Growth diagrams, crystal operators and Cauchy kernel expansions in type A

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Ellwagen

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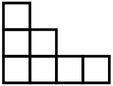
SSYTableaux

Ferrers diagram

Each partition λ of n is associated to a collection of squares (or cells) called a Ferrers diagram, $dg(\lambda)$ or Young diagram. The i-th row of the Ferrers diagram consists of λ_i cells.



$\lambda = (4, 2, 1)$ $\quad\quad$ $|\lambda| = 4 + 2 + 1 = 7$

$\lambda = 421$

SSYTableaux

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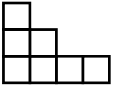

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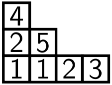
SSYT

A filling of shape λ is a map $T : dg(\lambda) \to \mathcal{A} = \{1, \ldots, n\}$, where $n \geq |\lambda|$. A semi-standard Young tableau (SSYT) of shape λ is a filling of λ such that T is weakly increasing along each row and strictly increasing along each column.



RSSYTableaux

RSSYT

A reverse semi-standard tableau (RSSYT) (or strict-column plane partition) is a filling of a Ferrers diagram such that the entries in each row are weakly decreasing from left to right, and strictly decreasing from bottom to top.

2			
4	1		
5	5	4	3

$$cont(T) = (1, 1, 3, 1, 1)$$

A bijection between RSSYT and semi-skyline augmented fillings (SSAF) preserving the weight (S. Mason, 2008)

$$\widetilde{P} = \begin{matrix} 1 & & \\ 3 & & \\ 4 & 1 & \\ 5 & 2 & \\ 7 & 3 & 2 \end{matrix}$$

A bijection between RSSYT and semi-skyline augmented fillings (SSAF) preserving the weight (S. Mason, 2008)

$$\widetilde{P} = \begin{array}{lll} 1 & & \\ 3 & & \\ 4 & 1 & \\ 5 & 2 & \\ 7 & 3 & 2 \end{array} \qquad R_1 = \{7, 5, 4, 3, 1\}$$

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1		3	4	5		7
1	2	3	4	5	6	7

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$R_1 = \{7, 5, 4, 3, 1\}$

$R_2 = \{3, 2, 1\}$

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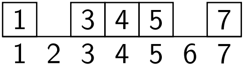

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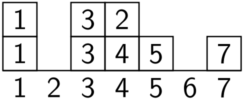

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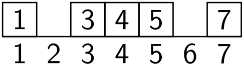
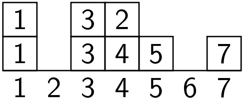


$R_1 = \{7, 5, 4, 3, 1\}$



$R_2 = \{3, 2, 1\}$

$R_3 = \{2\}$

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$$\widetilde{P} = \begin{array}{lll} 1 & & \\ 3 & & \\ 4 & 1 & \\ 5 & 2 & \\ 7 & 3 & 2 \end{array}$$

1		3	4	5		7
1	2	3	4	5	6	7

$R_1 = \{7, 5, 4, 3, 1\}$

1		3	2			
1		3	4	5		7
1	2	3	4	5	6	7

$R_2 = \{3, 2, 1\}$

		2				
1		3	2			
1		3	4	5		7
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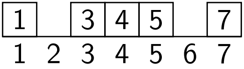

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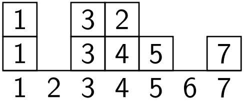
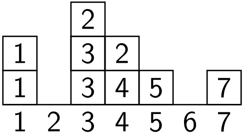


$R_2 = \{3, 2, 1\}$



$R_3 = \{2\}$

$sh(\widetilde{P}) = (3, 2, 2, 1, 1, 0, 0)$

$sh(F) = (2, 0, 3, 2, 1, 0, 1)$

A weight and shape preserving bijection between RSSYT and SSYT

The reverse Schensted insertion applied to $b_1 \dots b_m$ consists of reversing the roles of $\leq$ and $\geq$ in defining the Schensted insertion of $b_1 \dots b_m$, to get the reverse tableau. (Equivalently, apply Schensted insertion to $-b_m, \dots, -b_1$ instead of $b_1, \dots, b_m$ and then change the sign back to positive all entries of the SSYT $P(-b_m, \dots, -b_1)$, to obtain a reverse SSYT $\tilde{P}$.

SSYT $T \rightarrow$ RSSYT $\widetilde{T} =$ reverse Schensted insertion of the column word of T

Two equivalent weight preserving, shape rearranging bijections between SSYT and SSAF (S. Mason, 2008)

$$T \to \rho(\widetilde{T})\ SSAF$$

$T \to \Psi(T) =$ Schensted insertion analogue applied to column word of T

to obtain a SSAF

$$= \rho(\widetilde{T})$$

Right key of SSYT

Right key of SSYT (S.Mason 2009).

$$T = \begin{array}{llll} 4 & & & \\ 2 & 5 & & \\ 1 & 3 & 3 & 3 \end{array} \qquad \Psi(T) = \begin{array}{ccccc} & & \boxed{2} & & \\ & & \boxed{3} & & \\ & & \boxed{3} & & \boxed{4} \\ \boxed{1} & & \boxed{3} & & \boxed{5} \\ \hline 1 & 2 & 3 & 4 & 5 \end{array}$$

$$sh(\Psi(T)) = (1, 0, 4, 0, 2) \qquad key(sh(\Psi(T))) = \begin{array}{llll} 5 & & & \\ 3 & 5 & & \\ 1 & 3 & 3 & 3 \end{array}$$

Remark.

The original definition of right key of a tableau is due to Lascoux and Schützenberger (1988).

Robinson-Schensted-Knuth(RSK) correspondences for pairs of SSYT, RSSYT, SSAF

RSK.

The Robinson-Schensted-Knuth(RSK) algorithm gives a bijection between biwords $w = \begin{pmatrix} a_n & \dots & a_1 \\ b_n & \dots & b_1 \end{pmatrix}$ in lexicographic order and pairs of SSYTs of same shape.

Robinson-Schensted-Knuth(RSK) correspondences for pairs of SSYT, RSSYT, SSAF

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Reverse RSK

The reverse RSK algorithm is the obvious variant of the RSK algorithm, where we apply RSK for

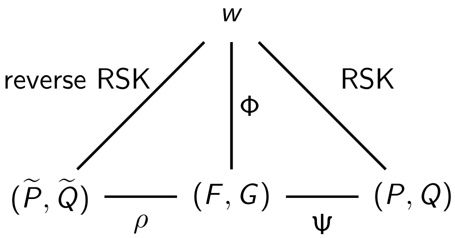
$$\widetilde{w} = \begin{pmatrix} -a_1 & \dots & -a_n \\ -b_1 & \dots & -b_n \end{pmatrix}$$

instead of

$$w = \begin{pmatrix} a_n & \dots & a_1 \\ b_n & \dots & b_1 \end{pmatrix}$$

Then change the sign back to positive of all entries of the pair.

A triangle of RSK's (S.Mason 2008)



$$sh(F)^{+} = sh(G)^{+} = sh(P) = sh(Q) = sh(\widetilde{P}) = sh(\widetilde{Q})$$

$$key(sh(F)) = k_{+}(P), \; key(sh(G)) = k_{+}(Q)$$

$$cont(F) = cont(P) = cont(\widetilde{P}), \; cont(G) = cont(Q) = cont(\widetilde{Q})$$

Representation of a biword w in the $n \times n$ square.

$$w = \begin{pmatrix} 1 & 1 & 2 & 3 & 4 & 4 & 5 & 7 & 7 \\ 2 & 7 & 2 & 4 & 1 & 3 & 3 & 1 & 1 \end{pmatrix}$$

1						
		1				
			1	1		
1	1					
			1			2

Representation of a biword w in the $n \times n$ square.

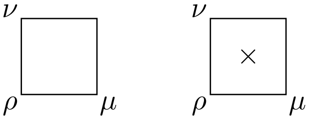
$$w = \begin{pmatrix} 1 & 1 & 2 & 3 & 4 & 4 & 5 & 7 & 7 \\ 2 & 7 & 2 & 4 & 1 & 3 & 3 & 1 & 1 \end{pmatrix}$$

1						
		1				
			1	1		
1	1					
			1			2

	×								
			×						
						×			
					×				
		×							
×									
									×
								×	
				×					

Fomin's growth diagram:Local rules

- If $\rho = \mu = \nu$, and if there is no cross in the cell, then $\lambda = \rho$.
- If $\rho = \mu \neq \nu$, then $\lambda = \nu$.
- If $\rho = \nu \neq \mu$, then $\lambda = \mu$.
- If ρ, μ, ν are pairwise different, then $\lambda = \mu \cup \nu$.
- If $\rho \neq \mu = \nu$, then λ is formed by adding a square to the $(k+1)$-st row of $\mu = \nu$, given that $\mu = \nu$ and ρ differ in the k-th row.
- If $\rho = \mu = \nu$, and if there is a cross in the cell, then λ is formed by adding a square to the first row of $\rho = \mu = \nu$.



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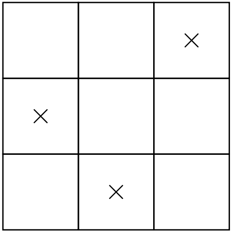

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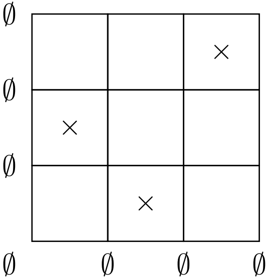

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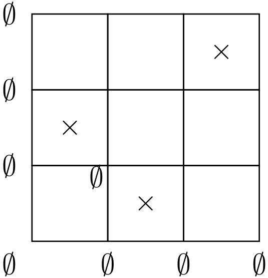

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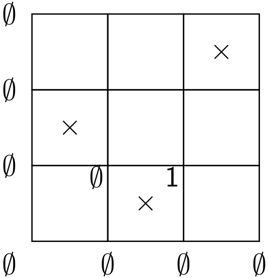

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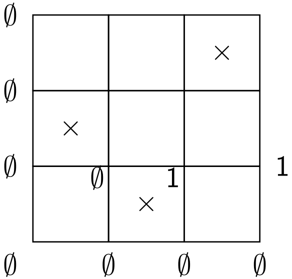

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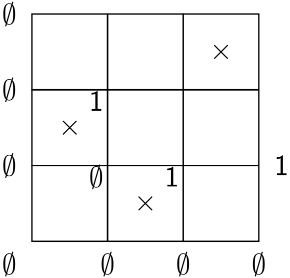

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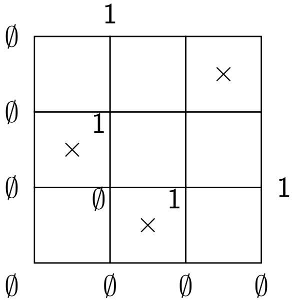

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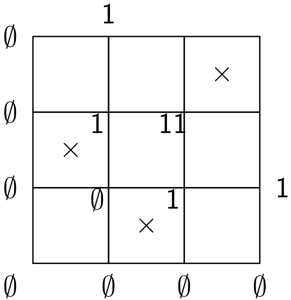

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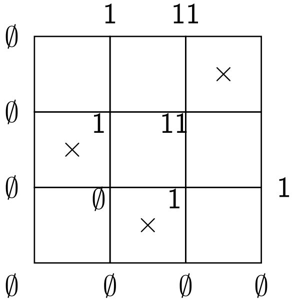

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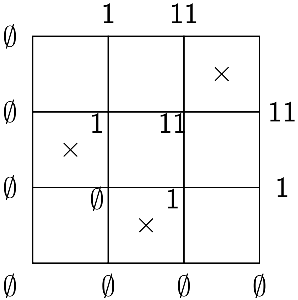

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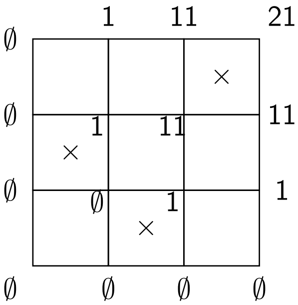
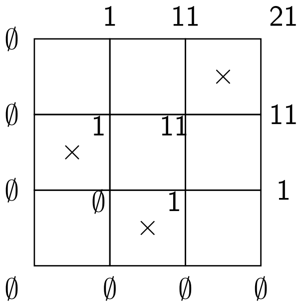
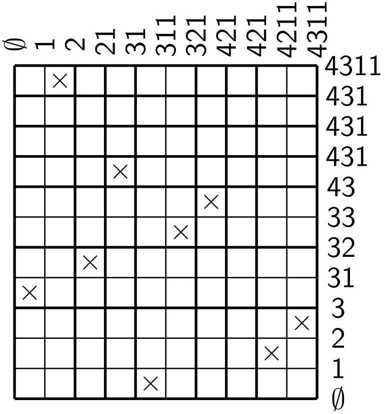




Greene's invariants: the maximal length of a NE chain is the length of the first row of the partition.
The maximum of the sum of k disjoint NE chains -The maximum of the sum of $(k-1)$ disjoint NE chains $=$ the length of the kth row of the partition.

Fomin's growth diagram for RSK



P=

7			
4			
2	2	3	
1	1	1	3

Q=

7			
4			
2	4	7	
1	1	3	5

Greene's invariants: the maximal length of a NE chain is the length of the first row.
The maximum of the sum of k disjoint NE chains -The maximum of the sum of $(k-1)$ disjoint NE chains = the length of the kth row.

4311

431

431

431

43

32

3

4311

431

431

431

43

32

3

1	1	1

4311

431

431

431

43

32

2	2	
1	1	1

3

1	1	1

4311

431

431

431

43

2	2	3	
1	1	1	3

32

2	2	
1	1	1

3

1	1	1

4311

431

431

431

4			
2	2	3	
1	1	1	3

43

2	2	3	
1	1	1	3

32

2	2	
1	1	1

3

1	1	1

4311

431 —

431 —

431

4			
2	2	3	
1	1	1	3

43

2	2	3	
1	1	1	3

32

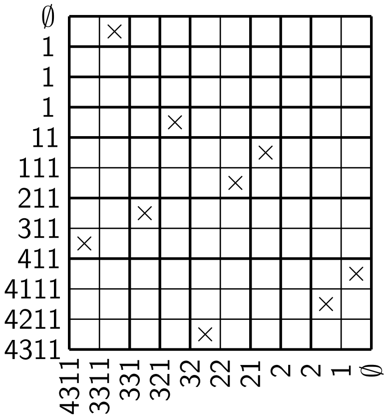
2	2	
1	1	1

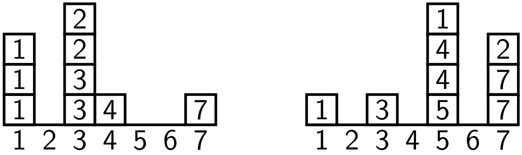
3

1	1	1

4311	7 4 2 2 3 1 1 1 3
431	—
431	—
431	4 2 2 3 1 1 1 3
43	2 2 3 1 1 1 3
32	2 2 1 1 1
3	1 1 1

Fomin's growth diagram for RSK analogue (SSAF)





1

1

1

11

111

4111

211

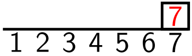
4211

311

411

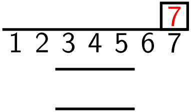
4311

7	1		
6	1		
5	1		
4	11		
3	111		
		1	4111
3	211		
		1	4211
2	311		
2	411	1	4311



7	1		
6	1		
5	1		
4	11		
3	111		
		1	4111
3	211		
		1	4211
2	311		
2	411	1	4311

7 1



6 1

5 1

4 11

3 111

1 4111

3 211

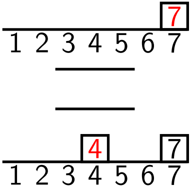
1 4211

2 311

2 411

1 4311

7 1
6 1
5 1
4 11



3 111

1 4111

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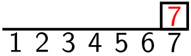
2 311

2 411

1 4311

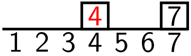
7 1



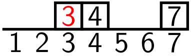
6 1

5 1

4 11



3 111



1 4111

3 211

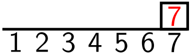
1 4211

2 311

1 4311

2 411

7 1



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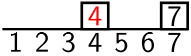
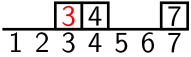
5 1

4 11

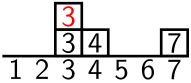


3 111



1 4111

3 211



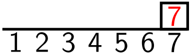
2 311

1 4211

2 411

1 4311

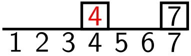
7 1



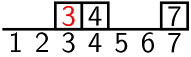
6 1

5 1

4 11



3 111



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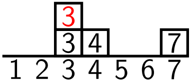

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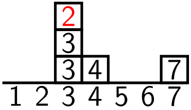

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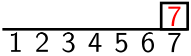
7	1			
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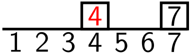
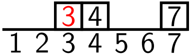
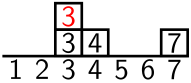
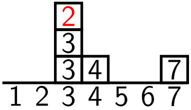
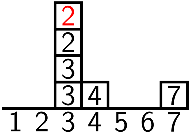
5	1			
4	11			
3	111			
			1	4111
3	211			
2	311		1	4211
2	411		1	4311

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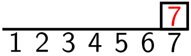

6 1 ——

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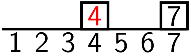

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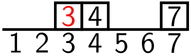

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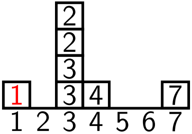

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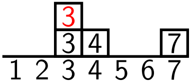

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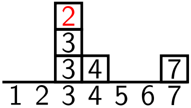

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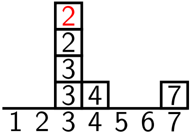

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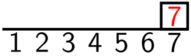

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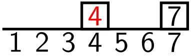

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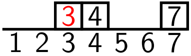

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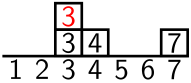

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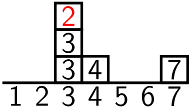

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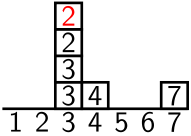

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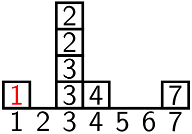
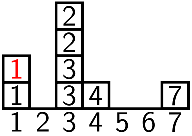
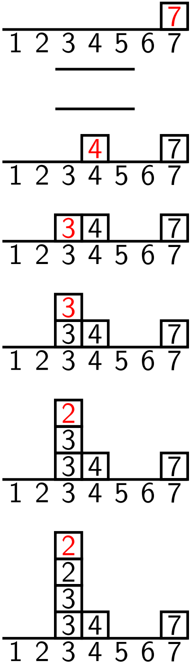


1 4211



1 4311

7	1
6	1
5	1
4	11
3	111
3	211
2	311
2	411



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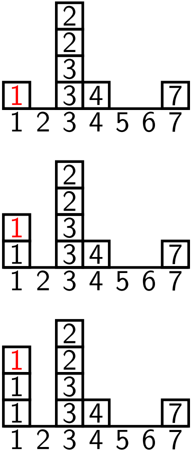

A symmetric Cauchy identity

Cauchy identity.

$$\prod_{(i,j)\in[n]\times[n]} (1 - x_i y_j)^{-1} = \sum_{\lambda \ \textit{partition} \ \in\mathbb{N}^n} s_\lambda(x) s_\lambda(y).$$

$x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$

A non-symmetric Cauchy identity.

$$\prod_{(i,j)\in dg(n,n-1,\dots,1)}(1-x_iy_j)^{-1}=\prod_{i+j\leq n+1}(1-x_iy_j)^{-1}=\sum_{\nu\in\mathbb{N}^n}\widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y).$$

A non-symmetric Cauchy identity.

$$\prod_{(i,j)\in dg(n,n-1,\dots,1)} (1-x_i y_j)^{-1} = \prod_{i+j\leq n+1} (1-x_i y_j)^{-1} = \sum_{\nu\in\mathbb{N}^n} \widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y).$$

- Amy M. Fu, Alain Lascoux (2009) algebraic proof using Demazure operators in type A.

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- Alain Lascoux (2003) crystal version of RSK and combinatorial interpretation of Demazure operators in terms of crystal operators.

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- Alain Lascoux (2003) crystal version of RSK and combinatorial interpretation of Demazure operators in terms of crystal operators.
- O. Azenhas, A. Emami, *Semi-skyline augmented fillings and non-symmetric Cauchy kernels for stair-type shapes*. (FPSAC13, to appear in DMTCS Proceedings).

Demazure operators

Demazure operators (isobaric divided differences) in type A.
$\pi_i,\ \widehat{\pi}_i : \mathbb{Z}[x_1,\dots,x_n] \to \mathbb{Z}[x_1,\dots,x_n]$

$$\pi_i : f \mapsto \pi_i(f) := \frac{x_i f - x_{i+1} s_i(f)}{x_i - x_{i+1}}, \quad 1 \leq i < n, \ \ \widehat{\pi}_i := \pi_i - 1.$$

Demazure operators (isobaric divided differences) in type A.
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Braid relations and quadratic relations

$$\pi_i \pi_j = \pi_j \pi_i \quad \textit{for} \ \ |i-j| > 1 \qquad \widehat{\pi}_i \widehat{\pi}_j = \widehat{\pi}_j \widehat{\pi}_i \quad \textit{for} \ \ |i-j| > 1$$

$$\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1} \qquad \widehat{\pi}_i \widehat{\pi}_{i+1} \widehat{\pi}_i = \widehat{\pi}_{i+1} \widehat{\pi}_i \widehat{\pi}_{i+1}$$

$$\pi_i \pi_i = \pi_i \qquad \widehat{\pi}_i \widehat{\pi}_i = -\widehat{\pi}_i$$

Let $\sigma \in \mathfrak{S}_n$ be a permutation. Define $\pi_\sigma = \pi_{i_1} \pi_{i_2} \ldots \pi_{i_k}$, and $\widehat{\pi}_\sigma = \widehat{\pi}_{i_1} \widehat{\pi}_{i_2} \ldots \widehat{\pi}_{i_k}$, where $s_{i_1} \ldots s_{i_k}$ is a reduced decomposition of σ.

(Strong) Bruhat order

(Strong) Bruhat order in the symmetric group ($\mathfrak{S}_n$)

Let σ, $\mu \in \mathfrak{S}_n$. We say that σ is less or equal than μ in the Bruhat order and we write $\sigma \leq \mu$ if some subword of some reduced decomposition of μ is a reduced decomposition of σ.

(Strong) Bruhat order in compositions

Let α_1 and α_2 be two rearrangements of a partition λ in $\mathbb{N}^n$. Then $\alpha_1 \leq \alpha_2$ in Bruhat order if and only if $key(\alpha_1) \leq key(\alpha_2)$.

Demazure character/ key polynomial

Given the partition λ and $\alpha \in \mathbb{N}^n$ a rearrangement of λ, let $\sigma \in \mathfrak{S}_n$ be the shortest permutation such that $\sigma\lambda = \alpha$. Then

$$\kappa_\alpha(x) = \pi_\sigma(x^\lambda).$$

Demazure character/ key polynomial

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Demazure atom

Given the partition λ and $\alpha \in \mathbb{N}^n$ a rearrangement of λ, let $\sigma \in \mathfrak{S}_n$ be the shortest permutation such that $\sigma\lambda = \alpha$. Then

$$\widehat{\kappa}_\alpha(x) = \widehat{\pi}_\sigma(x^\lambda).$$

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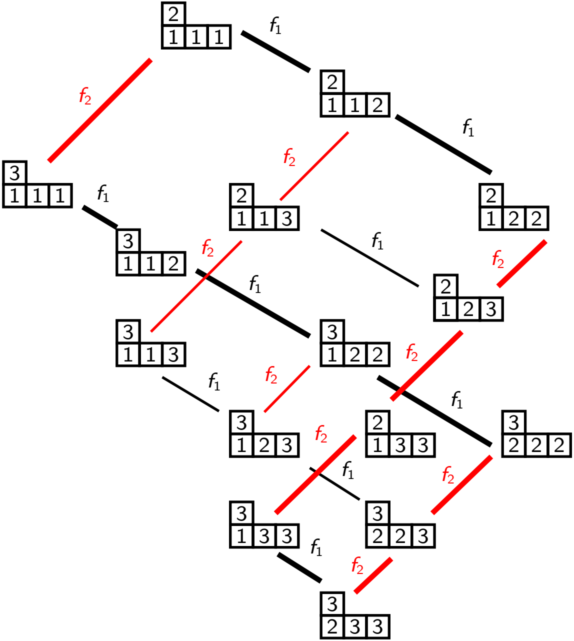
$$\widehat{\kappa}_\alpha(x) = \widehat{\pi}_\sigma(x^\lambda).$$

Properties of Key polynomials.

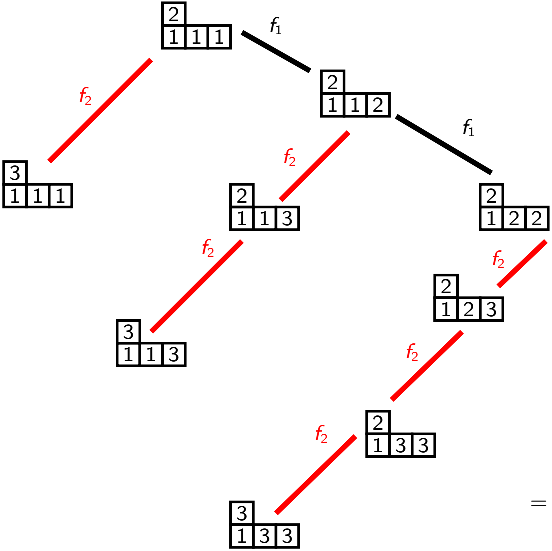
$$\pi_i\kappa_\alpha = \begin{cases} \kappa_{s_i\alpha} & \text{if } \alpha_i > \alpha_{i+1} \\ \kappa_\alpha & \text{otherwise} \end{cases}$$

$$\widehat{\pi}_i\widehat{\kappa}_\alpha = \begin{cases} \widehat{\kappa}_{s_i\alpha} & \text{if } \alpha_i > \alpha_{i+1} \\ -\widehat{\kappa}_\alpha & \alpha_i < \alpha_{i+1} \\ 0 & \alpha_i = \alpha_{i+1}. \end{cases}$$

The crystal graph $\mathfrak{B}_\lambda$ corresponding to partition $\lambda = (3, 1, 0)$.



The Demazure crystal graph $\mathfrak{B}_{s_2 s_1(\lambda)}$ corresponding to vector $s_2 s_1(\lambda) = (1, 0, 3)$.



$$\kappa_{103}(x_1, x_2, x_3) = \pi_2\pi_1(x^{310})$$

$$= \pi_2(x^{220} + x^{130} + x^{310})$$

$$= x^{220} + x^{130} + x^{310} + x^{301} + x^{211}$$

$$= x^{202} + x^{121} + x^{112} + x^{103}$$

Crystal operators for SSYT/SSAF

- T a SSYT tableau at the begining of an i-string in the crystal graph

$$f_{s_i}(T) := \{f_i^{m_i}(T) : m_i \geq 0\} \setminus \{0\} = \{T\} \cup \{f_i^{m_i}(T) : m_i > 0\} \setminus \{0\}$$

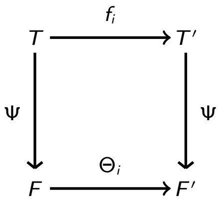
$$\pi_i(x^T) = \sum_{U \in f_{s_i}(T)} x^U = x^T + \hat{\pi}_i(x^T)$$

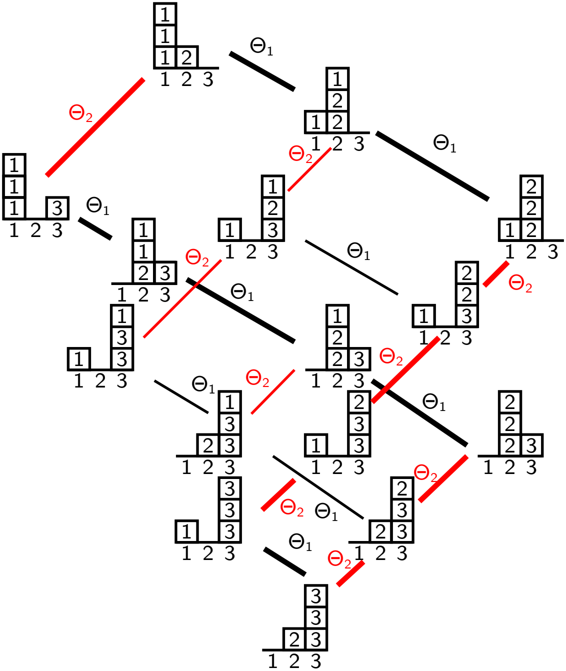
- $\sigma = s_{i_N} \ldots s_{i_1}$ a reduced decomposition

$$f_\sigma(T) := \{f_{i_N}^{m_N} \ldots f_{i_2}^{m_2} f_{i_1}^{m_1}(T) : m_i \geq 0\} \setminus \{0\}$$

$$\pi_\sigma(x^T) = \sum_{U \in f_\sigma(T)} x^U = \sum_{\mu \leq \sigma} \hat{\pi}_\mu(x^T)$$

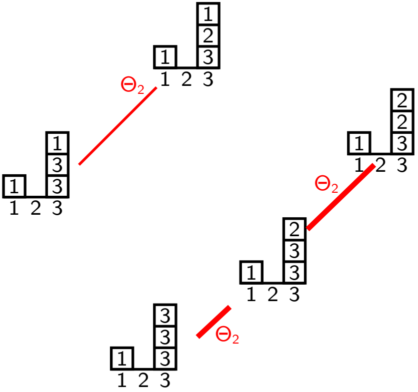
An analogue of crystal operator for SSAF (S.Mason 2009).





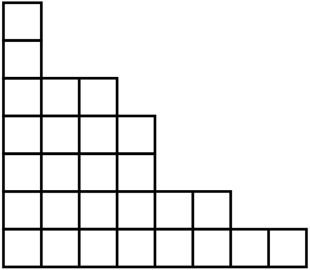
$$\widehat{\kappa}_\nu(x) = \sum_{\substack{\nu\in\mathbb{N}^n\\ K_+(T)=key(\nu)}} x^T$$

$$\kappa_\nu(x) = \sum_{\substack{\nu\in\mathbb{N}^n\\ K_+(T)\leq key(\nu)}} x^T$$



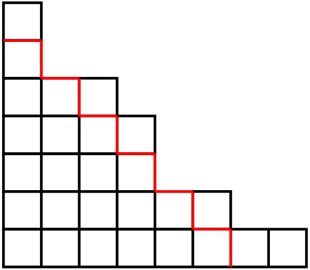
$$\widehat{\kappa}_{103}(x) = x^{202} + x^{211} + x^{103} + x^{112} + x^{121}$$

Lascoux's Cauchy kernel expansion over Ferrers shapes.



$$F_\lambda(x,y) := \prod_{(i,j)\in\lambda} (1-x_i y_j)^{-1} = \sum_{\nu\in\mathbb{N}^k} (\pi_{\sigma(\lambda,NW)}\widehat{\kappa}_\nu(x))(\pi_{\sigma(\lambda,SE)}\kappa_{\omega\nu}(y)).$$

Lascoux's Cauchy kernel expansion over Ferrers shapes.



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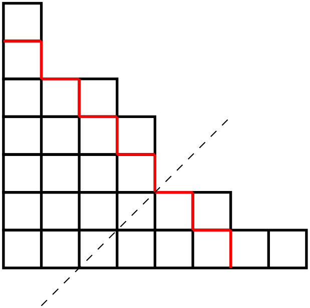

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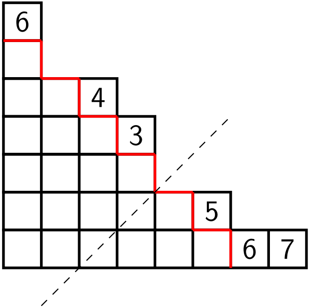

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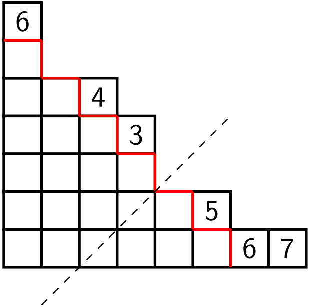
Lascoux's Cauchy kernel expansion over Ferrers shapes.



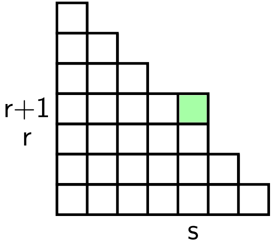
$$\sigma(\lambda, NW) = s_3 s_4 s_6 \qquad \sigma(\lambda, SE) = s_5 s_7 s_6$$

$$\pi_{\sigma(\lambda,NW)} = \pi_3\pi_4\pi_6 \qquad \pi_{\sigma(\lambda,SE)} = \pi_5\pi_7\pi_6$$

$$F_\lambda(x,y) := \prod_{(i,j)\in\lambda}(1-x_i y_j)^{-1} = \sum_{\nu\in\mathbb{N}^k}(\pi_{\sigma(\lambda,NW)}\widehat{\kappa}_\nu(x))(\pi_{\sigma(\lambda,SE)}\kappa_{\omega\nu}(y)).$$

Crystal operators and growth diagrams

$$w = \begin{pmatrix} 1 & 1 & 2 & 3 & 4 & 4 & 5 & 5 & 7 \\ 2 & 7 & 2 & 4 & 1 & 3 & 3 & 4 & 1 \end{pmatrix} \quad i + j \leq 7 + 1, \quad 5 + 4 = 9$$



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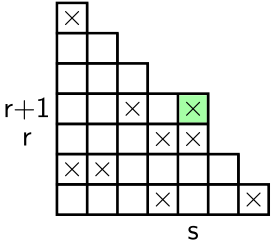

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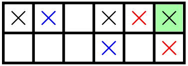

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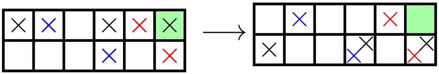
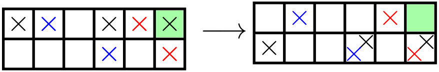
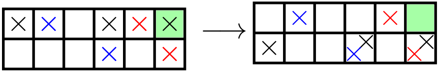


Crystal operators and growth diagrams



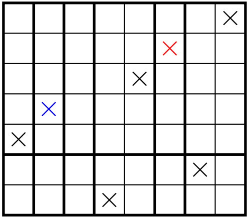
$$\begin{pmatrix} 1 & 2 & 4 & 4 & 5 & s & s \\ r+1 & r+1 & r & r+1 & r+1 & r & r+1 \end{pmatrix} \underset{f_r}{\overset{e_r}{\rightleftarrows}} \begin{pmatrix} 1 & 2 & 4 & 4 & 5 & s & s \\ r & r+1 & r & r & r+1 & r & r \end{pmatrix}$$

Crystal operators and growth diagrams



$$\begin{pmatrix} 1 & 2 & 4 & 4 & 5 & s & s \\ r+1 & \color{blue}{r+1} & \color{blue}{r} & r+1 & \color{red}{r+1} & \color{red}{r} & r+1 \end{pmatrix} \underset{f_r}{\overset{e_r}{\rightleftarrows}} \begin{pmatrix} 1 & 2 & 4 & 4 & 5 & s & s \\ r & \color{blue}{r+1} & \color{blue}{r} & r & \color{red}{r+1} & \color{red}{r} & r \end{pmatrix}$$

w_r



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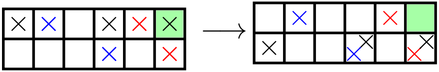
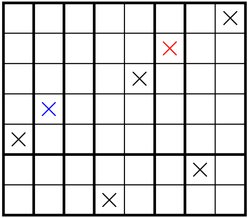
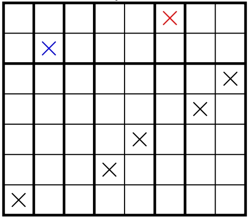


$$\begin{pmatrix} 1 & 2 & 4 & 4 & 5 & s & s \\ r+1 & r+1 & r & r+1 & r+1 & r & r+1 \end{pmatrix} \underset{f_r}{\overset{e_r}{\rightleftarrows}} \begin{pmatrix} 1 & 2 & 4 & 4 & 5 & s & s \\ r & r+1 & r & r & r+1 & r & r \end{pmatrix}$$

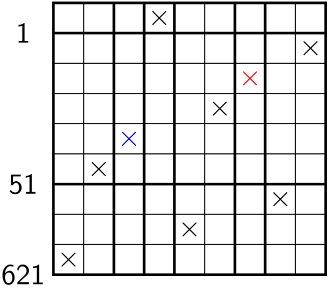
w_r

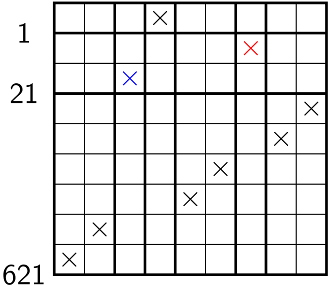


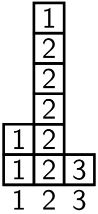
$\widetilde{w}_r$

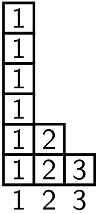


After the matching the size of the SW chain in row $r+1$ is the number of matched crosses and the size of the SW chain in row r is the number of crosses in that row plus the number of unmatched crosses in row $r+1$.


1
51
621


1
21
621


1
2
2
2
1 2
1 2 3
1 2 3


1
1
1
1
1 2
1 2 3
1 2 3

Theorem

Let w be a biword in lexicographic order and $\widetilde{w}$ the biword obtained from w by applying the crystal operator e_r to the second row of w. Let $\Phi(w) = (F, G)$, and $\Phi(\widetilde{w}) = (\widetilde{F}, \widetilde{G})$. Then $G = \widetilde{G}$ and $F = \Theta_r^m \widetilde{F}$, where m is the number of unmatched $r + 1$ in F.

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Corollary

Assume that the biletters of w are coordinates of a Ferrers shape λ where $\lambda_r = \lambda_{r+1}$, and w contains the biletter $\binom{s}{r+1}$ with $s = \lambda_{r+1}$ satisfying $r + s \geq n + 1$ with $1 \leq r, s \leq n$. If $sh(\widetilde{F}) = \nu$ then $sh(F) = s_r \nu$ and $\nu_r > \nu_{r+1}$.

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Let F be a SSAF with shape ν. Then $sh(\Theta_r F) = s_r \nu$ only if $\nu_r > \nu_{r+1}$.

One extra box above stair shape partition in position $(r+1, s+1)$, for $r, s \geq 0$.

$$\prod_{(i,j)\in\lambda} (1 - x_i y_j)^{-1} = \sum_{\nu\in\mathbb{N}^n} \pi_r \widehat{\kappa}_\nu(x) \kappa_{\omega\nu}(y)$$

or

$$\prod_{(i,j)\in\lambda} (1 - x_i y_j)^{-1} = \sum_{\nu\in\mathbb{N}^n} \widehat{\kappa}_\nu(x) \pi_s \kappa_{\omega\nu}(y)$$

$$\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1}=\sum_{c\geq 0}x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c}+\sum_{c\geq 0}\sum_{d>0}x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c}x_{r+1}^{d}y_{s+1}^{d}$$

$$\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1}=\sum_{c\geq 0}x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c}+\sum_{c\geq 0}\sum_{d>0}x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c}x_{r+1}^d y_{s+1}^d$$

$$:=\sum_{c\geq 0}\begin{pmatrix} j_1 & \dots & j_c \\ i_1 & \dots & i_c \end{pmatrix}+\sum_{c\geq 0}\sum_{d>0}\begin{pmatrix} j_1 & \dots & (s+1)^d & \dots & j_c \\ i_1 & \dots & (r+1)^d & \dots & i_c \end{pmatrix}$$

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$$:=\sum_{c\geq 0}\begin{pmatrix} j_1 & \dots & j_c\\ i_1 & \dots & i_c\end{pmatrix}+\sum_{c\geq 0}\sum_{d>0}\begin{pmatrix} j_1 & \dots & (s+1)^d & \dots & j_c\\ i_1 & \dots & (r+1)^d & \dots & i_c\end{pmatrix}$$

$$=\sum_{m\geq 0}\sum_{c\geq 0}\begin{pmatrix} j_1 & \dots & j_c\\ f_r^m(i_1 & \dots & i_c)\end{pmatrix}=\sum_{m\geq 0}\sum_{\nu\in\mathbb{N}^n}\sum_{\substack{(F,G)\in SSAF\\ sh(F)=\nu\\ sh(G)\leq\omega\nu}}(\Theta_r^m F,G)$$

$$\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1} = \sum_{c\geq 0} x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c} + \sum_{c\geq 0}\sum_{d>0} x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c}x_{r+1}^d y_{s+1}^d$$

$$:= \sum_{c\geq 0}\begin{pmatrix} j_1 & \dots & j_c \\ i_1 & \dots & i_c \end{pmatrix} + \sum_{c\geq 0}\sum_{d>0}\begin{pmatrix} j_1 & \dots & (s+1)^d & \dots & j_c \\ i_1 & \dots & (r+1)^d & \dots & i_c \end{pmatrix}$$

$$= \sum_{m\geq 0}\sum_{c\geq 0}\begin{pmatrix} j_1 & \dots & j_c \\ f_r^m(i_1 & \dots & i_c) \end{pmatrix} = \sum_{m\geq 0}\sum_{\nu\in\mathbb{N}^n}\sum_{\substack{(F,G)\in SSAF \\ sh(F)=\nu \\ sh(G)\leq\omega\nu}} (\Theta_r^m F, G)$$

$$:= \sum_{\nu\in\mathbb{N}^n}\Big(\sum_{\substack{(F,G)\in SSAF \\ sh(F)=\nu \\ sh(G)\leq\omega\nu}} x^F y^G + \sum_{\substack{(F,G)\in SSAF \\ sh(F)=s_r\nu \\ sh(G)\leq\omega\nu \\ \nu_r>\nu_{r+1}}} x^F y^G\Big)$$

$$\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1}=\sum_{c\geq 0}x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c}+\sum_{c\geq 0}\sum_{d>0}x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c}x_{r+1}^{d}y_{s+1}^{d}$$

$$:=\sum_{c\geq 0}\begin{pmatrix} j_1 & \cdots & j_c \\ i_1 & \cdots & i_c \end{pmatrix}+\sum_{c\geq 0}\sum_{d>0}\begin{pmatrix} j_1 & \cdots & (s+1)^d & \cdots & j_c \\ i_1 & \cdots & (r+1)^d & \cdots & i_c \end{pmatrix}$$

$$=\sum_{m\geq 0}\sum_{c\geq 0}\begin{pmatrix} j_1 & \cdots & j_c \\ f_r^m(i_1 & \cdots & i_c) \end{pmatrix}=\sum_{m\geq 0}\sum_{\nu\in\mathbb{N}^n}\sum_{\substack{(F,G)\in SSAF\\ sh(F)=\nu\\ sh(G)\leq\omega\nu}}(\Theta_r^m F,G)$$

$$:=\sum_{\nu\in\mathbb{N}^n}\Big(\sum_{\substack{(F,G)\in SSAF\\ sh(F)=\nu\\ sh(G)\leq\omega\nu}}x^F y^G+\sum_{\substack{(F,G)\in SSAF\\ sh(F)=s_r\nu\\ sh(G)\leq\omega\nu\\ \nu_r>\nu_{r+1}}}x^F y^G\Big)=\sum_{\nu\in\mathbb{N}^n}\Big(\sum_{\substack{F\in SSAF\\ sh(F)=\nu}}x^F\sum_{\substack{G\in SSAF\\ sh(G)\leq\omega\nu}}y^G$$

$$+\sum_{sh(F)=s_r\nu}x^F\sum_{sh(G)\leq\omega\nu}y^G\Big)$$

$$\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1}=\sum_{c\geq 0}x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c}+\sum_{c\geq 0}\sum_{d>0}x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c}x_{r+1}^{d}y_{s+1}^{d}$$

$$:=\sum_{c\geq 0}\begin{pmatrix} j_1 & \cdots & j_c \\ i_1 & \cdots & i_c\end{pmatrix}+\sum_{c\geq 0}\sum_{d>0}\begin{pmatrix} j_1 & \cdots & (s+1)^d & \cdots & j_c \\ i_1 & \cdots & (r+1)^d & \cdots & i_c\end{pmatrix}$$

$$=\sum_{m\geq 0}\sum_{c\geq 0}\begin{pmatrix} j_1 & \cdots & j_c \\ f_r^m(i_1 & \cdots & i_c)\end{pmatrix}=\sum_{m\geq 0}\sum_{\nu\in\mathbb{N}^n}\sum_{\substack{(F,G)\in SSAF\\ sh(F)=\nu\\ sh(G)\leq\omega\nu}}(\Theta_r^m F,G)$$

$$:=\sum_{\nu\in\mathbb{N}^n}\Big(\sum_{\substack{(F,G)\in SSAF\\ sh(F)=\nu\\ sh(G)\leq\omega\nu}}x^Fy^G+\sum_{\substack{(F,G)\in SSAF\\ sh(F)=s_r\nu\\ sh(G)\leq\omega\nu\\ \nu_r>\nu_{r+1}}}x^Fy^G\Big)=\sum_{\nu\in\mathbb{N}^n}\Big(\sum_{\substack{F\in SSAF\\ sh(F)=\nu}}x^F\sum_{\substack{G\in SSAF\\ sh(G)\leq\omega\nu}}y^G$$

$$+\sum_{sh(F)=s_r\nu}x^F\sum_{sh(G)\leq\omega\nu}y^G\Big)=\sum_{\nu\in\mathbb{N}^n}\Big(\sum_{\substack{P\in SSYT\\ sh(P)=\nu^+\\ K_+(P)=key(\nu)}}x^P\sum_{\substack{Q\in SSYT\\ sh(Q)=\nu^+\\ K_+(Q)=key(\beta)\\ \beta\leq\omega\nu}}y^Q$$

$$+\sum_{\substack{P\in SSYT\\ sh(P)=\nu^+\\ K_+(P)=key(s_r\nu)\\ \nu_r>\nu_{r+1}}}x^P\sum_{\substack{Q\in SSYT\\ sh(Q)=\nu^+\\ K_+(Q)=key(\beta)\\ \beta\leq\omega\nu}}y^Q\Big)$$

$$\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1}=\sum_{c\geq 0}x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c}+\sum_{c\geq 0}\sum_{d>0}x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c}x_{r+1}^d y_{s+1}^d$$

$$:=\sum_{c\geq 0}\begin{pmatrix} j_1 & \cdots & j_c \\ i_1 & \cdots & i_c\end{pmatrix}+\sum_{c\geq 0}\sum_{d>0}\begin{pmatrix} j_1 & \cdots & (s+1)^d & \cdots & j_c \\ i_1 & \cdots & (r+1)^d & \cdots & i_c\end{pmatrix}$$

$$=\sum_{m\geq 0}\sum_{c\geq 0}\begin{pmatrix} j_1 & \cdots & j_c \\ f_r^m(i_1 & \cdots & i_c)\end{pmatrix}=\sum_{m\geq 0}\sum_{\nu\in\mathbb{N}^n}\sum_{\substack{(F,G)\in SSAF\\ sh(F)=\nu\\ sh(G)\leq\omega\nu}}(\Theta_r^m F,G)$$

$$:=\sum_{\nu\in\mathbb{N}^n}\Big(\sum_{\substack{(F,G)\in SSAF\\ sh(F)=\nu\\ sh(G)\leq\omega\nu}}x^Fy^G+\sum_{\substack{(F,G)\in SSAF\\ sh(F)=s_r\nu\\ sh(G)\leq\omega\nu\\ \nu_r>\nu_{r+1}}}x^Fy^G\Big)=\sum_{\nu\in\mathbb{N}^n}\Big(\sum_{\substack{F\in SSAF\\ sh(F)=\nu}}x^F\sum_{\substack{G\in SSAF\\ sh(G)\leq\omega\nu}}y^G$$

$$+\sum_{sh(F)=s_r\nu}x^F\sum_{sh(G)\leq\omega\nu}y^G\Big)=\sum_{\nu\in\mathbb{N}^n}\Big(\sum_{\substack{P\in SSYT\\ sh(P)=\nu^+\\ K_+(P)=key(\nu)}}x^P\sum_{\substack{Q\in SSYT\\ sh(Q)=\nu^+\\ K_+(Q)=key(\beta)\\ \beta\leq\omega\nu}}y^Q$$

$$+\sum_{\substack{P\in SSYT\\ sh(P)=\nu^+\\ K_+(P)=key(s_r\nu)\\ \nu_r>\nu_{r+1}}}x^P\sum_{\substack{Q\in SSYT\\ sh(Q)=\nu^+\\ K_+(Q)=key(\beta)\\ \beta\leq\omega\nu}}y^Q\Big)=\sum_{\nu\in\mathbb{N}^n}\widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y)+\sum_{\substack{\nu\in\mathbb{N}^n\\ \nu_r>\nu_{r+1}}}\widehat{\kappa}_{s_r\nu}(x)\kappa_{\omega\nu}(y)$$

$$\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1}=\sum_{c\geq 0}x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c}+\sum_{c\geq 0}\sum_{d>0}x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c}x_{r+1}^d y_{s+1}^d$$

$$:=\sum_{c\geq 0}\begin{pmatrix} j_1 & \cdots & j_c \\ i_1 & \cdots & i_c \end{pmatrix}+\sum_{c\geq 0}\sum_{d>0}\begin{pmatrix} j_1 & \cdots & (s+1)^d & \cdots & j_c \\ i_1 & \cdots & (r+1)^d & \cdots & i_c \end{pmatrix}$$

$$=\sum_{m\geq 0}\sum_{c\geq 0}\begin{pmatrix} j_1 & \cdots & j_c \\ f_r^m(i_1 & \cdots & i_c) \end{pmatrix}=\sum_{m\geq 0}\sum_{\nu\in\mathbb{N}^n}\sum_{\substack{(F,G)\in SSAF\\ sh(F)=\nu\\ sh(G)\leq\omega\nu}}(\Theta_r^m F,G)$$

$$:=\sum_{\nu\in\mathbb{N}^n}\Big(\sum_{\substack{(F,G)\in SSAF\\ sh(F)=\nu\\ sh(G)\leq\omega\nu}}x^Fy^G+\sum_{\substack{(F,G)\in SSAF\\ sh(F)=s_r\nu\\ sh(G)\leq\omega\nu\\ \nu_r>\nu_{r+1}}}x^Fy^G\Big)=\sum_{\nu\in\mathbb{N}^n}\Big(\sum_{\substack{F\in SSAF\\ sh(F)=\nu}}x^F\sum_{\substack{G\in SSAF\\ sh(G)\leq\omega\nu}}y^G$$

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$$=\sum_{\nu\in\mathbb{N}^n}(1+\widehat{\pi}_r)\widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y)=\sum_{\nu\in\mathbb{N}^n}\pi_r\widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y).$$

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