On the Roots of Generalized Eulerian Polynomials

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Outline

1. Introduction
   - Eulerian polynomials
   - Permutations and inversion sequences
   - An Eulerian statistic on inversion sequences

2. A novel approach to Eulerian polynomials
   - s-inversion sequences and s-Eulerian polynomials
   - Our main result
   - The proof using compatible polynomials

3. Applications
   - h^*-polynomials of s-lecture hall polytope
   - Generalized Eulerian polynomials and q-analogs

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For a permutation $\pi = \pi_1 \ldots \pi_n$ in $\mathfrak{S}_n$, let

$$\text{des}(\pi) = |\{i \mid \pi_i > \pi_{i+1}\}|$$

denote the number of *descents* in π.

The Eulerian polynomial

$$\mathfrak{S}_n(x) := \sum_{\pi \in \mathfrak{S}_n} x^{\text{des}(\pi)} = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle x^k,$$

where $\left\langle {n \atop k} \right\rangle$ is the number of permutations in $\mathfrak{S}_n$ with k descents.

Eulerian numbers: $\left\langle {n \atop k} \right\rangle$

Euler's triangle

		k:					
		0	1	2	3	4	5
n:	1	1					
	2	1	1				
	3	1	4	1			
	4	1	11	11	1		
	5	1	26	66	26	1	
	6	1	57	302	302	57	1

		k:					
		0	1	2	3	4	5
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	5	1	26	66	26	1	
	6	1	57	302	302	57	1

- $\mathfrak{S}_1(x) = 1,$
- $\mathfrak{S}_2(x) = 1 + x,$
- $\mathfrak{S}_3(x) = 1 + 4x + x^2,$
- $\mathfrak{S}_4(x) = 1 + 11x + 11x^2 + x^3, \ldots$

The roots of $\mathfrak{S}_n(x)$

Theorem (Frobenius)

$\mathfrak{S}_n(x)$ *has only (negative and simple) real roots.*

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Corollary

For all $n \geqslant 1$, *the Eulerian numbers*

$$\left\langle {n \atop 0} \right\rangle, \left\langle {n \atop 1} \right\rangle, \ldots, \left\langle {n \atop n-1} \right\rangle$$

form a (strictly) log-concave, and hence unimodal sequence.

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form a (strictly) log-concave, and hence unimodal sequence.

Most proofs of the theorem rely on the recurrence:

$$\mathfrak{S}_n(x) = (1+nx)\mathfrak{S}_{n-1}(x) + x(1-x)\mathfrak{S}'_{n-1}(x).$$

Plan: Generalize Frobenius' theorem

- Various algebraic and enumerative generalizations of $\mathfrak{S}_n(x)$ have been studied. For example:
  - the descent generating function for Coxeter groups,
  - the second-order Eulerian polynomial.
- Does the property of having only real roots hold for these generating functions?
- How far can this be extended?

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Inversion sequences

Let π be a permutation in the symmetric group $\mathfrak{S}_n$.

Definition

The inversion sequence $\boldsymbol{e} = (e_1, \ldots, e_n)$ for a permutation π is defined as

$$e_j = \left|\{i \mid \pi^{-1}(i) > \pi^{-1}(j), i < j\}\right| .$$

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Alternative way to represent permutations.

Example ($n = 3$)

$e_1 e_2 e_3$	$\pi_1 \pi_2 \pi_3$
0 0 0	1 2 3
0 0 1	1 3 2
0 0 2	3 1 2
0 1 0	2 1 3
0 1 1	2 3 1
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Eulerian statistic

Recall that for a permutation π in $\mathfrak{S}_n$,

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Example

$$\mathfrak{S}_n(x) := \sum_{\pi \in \mathfrak{S}_n} x^{\operatorname{des}(\pi)}.$$

Surprise!

The ascent statistics over inversion sequences is Eulerian.

Theorem (Savage, Schuster)

For $e \in \mathrm{I}_n$, *let* $\mathrm{asc}_{\mathrm{I}}(e) = |\{i \mid e_i < e_{i+1}\}|$. *Then*

$$\sum_{e \in \mathrm{I}_n} x^{\mathrm{asc}_{\mathrm{I}}(e)} = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{des}(\pi)}.$$

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Example ($n = 3$)

$e_1 e_2 e_3$	$\text{asc}_I(e)$	$\pi_1 \pi_2 \pi_3$	$\text{des}(\pi)$
0 0 0	0	1 2 3	0
0 0 1	1	1 3 2	1
0 0 2	1	3 1 2	1
0 1 0	1	2 1 3	1
0 1 1	1	2 3 1	1
0 1 2	2	3 2 1	2

Outline

Generalized inversion sequences

Recall some facts about the inversion sequences:

- $I_n = \{(e_1, \ldots, e_n) \in \mathbb{Z}^n \mid 0 \leqslant e_i < i\}$.
- $I_n = \{0\} \times \{0, 1\} \times \cdots \times \{0, 1, \ldots, n-1\}$.
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Definition

For a given sequence $s = (s_1, \ldots, s_n) \in \mathbb{N}^n$, let $I_n^{(s)}$ denote the set of s-*inversion sequences* by

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$$I_n^{(s)} = \{(e_1, \ldots, e_n) \in \mathbb{Z}^n \mid 0 \leqslant e_i < s_i\}.$$

$$I_n = \{0, \ldots, s_1 - 1\} \times \{0, \ldots, s_2 - 1\} \times \cdots \times \{0, \ldots, s_n - 1\}.$$

$$\left|I_n^{(s)}\right| = \prod_{i=1}^{n} s_i .$$

The *ascent* statistic on s-inversion sequences

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Definition

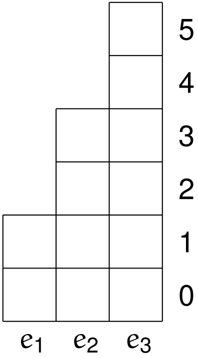
For $e = (e_1, \ldots, e_n) \in I_n^{(s)}$, let

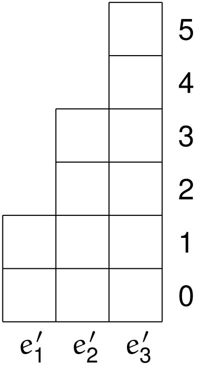
$$\text{asc}_I(e) = \left| \left\{ i \in \{0, \ldots, n-1\} : \frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}} \right\} \right| ,$$

where we use the convention $e_0 = 0$ (and $s_0 = 1$).

The ascent statistic on s-inversion sequences

Two examples for the sequence $s = (2, 4, 6)$





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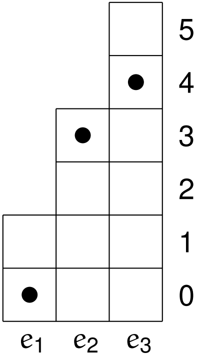
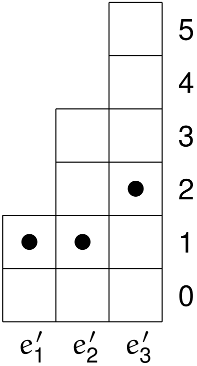


$e = (0, 3, 4)$



$e' = (1, 1, 2)$

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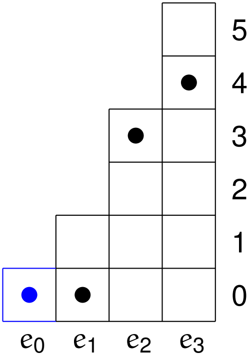

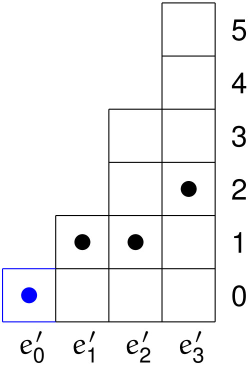

$e = (0, 3, 4)$ $e' = (1, 1, 2)$

The ascent statistic on s-inversion sequences

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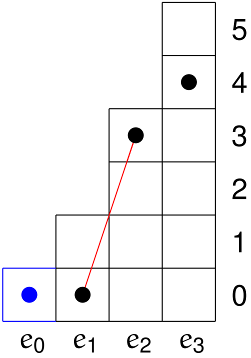

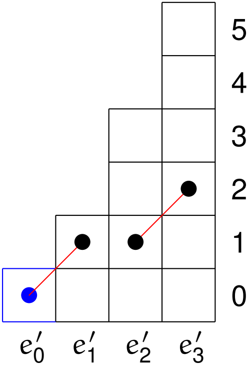

$e = (0, 3, 4)$ with

- $\text{asc}_I(e) = 1$.

$e' = (1, 1, 2)$ with

- $\text{asc}_I(e') = 2$.

s-Eulerian polynomials

Theorem (Savage, Schuster)

$$\mathfrak{S}_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{des}(\pi)} \qquad (1)$$

$$= \sum_{e \in I_n^{(s)}} x^{\mathrm{asc}_I(e)}, \qquad (2)$$

when $s = 1, 2, \ldots, n$.

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when $s = 1, 2, \ldots, n$.

Definition (s-Eulerian polynomials)

For an arbitrary sequence $s = s_1, s_2, \ldots,$ let

$$\mathcal{E}_n^{(s)}(x) := \sum_{e \in I_n^{(s)}} x^{\mathrm{asc}_I(e)}.$$

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On the roots of s-Eulerian polynomials

Theorem (Frobenius)

The Eulerian polynomials

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This can be generalized to the following.

Theorem (Savage, V.)

For any sequence s of nonnegative integers, the s-Eulerian polynomials

$$\mathcal{E}_n^{(s)}(x) = \sum_{e \in I_n^{(s)}} x^{\mathrm{asc}_I(e)}$$

have only real roots.

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Compatible polynomials

Definition

Polynomials $f_1(x), \ldots, f_m(x)$ over $\mathbb{R}$ are *compatible*, if all their conic combinations, i.e., the polynomials

$$\sum_{i=1}^{m} c_i f_i(x) \quad \text{with } c_1, \ldots, c_m \geqslant 0$$

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Fact: A real-rooted polynomial is compatible with itself.

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Fact: A real-rooted polynomial is compatible with itself.

Definition

The polynomials $f_1(x), \ldots, f_m(x)$ are *pairwise compatible* if for all $i, j \in \{1, 2, \ldots, m\}$, $f_i(x)$ and $f_j(x)$ are compatible.

Compatible polynomials

Remark

Polynomials $f(x)$ *and* $g(x)$ *are compatible if and only if each of the following pairs is compatible*

- $af(x)$ *and* $bg(x)$ *for any positive* $a, b \in \mathbb{R}$,
- $xf(x)$ *and* $xg(x)$
- $(c + dx)f(x)$ *and* $(c + dx)g(x)$ *for any positive* $c, d \in \mathbb{R}$.

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A key tool in our proof is the following.

Lemma (Chudnovsky–Seymour)

The polynomials $f_1(x), \ldots, f_m(x)$ *are compatible if and only if they are pairwise compatible.*

Proving more is sometimes easier

Instead of working with

$$\mathcal{E}_n^{(s)}(x) = \sum_{e \in I_n^{(s)}} x^{\mathrm{asc}_I(e)}$$

we will work with the partial sums

$$P_{n,i}^{(s)}(x) := \sum_{\{e \in I_n^{(s)} \mid e_n = i\}} x^{\mathrm{asc}_I(e)}.$$

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Clearly,

$$\mathcal{E}_n^{(s)}(x) = \sum_{i=0}^{s_n - 1} P_{n,i}^{(s)}(x).$$

Thus, $P_{n,i}^{(s)}(x)$ compatible $\Longrightarrow \mathcal{E}_n^{(s)}(x)$ has only real roots.

A simple recurrence

$$P_{n,i}^{(s)}(x) = \sum_{\{e \in I_n^{(s)} \mid e_n = i\}} x^{\mathrm{asc}_I(e)}.$$

Lemma

Given a sequence $s = \{s_i\}_{i \geqslant 1}$ *of positive integers, let* $n \geqslant 1$ *and* $0 \leqslant i < s_n$. *Then for* $n > 1$,

$$P_{n,i}^{(s)}(x) = \sum_{j=0}^{\ell-1} x P_{n-1,j}^{(s)}(x) + \sum_{j=\ell}^{s_{n-1}-1} P_{n-1,j}^{(s)}(x),$$

where

$$\ell = \lceil i s_{n-1}/s_n \rceil.$$

When $n = 1$, $P_{1,0}^{(s)}(x) = 1$ *and* $P_{1,i}^{(s)}(x) = x$ *for* $i > 0$.

A simple recurrence

$$\mathrm{asc}_I(e) = \left| \left\{ i \in \{0, \ldots, n-1\} : \frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}} \right\} \right| .$$

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When $n = 1$, $P_{1,0}^{(s)}(x) = 1$ *and* $P_{1,i}^{(s)}(x) = x$ *for* $i > 0$.

Again: prove something *even* stronger

Theorem (Savage, V.)

Given a sequence $s = \{s_i\}_{i \geqslant 1}$, *for all* $0 \leqslant i \leqslant j < s_n$,

(i) $P_{n,i}^{(s)}(x)$ *and* $P_{n,j}^{(s)}(x)$ *are compatible, and*

(ii) $xP_{n,i}^{(s)}(x)$ *and* $P_{n,j}^{(s)}(x)$ *are compatible.*

Corollary

The polynomials $P_{n,0}^{(s)}(x), P_{n,1}^{(s)}(x) \ldots, P_{n,s_n-1}^{(s)}(x)$ *are compatible.*

Proof of real-rootedness

Use induction. Base case: $(x, 1)$ or (x, x) or (x^2, x).

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For $i < j$, we have $\ell \leqslant k$.

$$P^{(s)}_{n+1,i} = x\,\underbrace{(P^{(s)}_{n,0} + \cdots + P^{(s)}_{n,\ell-1})}_{\ell} + \cdots + P^{(s)}_{n,k-1} + \cdots + P^{(s)}_{n,s_n-1}\,,$$

$$P^{(s)}_{n+1,j} = x\,\underbrace{(P^{(s)}_{n,0} + \cdots + P^{(s)}_{n,\ell-1} + \cdots + P^{(s)}_{n,k-1})}_{k} + \cdots + P^{(s)}_{n,s_n-1}\,.$$

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$$P^{(s)}_{n+1,j} = x\,\underbrace{(P^{(s)}_{n,0} + \cdots + P^{(s)}_{n,\ell-1} + \cdots + P^{(s)}_{n,k-1})}_{k} + \cdots + P^{(s)}_{n,s_n-1}.$$

(i) $P^{(s)}_{n+1,i}(x)$ and $P^{(s)}_{n+1,j}(x)$ are compatible because

$$\left\{xP^{(s)}_{n,\alpha}\right\}_{0\leqslant\alpha<\ell} \cup \left\{(c+dx)P^{(s)}_{n,\beta}\right\}_{\ell\leqslant\beta<k} \cup \left\{P^{(s)}_{n,\gamma}\right\}_{k\leqslant\gamma<s_n}$$

are pairwise compatible.

Proof of real-rootedness (cont'd)

Now

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- Two polynomials from the same set are compatible by IH(i).

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- $xP_{n,\alpha}^{(s)}$ and $P_{n,\gamma}^{(s)}$ is compatible by IH(ii).

Now

$$\left\{xP_{n,\alpha}^{(s)}\right\}_{0\leqslant\alpha<\ell} \cup \left\{(c+dx)P_{n,\beta}^{(s)}\right\}_{\ell\leqslant\beta<k} \cup \left\{P_{n,\gamma}^{(s)}\right\}_{k\leqslant\gamma<s_n}$$

are parwise compatible because of the following:

- Two polynomials from the same set are compatible by IH(i).
- $xP_{n,\alpha}^{(s)}$ and $P_{n,\gamma}^{(s)}$ is compatible by IH(ii).
- $xP_{n,\alpha}^{(s)}$ and $(c+dx)P_{n,\beta}^{(s)}$ are compatible because
  - $xP_{n,\alpha}^{(s)}, xP_{n,\beta}^{(s)}, P_{n,\beta}^{(s)}$ are pairwise compatible.

Proof of real-rootedness (cont'd)

Now

$$\left\{xP^{(s)}_{n,\alpha}\right\}_{0\leqslant\alpha<\ell} \cup \left\{(c+dx)P^{(s)}_{n,\beta}\right\}_{\ell\leqslant\beta<k} \cup \left\{P^{(s)}_{n,\gamma}\right\}_{k\leqslant\gamma<s_n}$$

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- $xP^{(s)}_{n,\alpha}$ and $P^{(s)}_{n,\gamma}$ is compatible by IH(ii).
- $xP^{(s)}_{n,\alpha}$ and $(c+dx)P^{(s)}_{n,\beta}$ are compatible because
  - $xP^{(s)}_{n,\alpha}, xP^{(s)}_{n,\beta}, P^{(s)}_{n,\beta}$ are pairwise compatible.
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Proof of real-rootedness (cont'd)

(i) Thus, $P_{n+1,i}^{(s)}(x)$ and $P_{n+1,j}^{(s)}(x)$ are compatible because

$$\left\{xP_{n,\alpha}^{(s)}\right\}_{0\leqslant\alpha<\ell} \cup \left\{(c+dx)P_{n,\beta}^{(s)}\right\}_{\ell\leqslant\beta<k} \cup \left\{P_{n,\gamma}^{(s)}\right\}_{k\leqslant\gamma<s_n}$$

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Outline

Geometric interpretation

The *Ehrhart series* of a polytope $\mathcal{P}$ in $\mathbb{R}^n$ is the series

$$\sum_{t \geqslant 0} i(\mathcal{P}, t)x^t,$$

where $t\mathcal{P}$ is the t-fold *dilation* of $\mathcal{P}$:

$$t\mathcal{P} = \{(t\lambda_1, t\lambda_2, \ldots, t\lambda_n) \mid (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{P}\},$$

and $i(\mathcal{P}, t)$ is the number of points in $t\mathcal{P}$, all of whose coordinates are integer:

$$i(\mathcal{P}, t) = |t\mathcal{P} \cap \mathbb{Z}^n|.$$

The h^*-polynomial of a polytope

If all vertices of $\mathcal{P}$ are integer, then $i(\mathcal{P}, t)$ is a polynomial in t and the Ehrhart series of $\mathcal{P}$ has the form

$$\sum_{t \geqslant 0} i(\mathcal{P}, t)x^t = \frac{h(x)}{(1-x)^n},$$

for a polynomial

$$h(x) = h_0 + h_1 x + \cdots h_d x^d$$

known as the h^*-polynomial of $\mathcal{P}$. Here d is the dimension of $\mathcal{P}$.

h^*-polynomial of the s-lecture hall polytope

Definition (s-lecture hall polytope)

$$\mathcal{P}_n^{(s)} = \left\{ \lambda \in \mathbb{R}^n : 0 \leqslant \frac{\lambda_1}{s_1} \leqslant \frac{\lambda_2}{s_2} \leqslant \cdots \leqslant \frac{\lambda_n}{s_n} \leqslant 1 \right\}.$$

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Theorem (Savage, Schuster)

For any sequence s of positive integers,

$$\sum_{t \geqslant 0} i(\mathcal{P}_n^{(s)}, t) x^t = \frac{\mathcal{E}_n^{(s)}(x)}{(1-x)^{n+1}}.$$

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Our theorem for $\mathcal{E}_n^{(s)}(x)$ implies:

Corollary (Savage, V.)

For any sequence s of positive integers, the h^*-polynomial of the s-lecture hall polytope has all roots real.

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Variations on a theme: Eulerian polynomials

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have only real roots.

Theorem (Savage, V.)

The MacMahon–Carlitz q-analog

$$\mathfrak{S}_n(x, q) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{des}(\pi)} q^{\text{maj}(\pi)}$$

has only real roots for $q \geqslant 0$.

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has only real roots for $q \geqslant 0$.

Our result also holds for

- the hyperoctahedral group (type B), and
- the generalized symmetric group (wreath product $\mathfrak{S}_n \wr C_r$), and
- other q-statistics (finv, comaj).

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Question:

- Is there an s-inversion sequence which will give the type D Eulerian polynomial?