Counting smaller trees in the Tamari order

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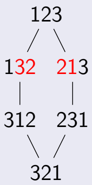
Permutations and the weak order

Permutations

A permutation σ is a word of size n where every letter of $\{1, \ldots, n\}$ appear only once.
Ex : 1234, 15234, 4231.

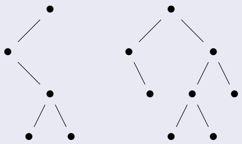
Weak order: partial order on permutations

At each step, we exchange two increasing consecutives values.

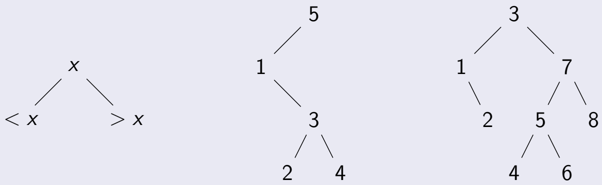


Binary trees

Binary trees



Canonical labelling



From permutations to binary trees

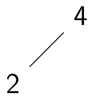
The binary search tree insertion

$$1532\textcolor{red}{4} \quad \rightarrow \quad 4$$

From permutations to binary trees

The binary search tree insertion

15324 $\rightarrow$



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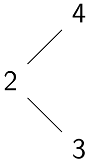

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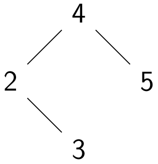

From permutations to binary trees

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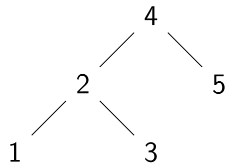
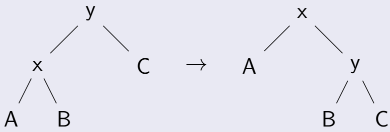


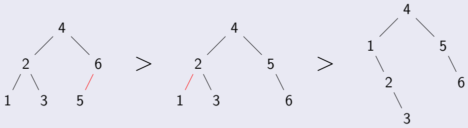
Order relation

Right rotation



This gives an order relation on binary trees.

Example



The Tamari lattice

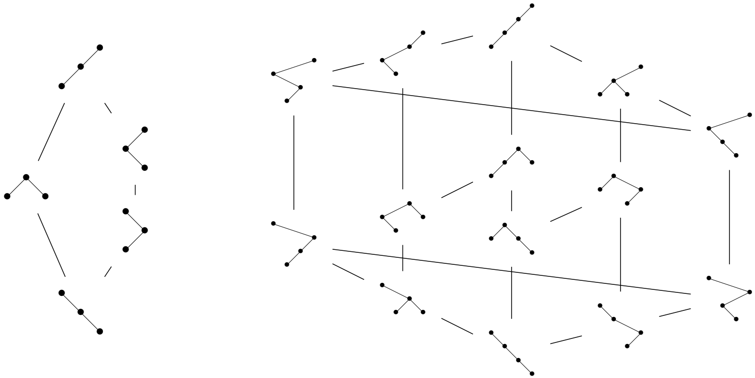
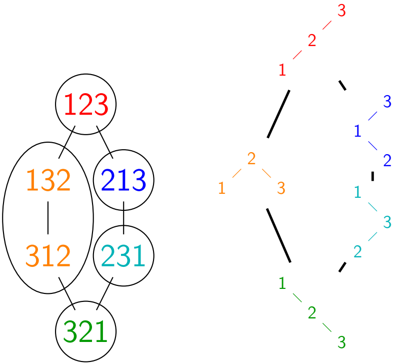
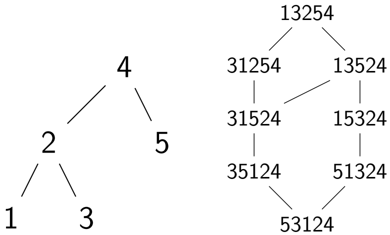


Figure: The Tamari lattices of size 3 and 4.

The Tamari lattice as a quotient of the weak order



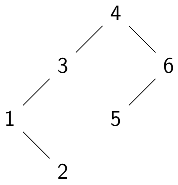


Our objective

Goal

We want a formula that computes, for any given tree T the number of trees smaller than T in the Tamari order.

Example : how many trees are smaller than or equal to this one ?



1. Introduction
   - Basic definitions
   - Tamari lattice
   - Goal
2. Our work
   - Main result
   - Example
   - Sketch of our proof

Tamari polynomials

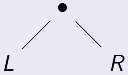
Tamari polynomials

Given a binary tree T, we define:

$$\mathcal{B}_\emptyset := 1 \tag{1}$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1} \tag{2}$$

with T =



Main theorem

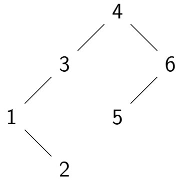
Theorem

Let T be a binary tree. Its Tamari polynomial $\mathcal{B}_T(x)$ counts the trees smaller than or equal to T in the Tamari order according to the number of nodes on their left border. In particular, $\mathcal{B}_T(1)$ is the number of trees smaller than T.

Example

$$\mathcal{B}_{\emptyset} := 1$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$



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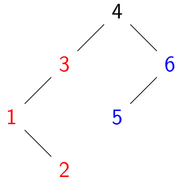

- $\mathcal{B}_T(x) = \mathcal{B}_4(x) = x\mathcal{B}_3(x)\frac{x\mathcal{B}_6(x)-\mathcal{B}_6(1)}{x-1}$

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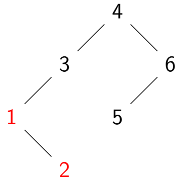

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Example

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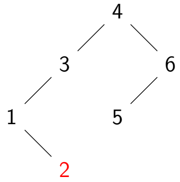

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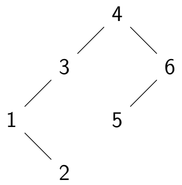

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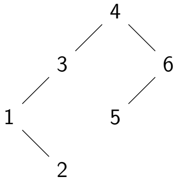

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- $\mathcal{B}_1(x) = x(1+x) = x + x^2$

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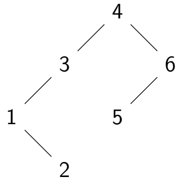

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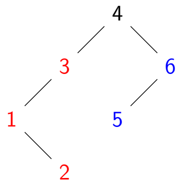

- $\mathcal{B}_T(x) = \mathcal{B}_4(x) = x(x^2 + x^3)\frac{x\mathcal{B}_6(x) - \mathcal{B}_6(1)}{x-1}$

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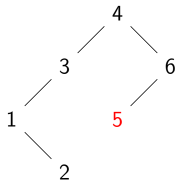

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- $\mathcal{B}_6(x) = x\mathcal{B}_5(x)$

Example

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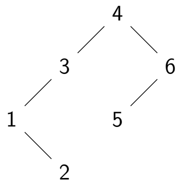

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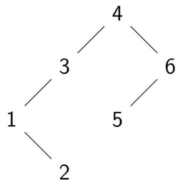

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- $\mathcal{B}_6(x) = x\mathcal{B}_5(x)$
- $\mathcal{B}_5(x) = x$
- $\mathcal{B}_6(x) = x \cdot x = x^2$

Example

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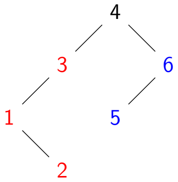

- $\mathcal{B}_4(x) = x(x^2 + x^3)(1 + x + x^2)$

Example

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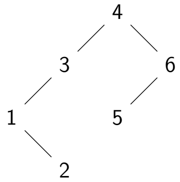
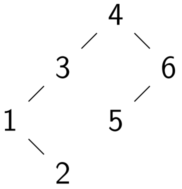
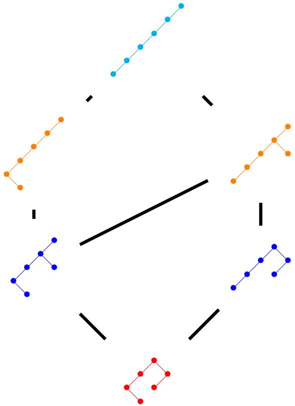


- $\mathcal{B}_4(x) = x(x^2 + x^3)(1 + x + x^2)$
- $\mathcal{B}_4(x) = x^6 + 2x^5 + 2x^4 + x^3$

Example



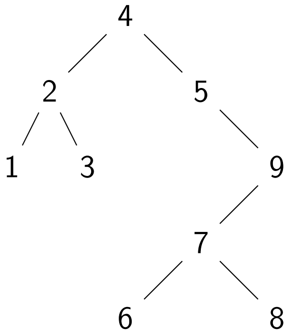


$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$

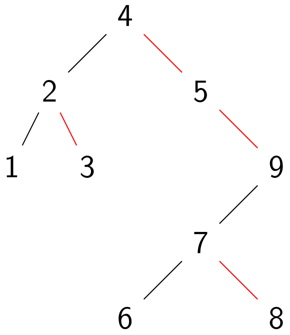
$$\mathcal{B}_T(1) = 6$$

Sketch of our proof

Increasing and decreasing forests



Sketch of our proof
Increasing and decreasing forests



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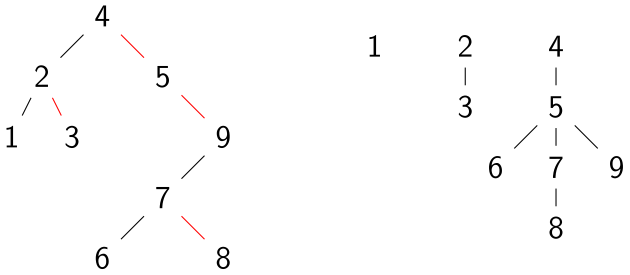

Sketch of our proof

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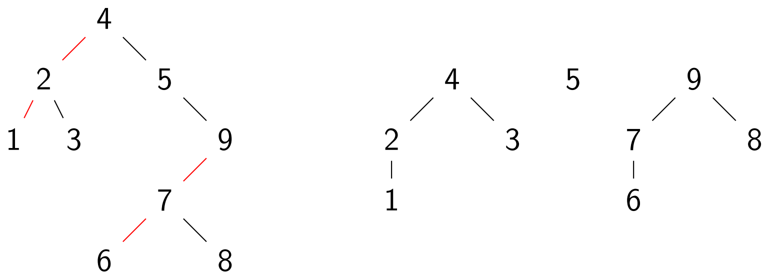
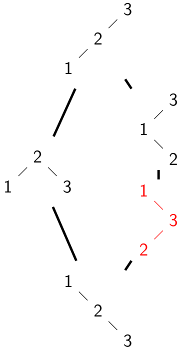
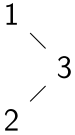
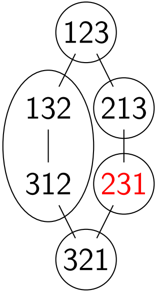


Initial and final intervals as linear extensions







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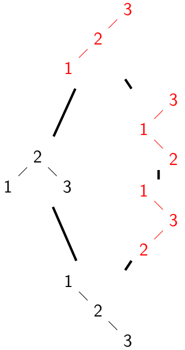

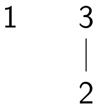

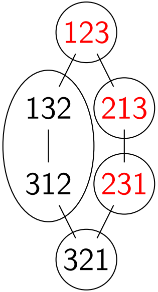

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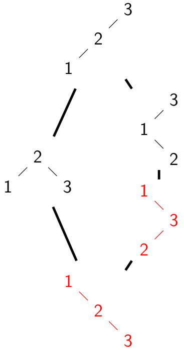
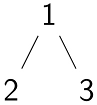
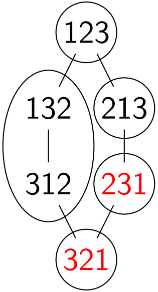
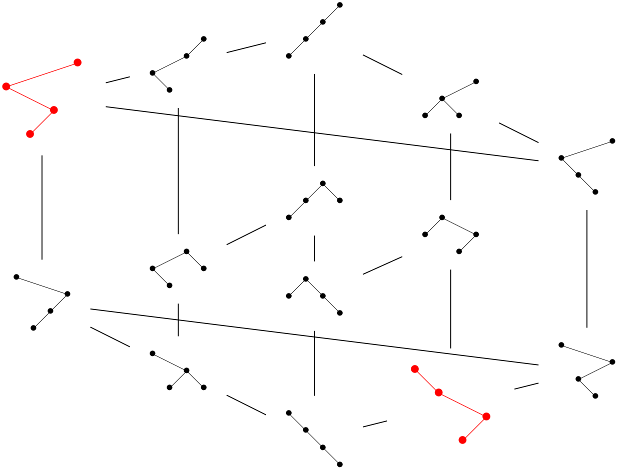






Linear extensions of any interval



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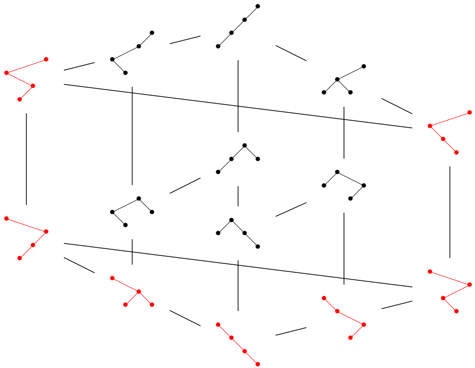

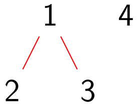

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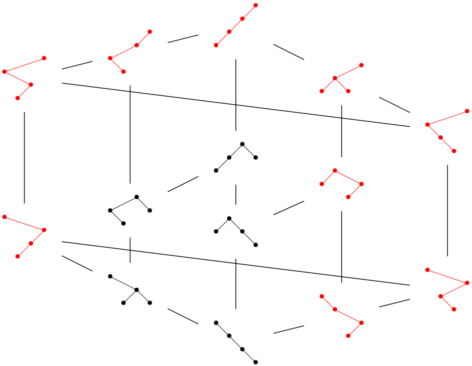

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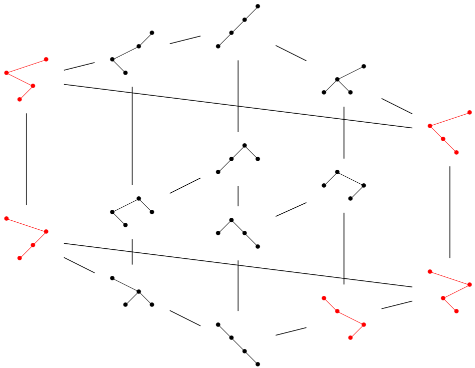
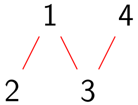




Sketch of proof

Back to the functional equation

$$\mathcal{B}_T(x) = x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

Since it is defined on *binary* trees, see it as *bilinear* map.

$$\mathcal{B}(f, g) = xf(x)\frac{xg(x) - g(1)}{x-1}$$

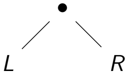
Sketch of proof

Combinatorial interpretation of $\mathcal{B}$

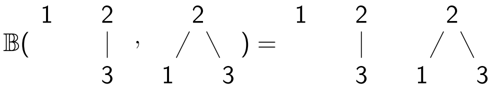
Lift the bilinear map to take intervals as arguments.

$$\sum_{T' \leq T} [T', T] = \mathbb{B}\Big(\sum_{L' \leq L} [L', L], \sum_{R' \leq R} [R', R]\Big)$$

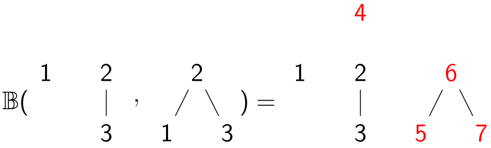
with T $=$



Definition of $\mathbb{B}$ on an example



Definition of $\mathbb{B}$ on an example



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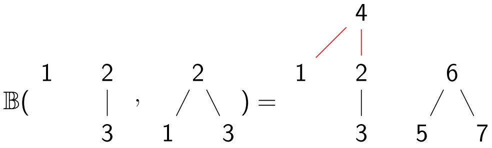

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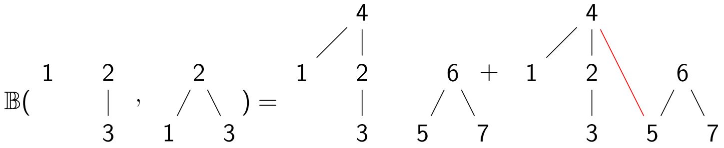
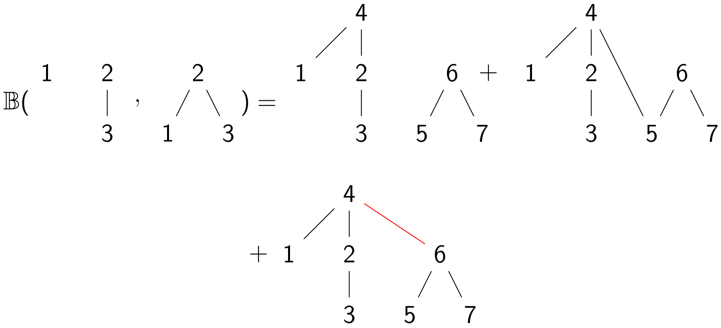
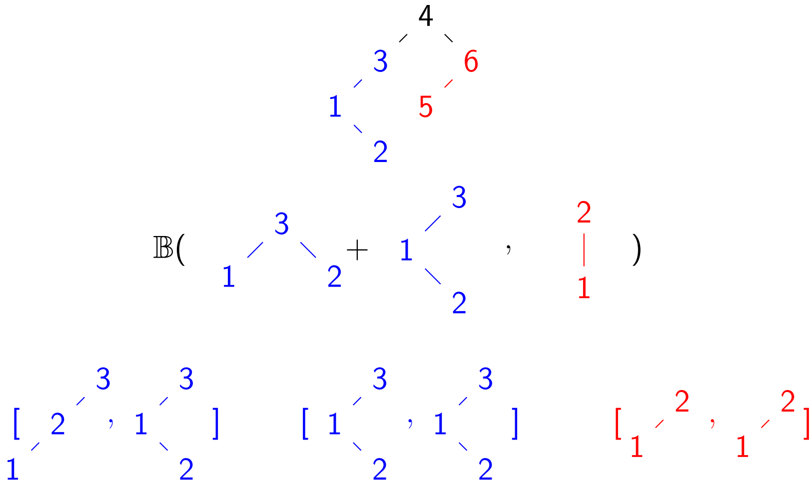
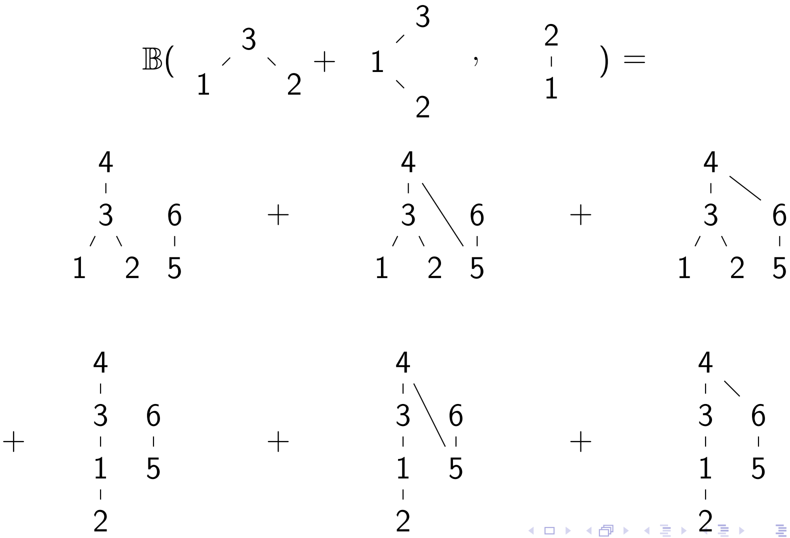
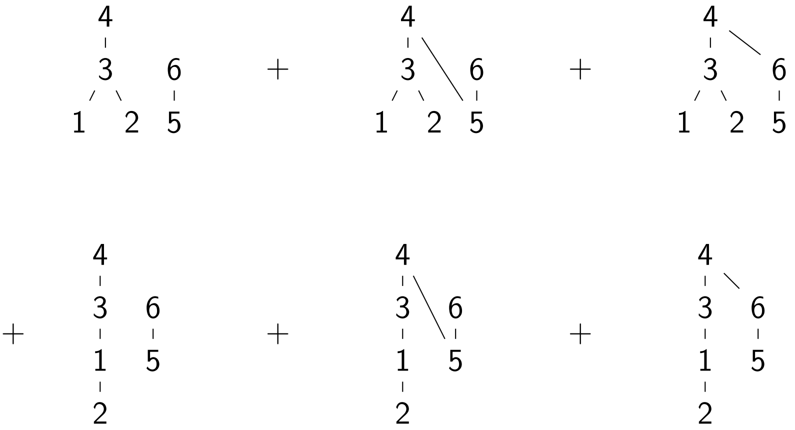


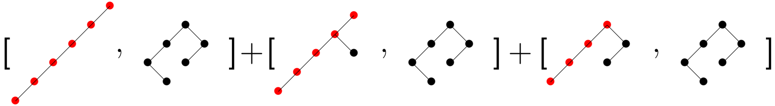
Definition of $\mathbb{B}$ on an example



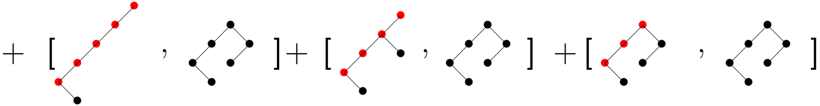








$$x^6 + x^5 + x^4$$



$$+ x^5 + x^4 + x^3$$