Sweedler's duals, Automata theory and Combinatorial Physics

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70th Séminaire Lotharingien de Combinatoire
March 26 2013

Plan

Introduction

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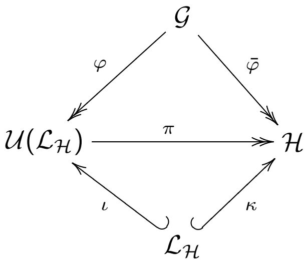
As it has been seen exponentials of infinitesimal characters and their convolution provide characters which have beautiful combinatorial counterparts. The set of characters in a Hopf algebra is a group which has a natural structure of (infinite dimensional) Lie group under certain conditions which are often fulfilled in practice (grading). All these elements belong to the biggest dual one can consider for a Hopf algebra, Sweedler's dual. This dual is, in a certain way, related to (finite state) automata theory. In the end of the talk, we study another case of convergence.

Some Hopf algebras relating to physics

These Hopf algebras are constructed on bases that are often graphic (contructions on drawings) or discrete (Hopf algebra - of classes - of matroids, see Hoang's talk).

- Connes-Kreimer (basis : non planar rooted trees, contruction $S(?)$)
- Non-commutative Connes-Kreimer (basis : planar rooted trees, contruction $T(?)$)
- Heisenberg-Weyl graphs



Combinatorial Algebra

Algebra

Lie Algebra

- The Hopf algebras **DIAG**, **LDIAG** of labeled diagrams in QFT of partitions (see details below for **LDIAG**)

(Non monoidal) Composition of Heisenberg-Weyl graphs

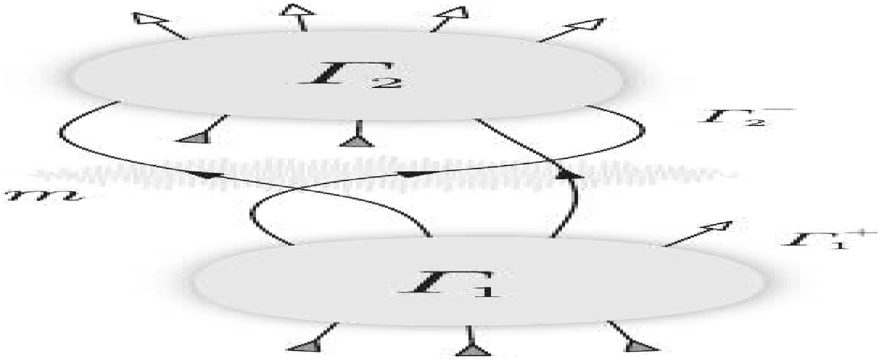


Figure: Composition of Heisenberg-Weyl graphs

How and why **LDIAG** diagrams arise 1/3

Let $\mathcal{H}(?,?)$ be the "Hadamard exponential coupling" of QFTP [1] defined by

$$\mathcal{H}(F,G) = F\left(z\frac{d}{dx}\right) G(x)\Big|_{x=0} . \qquad (1)$$

one can check that, with $F(z)=\sum_{n\geq 0} a_n \frac{z^n}{n!}$ and $G(z)=\sum_{n\geq 0} b_n \frac{z^n}{n!}$, one has

$$\mathcal{H}(F,G) = \sum_{n\geq 0} a_n b_n \frac{z^n}{n!} \qquad (2)$$

[1] C. M. Bender, D. C. Brody, and B. K. Meister, Quantum field theory of partitions, J. Math. Phys. Vol 40 (1999)

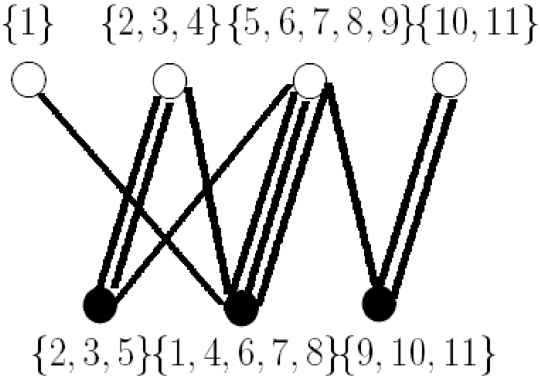
How and why **LDIAG** diagrams arise 2/3

$$F(z) = \exp\left(\sum_{n=1}^{\infty} L_n \frac{z^n}{n!}\right) \;;\; G(z) = \exp\left(\sum_{n=1}^{\infty} V_n \frac{z^n}{n!}\right) \tag{3}$$

$$\mathcal{H}(F,G) \quad = \quad \sum_{n\geq 0} \frac{z^n}{n!} \sum_{P_1,P_2\in UP_n} \mathbb{L}^{Type(P_1)}\mathbb{V}^{Type(P_2)} \tag{4}$$

where UP_n is the set of unordered partitions of $[1\cdots n]$. The type of $P \in UP_n$ (denoted above by $Type(P)$) is the multi-index $(\alpha_i)_{i\in\mathbb{N}^+}$ such that α_k is the number of k-blocks, that is the number of members of P with cardinality k.

How and why **LDIAG** diagrams arise 3/3



A template example : non-commutative polynomials

Example 3.1

Let X be an alphabet and X^*, the monoid freely generated by X (i.e. it is the set of all words with letters in X endowed with the concatenation product). Its algebra $k[X^*] = k\langle X\rangle = k^{(X^*)}$ has the dual k^{X^*} and the pairing is given by

$$\langle f|g\rangle = \sum_{w\in X^*} f(w)g(w) \tag{5}$$

for $f \in k^{X^*}$, $g \in k^{(X^*)}$. Identifying the words with their characteristic function, we get $f(w) = \langle f|w\rangle$ and $f = \sum_{w\in X^*} f(w)w$ (non-commutative series).
We describe the Hopf algebra $(k\langle X\rangle, conc, 1_{X^*}, \Delta_{⧢}, \epsilon, S)$, its (finely graded[a]) dual $(k\langle X\rangle, ⧢, 1_{X^*}, \Delta_{conc}, \epsilon, S)$ and the greatest dual

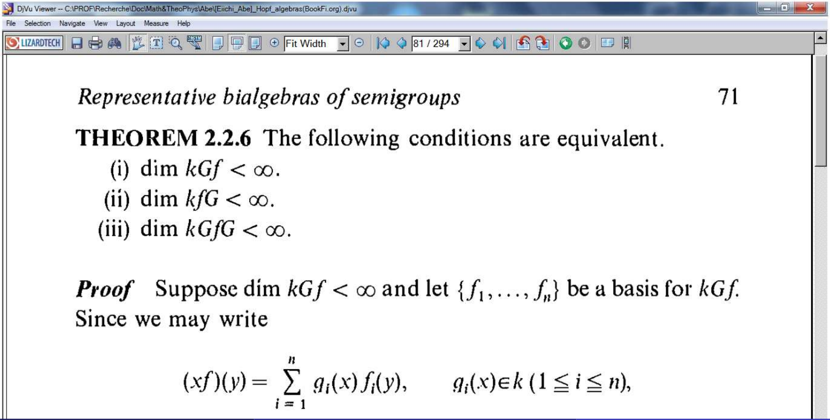
$$(k^{rat}\langle\langle X\rangle\rangle, ⧢, 1_{X^*}, \Delta_{conc}, \epsilon, S) \ .$$

[a] In the case when the alphabet is infinite, one has $\sum_{x\in X} x$ in the dual for the grading by total degrees.

Abe's theorem

In order to grasp what happens in all these situations, we need to go at a more general level.

Eiichi Abe, Hopf algebras, Cambridge University Press, 1977.



Representative bialgebras of semigroups 71

THEOREM 2.2.6 The following conditions are equivalent.

(i) $\dim kGf < \infty$.

(ii) $\dim kfG < \infty$.

(iii) $\dim kGfG < \infty$.

Proof Suppose $\dim kGf < \infty$ and let $\{f_1, \ldots, f_n\}$ be a basis for kGf. Since we may write

$$(xf)(y) = \sum_{i=1}^{n} g_i(x) f_i(y), \qquad g_i(x) \in k \ (1 \leq i \leq n),$$

Abe's theorem, cont'd

Let k be a field and $f \in k^S$ (S is a semigroup). One defines the shifts of f by

$$ {}_x f(y) = f(yx)\ ;\ f_z(y) = f(zy)\ ;\ {}_x f_z(y) = f(zyx)\ ; \tag{6}$$

Theorem 4.1

(Abe's theorem reformulation)
Let $f \in k^S$ TFAE
i) $({}_x f)_{x\in S}$ is of finite rank
ii) $(f_z)_{z\in S}$ is of finite rank
iii) $({}_x f_z)_{x,z\in S}$ is of finite rank
iv) There exists a matrix representation of S, $\rho : S \to k^{n\times n}$, a row vector $\lambda \in k^{1\times n}$ and a column vector $\gamma \in k^{n\times 1}$ s.t. for all $s \in S$

$$f(s) = \lambda\rho(s)\gamma \tag{7}$$

v) There are functions $f_i^{(1)}, f_i^{(2)}, i = 1..n$ s.t. for all $x, y \in S$

$$f(xy) = \sum_{i=1}^{n} f_i^{(1)}(x) f_i^{(2)}(y) \tag{8}$$

Remark 4.2

(iv) is the basis of automata theory with multiplicities in a field. One can (at the cost of the equivalence) reformulate (iv) for a semiring and as one can construct a (weighted) graph from λ, ρ, γ. This has very much to do with the theory of languages (where $k = \mathbb{B} = \{0, 1\}$, the boolean semiring).

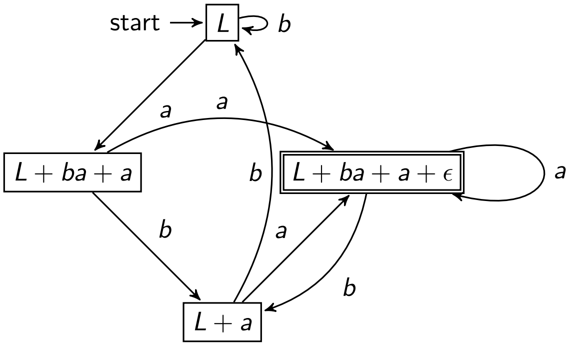
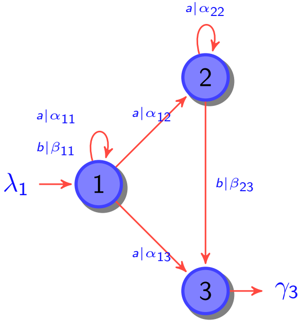


Figure: Automaton of (right) shifts of $L = A^*(aba + aa)$.



This (weighted) automaton above corresponds to the representation

$$a \mapsto \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ 0 & \alpha_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \ b \mapsto \begin{pmatrix} \beta_{11} & 0 & 0 \\ 0 & 0 & \beta_{23} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda = (\lambda_1, 0, 0) \ ; \ \gamma^T = (0, 0, \gamma_3) \tag{9}$$

Characters, Dual elements

1. (Convolution) Let $(\mathcal{H}, \mu, 1_{\mathcal{H}}, \Delta, \epsilon, S)$ be a Hopf algebra. For $f, g \in \mathcal{H}^*$,,one defines $f * g$ as the linear form such that

$$\langle f * g | x \rangle = \langle f \otimes g | \Delta(x) \rangle^{\otimes 2} \tag{10}$$

2. (Characters) A morphism of k-algebras $\mathcal{H} \to k$. We denote their set $\mathbb{X}(\mathcal{H})$.
3. (Group of characters) The set of characters with the convolution $(\mathbb{X}(\mathcal{H}), *)$ forms a group.
4. (Infinitesimal Characters) A linear form $\delta \in \mathcal{H}^*$ such that

$$\delta(xy) = \delta(x)\epsilon(y) + \epsilon(x)\delta(y) \tag{11}$$

their set is closed for the (convolutional) Lie bracket.

Remark 5.1

All these elements are in the Sweedler's dual $\mathcal{H}^0$: a character is of rank (the dimension of the orbit by shifts) one and an infinitesimal character is of rank less than two.

Hausdorff group, Direct elements

1. (Group-like elements) $x \in \mathcal{H}$ is said group-like iff

$$\Delta(x) = x \otimes x \ ; \ \epsilon(x) = 1 \qquad (12)$$

$\hookrightarrow$ they form a group $Haus(\mathcal{H})$.

2. (Primitive elements)

$$\Delta(x) = x \otimes 1 + 1 \otimes x \qquad (13)$$

$\hookrightarrow$ their set, a subspace, is closed by the Lie bracket $[x, y] = xy - yx$.

Results, log $\leftrightarrow$ exp correspondence

Let $(\mathcal{H}, \mu, 1_{\mathcal{H}}, \Delta, \epsilon, S)$ be a Hopf algebra, we define I_+ as the projector $x \mapsto x - \epsilon(x)1_{\mathcal{H}}$ and $\Delta_+ = (I_+ \otimes I_+)\Delta$. It is standard that Δ_+ is co-associative.

Theorem 5.2

Let $(\mathcal{H}, \mu, 1_{\mathcal{H}}, \Delta, \epsilon, S)$ be a Hopf algebra be a Hopf algebra as above. We suppose that Δ_+ is nilpotent[a] then,

i) given any infinitesimal character $\delta \in \mathbb{X}^{inf}(\mathcal{H})$, the sum

$$\sum_{n\geq 0} \frac{1}{n!}\, \delta^{*n}$$

converges locally (i.e. for all $x \in \mathcal{H}$, $n \mapsto \delta^{*n}(x)$ is finitely supported) and its sum (call it $\exp_*(\delta)$) is a character of $\mathcal{H}$.

ii) given a character $\chi \in \mathbb{X}(\mathcal{H})$, the series

$$\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} (\chi - \epsilon)^{*n}$$

converges locally in the preceding sense (i.e. for all $x \in \mathcal{H}$, $n \mapsto (\chi - \epsilon)^{*n}(x)$ is finitely supported) and the sum, which we could denote $\log_*(\chi)$ is an infinitesimal character of $\mathcal{H}$.

iii) The two correspondences are mutually inverse.

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[a] means that, for all $x \in \mathcal{H}$, it exists $N > 0$ such that $\Delta_+^{(N)}(x) = 0$

Concluding Remarks

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i) Provided that $ch(k) = 0$*, in the locally finite case (*Δ_+ *is nilpotent), which includes all the graded Hopf algebras and also the enveloping algebras (which may not be graded) we have a good*

$$\log \longleftrightarrow \exp$$

correspondence between characters and infinitesimal characters.
ii) In the other case, one must be careful with the convergence within the field of coefficients (see Laurent polynomials, $k[X, X^{-1}]$ *with* $\Delta(X) = X \otimes X$ *and* $\epsilon(X) = 1$*).*
iii) Group-like elements appear in Hopf algebras where k is not a field in the context of the theory of iterated integrals and to perform Schützenberger's factorization of the identity which gives the local coordinates of the Hausdorff group of noncommutative series.

Thank you for your attention!