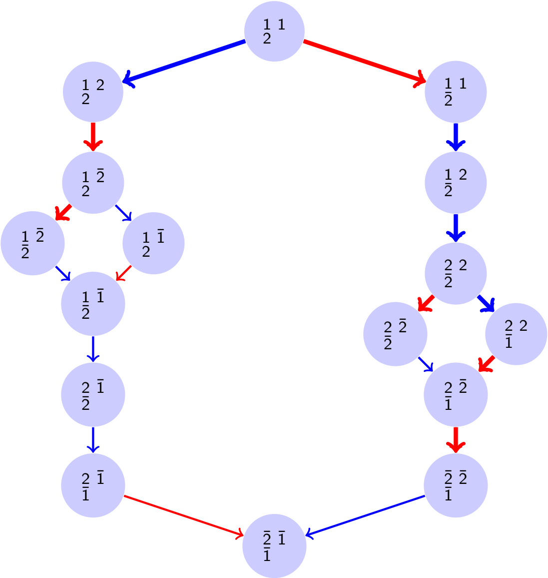

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Characterisation of the tableaux in the $\mathfrak{sp}_n$-Demazure crystal

- key polynomial of type C in n variables $x = (x_1, \ldots, x_n)$

$$\kappa^C_{w\lambda}(x) = \sum_{T \in B_w(\lambda)} x^{wt\,T} = \sum_{key(\beta) \leq key(w\lambda)} \widehat{\kappa}^C_\beta$$

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$$\widehat{\kappa}^C_\beta(x) = \sum_{\substack{symplectic T \\ wt T \in \mathbb{Z}^n \\ sh(T) = \lambda \\ K^+(T) = key(\beta)}} x^{wt T} ?$$

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