Dual braid monoids and Koszulity

Phillippe Nadeau (CNRS, Univ. Lyon 1)

SLC 69, Strobl, September 10th 2012

Goal

Investigate the **combinatorics** of a certain **graded algebra** associated to a **Coxeter system**, namely

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Most of the talk will be focused on **type** A for simplicity.

... and also because I cannot yet prove the main results in all generality.

Usual noncrossing partitions.

Let (W, S) the Coxeter system of type A_{n-1}.

So $W = S_n$ generated by $S = \{(i, i+1)\}$ for $i = 1, \ldots, n-1$.

- **Standard theory**

Length $\ell_S(w) =$ minimal k such that $w = s_{i_1} \cdots s_{i_k}$.

Bruhat order: $w \leq_S w'$ if $\ell_S(w) + \ell_S(w^{-1}w') = \ell_S(w')$

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- **Dual presentation**

W with **all** transpositions $T = \{(i, j)\}$ as generators.

Absolute length $\ell_T(w)$ = minimal k with $w = t_{i_1} \cdots t_{i_k}$.

$$= n - |\{ \text{ cycles of } w\}|$$

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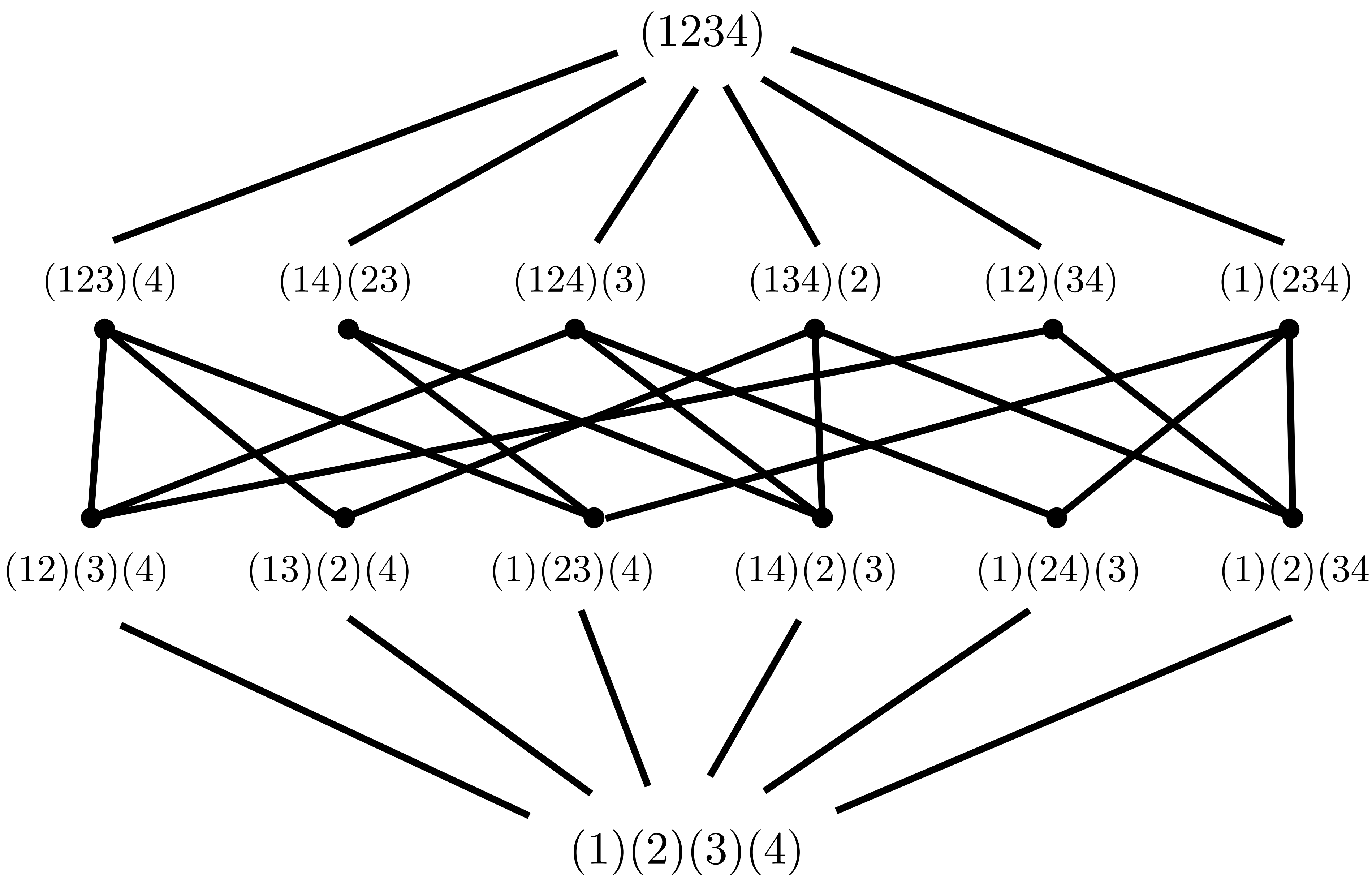
$$NC(n) = NC(A_{n-1}) = [id, (12 \cdots n)]_{\leq_T}$$

$NC(4)$ and its Möbius function



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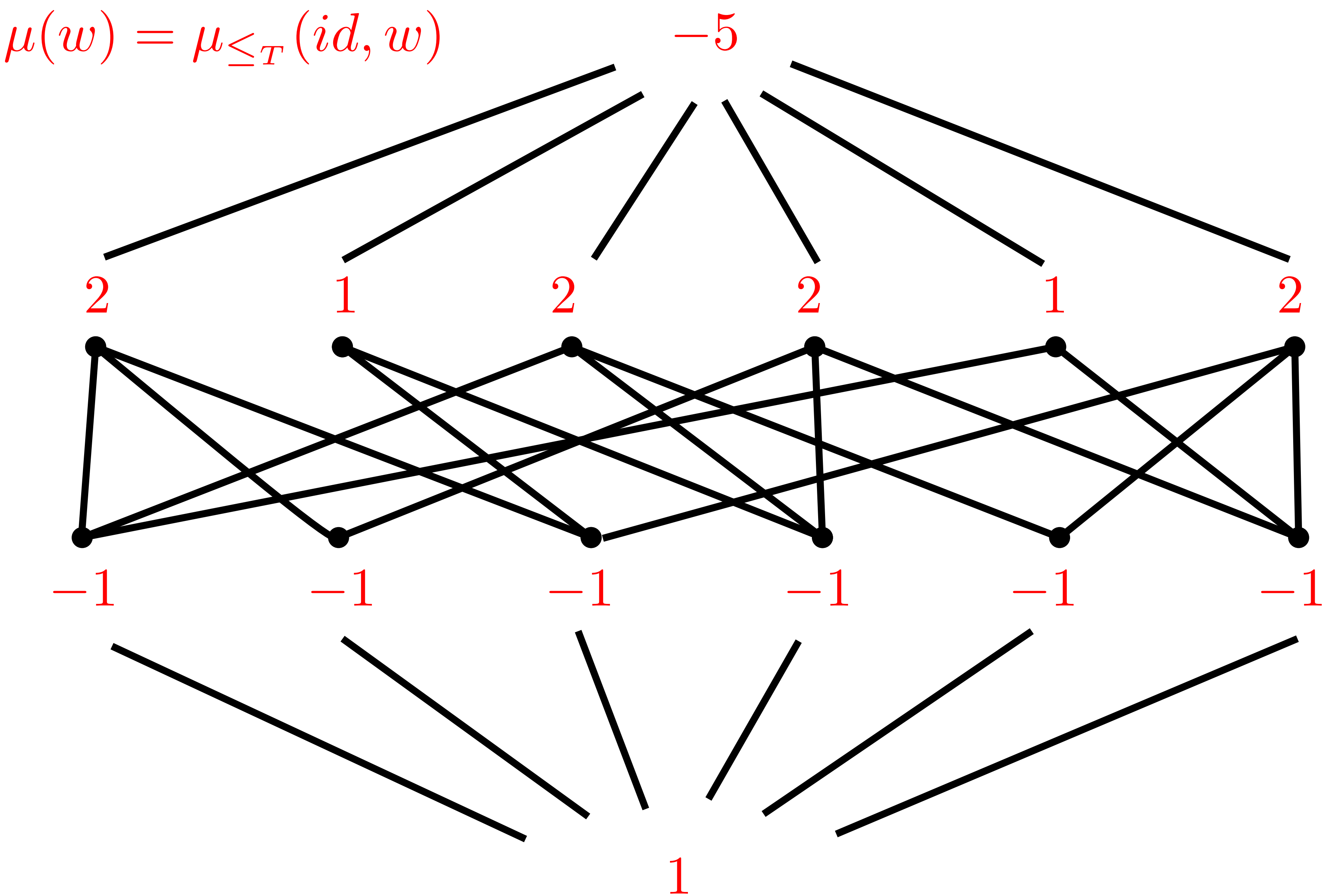

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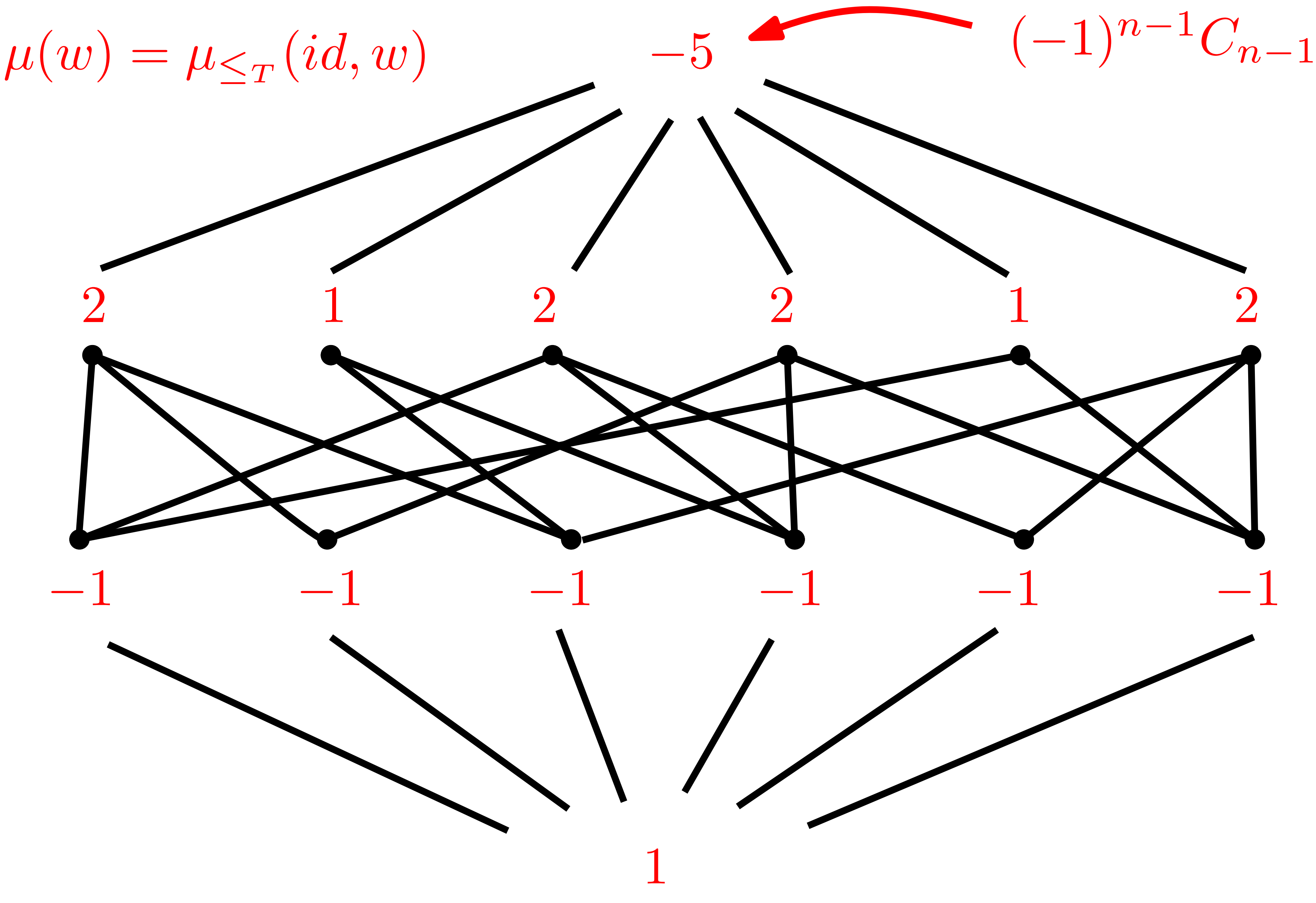
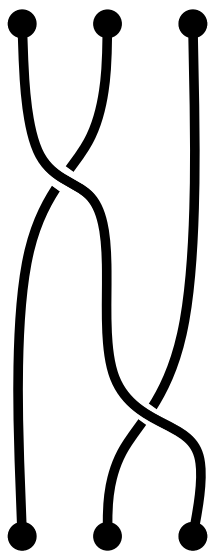


Braids



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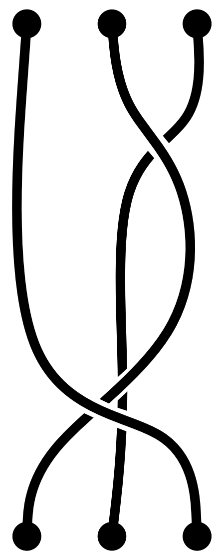

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Braids (up to isotopy) form a group wrt concatenation.

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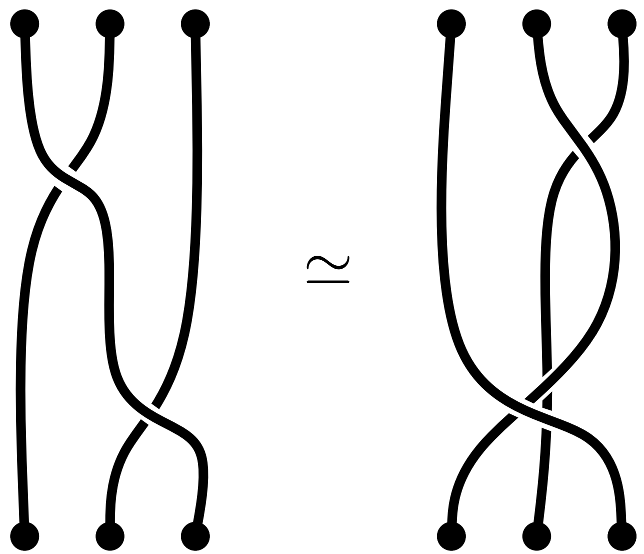

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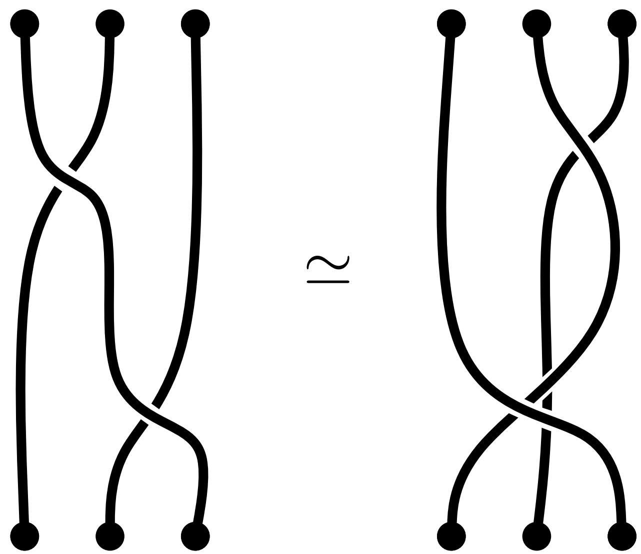

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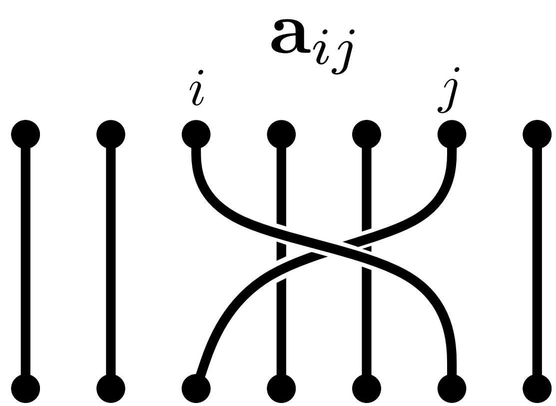

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(Usual generators of the braid group/monoid are $\mathbf{a}_{ii+1}$.)

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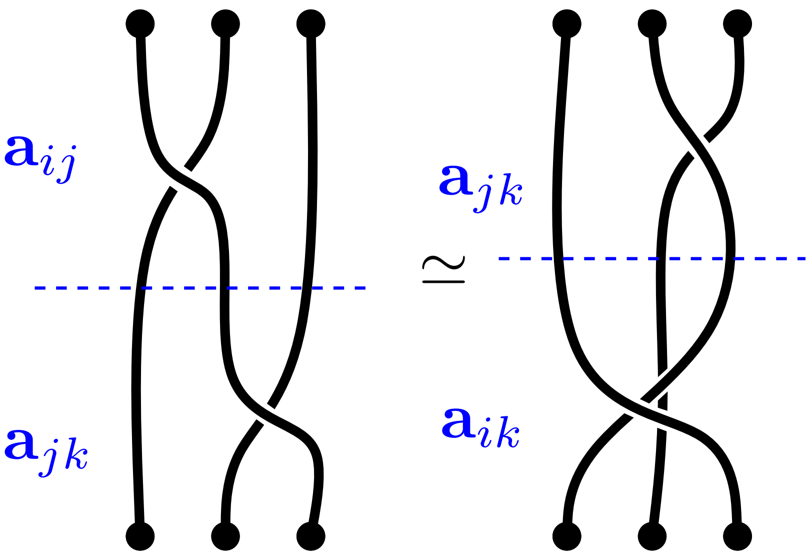

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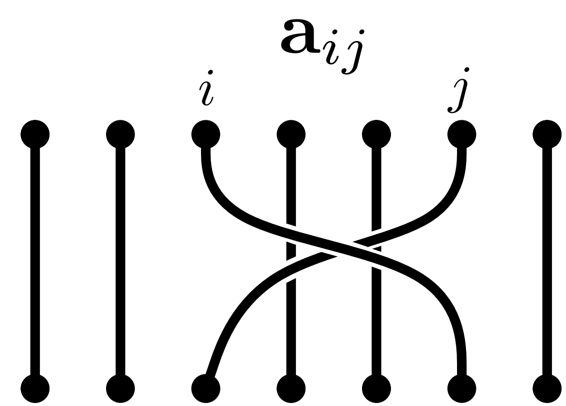

They verify certain relations, eg $\mathbf{a}_{ij}\mathbf{a}_{jk} = \mathbf{a}_{jk}\mathbf{a}_{ik}$.
One can characterize all such relations.

The Birman Ko Lee monoid

Proposition [BKL '98] The monoid BKL_n has generators $\mathbf{a}_{ij}$ for $1 \leq i < j \leq n$ and relations:

$$\mathbf{a}_{ij}\mathbf{a}_{jk} = \mathbf{a}_{jk}\mathbf{a}_{ik} = \mathbf{a}_{ik}\mathbf{a}_{ij} \quad \text{for } i < j < k;$$

$$\mathbf{a}_{ij}\mathbf{a}_{kl} = \mathbf{a}_{kl}\mathbf{a}_{ij} \quad \text{for } i < j < k < l \text{ or } i < k < l < j.$$

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Let $P_n(t) = \displaystyle\sum_{k=0}^{n-1} \frac{(n-1+k)!}{(n-1-k)!k!(k+1)!} t^k$.

Proposition $$\sum_{m \in BKL_n} t^{\ell(m)} = \frac{1}{P_n(-t)}.$$

For instance $$\sum_{m \in BKL_4} t^{\ell(m)} = \frac{1}{1 - 6t + 10t^2 - 5t^3}.$$

Proof of the evaluation of $P_n(t)$

In fact there is a well known relation between the length generating function and the Moebius function of the monoid ordered by divisibility (see [Cartier–Foata]).

$$P_n(t) = \sum_{m \in BKL_n} |\mu(m)| t^{\ell(m)}.$$

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But BKL_n is a Garside monoid with Garside element $C = \mathbf{a}_{12}\mathbf{a}_{23}\cdots\mathbf{a}_{n-1n} \Rightarrow \mu(m) = 0$ if m does not divide C. Furthermore $[1, C]_{\text{left divisibility}} \simeq NC(n)$, and so

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And then I talked to Vic Reiner.

The monoid algebra

We pass from the monoid to its k-algebra A:

Definition The algebra A is defined by the generators $\mathbf{a}_{ij}$ and relations

$$I =< \mathbf{a}_{ij}\mathbf{a}_{kl} - \mathbf{a}_{kl}\mathbf{a}_{ij} \quad \text{for } i < j < k < l \text{ or } i < k < l < j;$$

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Elements of BKL_n form a basis of A, which has a grading $A = \oplus_{k \geq 0} A_k$, with

$$Hilb_A(t) = \sum_{n \geq 0} \dim A_k \cdot t^k = \frac{1}{P_n(-t)}$$

We associate to A another algebra $A^\dagger$, the Koszul dual of A. This transformation $Q \mapsto Q^\dagger$ is defined more generally for all quadratic algebras Q.

Koszul duality of quadratic algebras

Definition A quadratic algebra Q is a graded algebra where the ideal of relations is generated by elements of degree 2.

$$Q = k\langle \mathbf{x}_1, \ldots, \mathbf{x}_m \rangle / Ideal(R)$$

with R vector subspace of $\mathbf{W}_2 := \{\sum_{i,j} \lambda_{ij} \mathbf{x}_i \mathbf{x}_j\}$.

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Examples

(a) $R = \text{span}\{\mathbf{x}_i\mathbf{x}_j - \mathbf{x}_j\mathbf{x}_i, i < j\}$ $R^\dagger = span\{\mathbf{x}_i^2, \mathbf{x}_i\mathbf{x}_j + \mathbf{x}_j\mathbf{x}_i, i < j\}$

Symmetric algebra Exterior algebra

(b) $R = \text{span}\{\mathbf{x}_i\mathbf{x}_j, (i,j) \in I \subseteq [m]^2\}$ $R^\dagger = \text{span}\{\mathbf{x}_i\mathbf{x}_j, (i,j) \in [m]^2 - I\}$

Monomial ideals

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- What is $R^\dagger$ in this case ?

It has a basis consisting of:

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Theorem [Albenque, N. '09; N.' 12]

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Main question: why did I reprove one of my own results ?

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We showed that A is a **Koszul algebra**, which can be defined as "A graded k-algebra Q such that the Q-module k admits a minimal graded free resolution which is **linear**".

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- New work: a nice basis of the algebra $A^\dagger$.

$$A^\dagger = \bigoplus_{w \in NC(n)} A^\dagger[w]$$

with an explicit basis of $A^\dagger[w]$ of cardinality $\mu(w)$.

Other Coxeter groups

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• [Bessis '00] Define the **dual braid monoid** as generated by $\mathbf{a}_t$ with $t \in T$ and relations $\mathbf{a}_t\mathbf{a}_u = \mathbf{a}_u\mathbf{a}_{utu}$ whenever $tu \leq_T c$.

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Its algebra $A(W)$ is clearly quadratic, we can therefore consider the dual algebra $A^\dagger(W)$ and explicit a presentation.

Same questions for $A(W)$ instead of $A = A(S_n)$

Questions for the future

1) Is $A(W)$ numerically Koszul, ie.

$$Hilb_{A(W)}(t) \cdot Hilb_{A^\dagger(W)}(-t) = 1?$$

2) Is $A(W)$ a Koszul algebra?

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Known and To do

- 3) or 2) imply 1).
- 2) is true for type B [Albenque, N. '09].
- Check 3) (or simply 1) for exceptional types by computer.
- Prove 3) by checking that a certain chain complex is exact (V. Féray).
- Use explicit EL-shelling of NC(W).