Recipe theorem for the Tutte polynomial for matroids, renormalization group-like approach

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Plan

1. Matroids: The Tutte polynomial and the Hopf algebra
2. Characters of the Hopf algebras of matroids
3. Convolution formula for the Tutte polynomials for matroids
4. Proof of the universality of the Tutte polynomials for matroids

Matroid theory - some definitions

Definition 1.1

A **matroid** $M = (E, \mathcal{I})$ is a pair $(E, \mathcal{I})$

- a finite set: E
- a collection of subsets of E: $\mathcal{I}$

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The set E : the ground set of the matroid
The members of $\mathcal{I}$: the independent sets of the matroid.

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Example 1.3

Let $E = \{1, 2\}$ and $\mathcal{I} = \{\emptyset, \{1\}, \{2\}\}$. One has the uniform matroid $U_{1,2}$.

Rank function

Definition 1.4

Let $M = (E, \mathcal{I})$ be a matroid and $A \subset E$. The rank function of A:

$$r(A) = max\{|B| : B \in \mathcal{I}, B \subset A\}. \quad (1)$$

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The nullity function of A:

$$n(A) = |A| - r(A). \quad (2)$$

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The element e is called a coloop if $r(E - \{e\}) = r(E) - 1$.

Two operations [1]

Definition 1.6 (Deletion)

One sets the collection of subsets that

$$\mathcal{I}' = \{I \subset E - T : I \in \mathcal{I}\}. \tag{3}$$

Then one has that the pair $(E - T, \mathcal{I}')$ is a matroid, called that the deletion of T from M.
$\hookrightarrow$ One denotes that $M\backslash_T$

Two operations [2]

Definition 1.7 (Contraction)

One sets the collection of subsets that

$$\mathcal{I}'' = \{I \subset E - T : I \cup B_T \in \mathcal{I}\}, \quad (4)$$

where B_T is a maximal independent subset of T.
Then one has that the pair $(E - T, \mathcal{I}'')$ is a matroid, called that the contraction of T from M.
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Tutte polynomial for matroids

Definition 1.8

The Tutte polynomial of matroid M:

$$T_M(x,y) = \sum_{A\subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{n(A)}. \tag{5}$$

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Theorem 1.10

Deletion-contraction relation:

$$T_M(x, y) = T_{M/e}(x, y) + T_{M \backslash e}(x, y). \tag{7}$$

Hopf algebra on matroids

(H. Crapo and W. Schmitt. A free subalgebra of the algebra of matroids. EJC, 26(7), 05.)

Coproduct

$$\Delta(M) = \sum_{A \subseteq E} M|A \otimes M/A. \qquad (8)$$

$$\epsilon(M) = \begin{cases} 1, & \text{if } M = U_{0,0}, \\ 0 & \text{otherwise.} \end{cases}$$

$$\Delta(U_{1,2}) = 1 \otimes U_{1,2} + 2U_{1,1} \otimes U_{0,1} + U_{1,2} \otimes 1.$$

$(k(\widetilde{\mathcal{M}}), \oplus, \mathbf{1}, \Delta, \epsilon)$ is bialgebra.
Moreover, this bialgebra is graded by the cardinal of the ground set, then it is a Hopf algebra.

Two infinitesimal characters

Let us define two linear forms.

$$\delta_{\mathrm{loop}}(M) = \begin{cases} 1_{\mathbb{K}} \text{ if } M = U_{0,1}, \\ 0_{\mathbb{K}} \text{ otherwise }. \end{cases} \tag{9}$$

$$\delta_{\mathrm{coloop}}(M) = \begin{cases} 1_{\mathbb{K}} \text{ if } M = U_{1,1}, \\ 0_{\mathbb{K}} \text{ otherwise }. \end{cases} \tag{10}$$

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One has

$$\delta_{\mathrm{loop}}(M_1 \oplus M_2) = \delta_{\mathrm{loop}}(M_1)\epsilon(M_2) + \epsilon(M_1)\delta_{\mathrm{loop}}(M_2).$$

$$\delta_{\mathrm{coloop}}(M_1 \oplus M_2) = \delta_{\mathrm{coloop}}(M_1)\epsilon(M_2) + \epsilon(M_1)\delta_{\mathrm{coloop}}(M_2).$$

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$$\delta_{\mathrm{loop}}(M_1\oplus M_2)=\delta_{\mathrm{loop}}(M_1)\epsilon(M_2)+\epsilon(M_1)\delta_{\mathrm{loop}}(M_2).$$
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Theorem 2.1

$exp_*\{a\delta_{\mathrm{coloop}}+b\delta_{\mathrm{loop}}\}$ is a Hopf algebra character.

$$exp_*\{a\delta_{\mathrm{coloop}}+b\delta_{\mathrm{loop}}\}(M)=a^{r(M)}b^{n(M)}. \tag{11}$$

A mapping α

Let us define

$$\alpha(x, y, s, M) := \exp_* s\{\delta_{\rm coloop} + (y-1)\delta_{\rm loop}\} * \exp_* s\{(x-1)\delta_{\rm coloop} + \delta_{\rm loop}\}(M). \quad (12)$$

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Proposition 3.1

α is a Hopf algebra character. Moreover, one has

$$\alpha(x, y, s, M) = s^{|E|} T_M(x, y). \tag{13}$$

A convolution formula for Tutte polynomials

The character α can be rewritten:

$$\begin{aligned} \alpha(x, y, s, M) &= \exp_* \left(s(\delta_{\mathrm{coloop}} + (y-1)\delta_{\mathrm{loop}})\right) * \exp_* \left(s(-\delta_{\mathrm{coloop}} + \delta_{\mathrm{loop}})\right) \\ &* \exp_* \left(s(\delta_{\mathrm{coloop}} - \delta_{\mathrm{loop}})\right) * \exp_* \left(s((x-1)\delta_{\mathrm{coloop}} + \delta_{\mathrm{loop}})\right) \end{aligned} \quad (14)$$

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Corollary 3.2 (Theorem 1 of [KRS99])

The Tutte polynomial satisfies

$$T_M(x,y) = \sum_{A \subset E} T_{M|A}(0,y)\, T_{M/A}(x,0). \quad (15)$$

W. Kook, V. Reiner, and D. Stanton. A Convolution Formula for the Tutte Polynomial. Journal of Combinatorial Series (99)

Differential equation of character α

Proposition 4.1

The character α is the solution of the differential equation:

$$\frac{d\alpha}{ds}(M) = (x\alpha * \delta_{\text{coloop}} + y\delta_{\text{loop}} * \alpha + [\delta_{\text{coloop}}, \alpha]_* - [\delta_{\text{loop}}, \alpha]_*)(M). \tag{16}$$

We take a four-variable matroid polynomial $Q_M(x, y, a, b)$ which has the following properties:

- a multiplicative law

$$Q_{M_1 \oplus M_2}(x, y, a, b) = Q_{M_1}(x, y, a, b) Q_{M_2}(x, y, a, b), \tag{17}$$

- if e is a coloop, then

$$Q_M(x, y, a, b) = x Q_{M \setminus e}(x, y, a, b), \tag{18}$$

- if e is a loop, then

$$Q_M(x, y, a, b) = y Q_{M/e}(x, y, a, b), \tag{19}$$

- if e is a nonseparating point, then

$$Q_M(x, y, a, b) = a Q_{M \setminus e}(x, y, a, b) + b Q_{M/e}(x, y, a, b). \tag{20}$$

A mapping β

$$\beta(x, y, a, b, s, M) := s^{|E|} Q_M(x, y, a, b). \tag{21}$$

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Proposition 4.3

The character β satisfies the following differential equation:

$$\frac{d\beta}{ds}(M) = (x\beta * \delta_{\mathrm{coloop}} + y\delta_{\mathrm{loop}} * \beta + b[\delta_{\mathrm{coloop}}, \beta]_* - a[\delta_{\mathrm{loop}}, \beta]_*)\,(M). \tag{22}$$

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Sketch of the proof: Using the definitions of the infitesimal character δ_{loop} and δ_{coloop} and the conditions of the polynomial $Q_M(x, y, a, b)$, one can get the result.

Main theorem

From the propositions 3.1, 4.1 and 4.3, one gets the result

Theorem 4.4

$$Q(x, y, a, b, M) = a^{n(M)} b^{r(M)} T_M(\frac{x}{b}, \frac{y}{a}). \tag{23}$$

Thank you for your attention!